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Lectures ULB

Content Part 2:

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I. Kaluza-Klein reduction on S^1

In this section we are working out the dimensional reduction of gravity in D+1 dimension down to D dimensions. As we will see, this provides a unification of the form:

D+1 Gravity $\Rightarrow$ Gravity + Maxwell + scalar in D

We will describe gravity in D+1 dimensions:

$$S_{D+1} = \frac{1}{2\kappa^2_{D+1}} \int d^{D+1}x \sqrt{-|\hat{g}|}\ \hat{R}$$

with $\hat{g}_{MN}$ and $\hat{R}_{MN}$ being the metric and Ricci scalar in a (D+1) dimensional space-time $\underbrace{M}_{x^M} = \underbrace{0, 1, \dots, D-1}_{x^\mu},\ \underbrace{z}_{z}$.

Let's take the z-coordinate to be S^1 $\Rightarrow$ Fourier expansion

$$\hat{g}_{MN}(x, z) = \sum_{n=0}^{\infty} \hat{g}^{(n)}_{MN}(x)\, e^{i\frac{n}{L}z}$$

$\Uparrow$ Fourier mode


S^1

$(z \to z + 2\pi L)$

$\Rightarrow$ The zero-mode $(n=0)$ is a massless mode whereas $n \neq 0$ corresponds to a tower of massive modes (KK tower).

Example: Scalar field $\hat{\phi}$ in D+1 dimensional flat space-time

$$S_\phi = \int d^{D+1}x\ \partial_M \hat{\phi}\, \partial^M \hat{\phi} \ \underset{\text{E.O.M}}{\Rightarrow}\ \hat{\Box}\hat{\phi} = \partial_M \partial^M \hat{\phi} = 0$$

Fourier expansion along S^1 : $\hat{\phi}(x,z) = \sum_{n=0}^{\infty} \phi^{(n)}(x)\, e^{i\frac{n}{L}z}$

so that

$$\hat{\Box}\hat{\phi} = (\underbrace{\partial_\mu \partial^\mu}_{\Box} + \partial_z^2)\hat{\phi} = \sum_{n=0}^{\infty} \underbrace{\left[\Box\phi^{(n)} - \frac{n^2}{L^2}\phi^{(n)}\right]}_{} e^{i\frac{n}{L}z} = 0$$

Each mode must satisfy

$$\Box\phi^{(n)} - \underbrace{\frac{n^2}{L^2}}_{}\phi^{(n)} = 0$$

$m^2 \equiv \frac{n^2}{L^2}$ => Massive modes !!

$m = \frac{|n|}{L}$

Important : The KK phylosophy is to assume a very small L (we don't observe S^1) so that all the modes with $n \neq 0$ are very massive $m = \frac{|n|}{L}$ and we have to reach very high energies in order to produce such KK modes :

$$L \sim m_p \sim 10^{-33}\,\text{cm} \Rightarrow m \sim 10^{-5}\,\text{gr}$$

(Higgs mass $\sim m_{top} \sim 10^{-22}$ gr)

Important : KK reduction = Truncation to $n=0$ massless modes

=> z - independence !!

$$\boxed{\hat{g}_{MN}(x) = \hat{g}^{(0)}_{MN}(x)}$$

The first step to perform a KK reduction is to split the (D+1)-metric in D-dimensional blocks:

$$\hat{g}_{MN}(x) = \begin{bmatrix} \hat{g}_{\mu\nu} & \hat{g}_{\mu z} \\ \hat{g}_{z\nu} & g_{zz} \end{bmatrix} \equiv \begin{bmatrix} g_{\mu\nu} & A_\mu \\ A_\nu & \phi \end{bmatrix} \Rightarrow \text{Too naive (big mess)}$$

$$\equiv \begin{bmatrix} e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ e^{2\beta\phi} A_\nu & e^{2\beta\phi} \end{bmatrix}$$

$$\Downarrow$$

Much more convenient !!

(see discussion on symmetries)

Therefore we parameterise the (D+1) metric $\hat{g}_{MN}$ as

$\phi \equiv$ "Dilaton"

$$\boxed{d\hat{s}^2_{D+1} = e^{2\alpha\phi} ds_D^2 + e^{2\beta\phi} (dz + A_\mu dx^\mu)^2}$$

with α and β being constants to be chosen later on.

The associated (D+1) frame then reads:

$$\hat{e}_M{}^A = \begin{bmatrix} e^{\alpha\phi} e_\mu{}^a & e^{\beta\phi} A_\mu \\ 0_{1\times 4} & e^{\beta\phi} \end{bmatrix} \qquad \boxed{\begin{array}{l} M = \mu, z \\ A = a, \underline{z} \end{array}}$$

Equivalently: $\hat{e}^a = e^{\alpha\phi} \underbrace{e^a}_{e_\mu{}^a dx^\mu}$ and $\hat{e}^{\underline{z}} = e^{\beta\phi}(dz + A)$ with $A \equiv A_\mu dx^\mu$

Ex: Check that $\hat{e}_M{}^A \; \hat{e}_N{}^B \; \hat{\eta}_{AB} = \hat{g}_{MN}$

$$\begin{bmatrix} e^{\alpha\phi} e_\mu{}^a & e^{\beta\phi} A_\mu \\ 0_{1\times4} & e^{\beta\phi} \end{bmatrix} \underbrace{\begin{bmatrix} \eta_{ab} & 0_{4\times1} \\ 0_{1\times4} & +1 \end{bmatrix} \begin{bmatrix} e^{\alpha\phi} e_\nu{}^b & 0_{4\times1} \\ e^{\beta\phi} A_\nu & e^{\beta\phi} \end{bmatrix}}_{\begin{bmatrix} e^{\alpha\phi} e_{\nu a} & 0_{4\times1} \\ e^{\beta\phi} A_\nu & e^{\beta\phi} \end{bmatrix}}$$

$$= \begin{bmatrix} e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ e^{2\beta\phi} A_\nu & e^{2\beta\phi} \end{bmatrix} = \hat{g}_{MN}(x)$$

In the following our goal will be to compute S_{D+1} using the $(D+1)$-dimensional frame $\hat{e}_M{}^A$ given above:

$$S_{D+1} = \frac{1}{2K^2_{D+1}} \int d^{D+1}x \;\; \hat{e} \;\; \hat{e}_A{}^M \; \hat{e}_B{}^N \; \hat{R}_{MN}{}^{AB}(\hat{e})$$

- $\hat{e} = e^{(\alpha D + \beta)\phi} \; e$

$A_a = e_a{}^\nu A_\nu$

- We need the inverse $(D+1)$-dim frame $\hat{e}_A{}^N$

$$\hat{e}_M{}^A \cdot \hat{e}_A{}^N = \delta_M{}^N \implies \hat{e}_A{}^N = \begin{bmatrix} e^{-\alpha\phi} e_a{}^\nu & -e^{-\alpha\phi} A_a \\ 0_{1\times4} & e^{-\beta\phi} \end{bmatrix}$$

Ex: check that $\hat{e}_M{}^A \; \hat{e}_A{}^N = \delta_M{}^N$

$$\begin{bmatrix} e^{\alpha\phi} e_\mu{}^a & e^{\beta\phi} A_\mu \\ 0_{1\times4} & e^{\beta\phi} \end{bmatrix} \begin{bmatrix} e^{-\alpha\phi} e_a{}^\nu & -e^{-\alpha\phi} A_a \\ 0_{1\times4} & e^{-\beta\phi} \end{bmatrix} = \begin{bmatrix} \delta_\mu{}^\nu & 0_{4\times1} \\ 0_{1\times4} & 1 \end{bmatrix}$$

⊙ Now we perform the computation of the Ricci scalar $\hat{R}$.

▲ First we compute the aholonomy coefficients $\hat{\Omega}$:

$$\hat{\Omega}_{[MN]P} = \left(\partial_M \hat{e}_N{}^A - \partial_N \hat{e}_M{}^A\right) \hat{e}_{PA}$$

- $\hat{\Omega}_{[\mu\nu]\rho} = (\partial_\mu \hat{e}_\nu{}^A - \partial_\nu \hat{e}_\mu{}^A) \hat{e}_{\rho A}$

$$= (\partial_\mu \hat{e}_\nu{}^a - \partial_\nu \hat{e}_\mu{}^a) \hat{e}_{\rho a} + (\partial_\mu \hat{e}_\nu{}^{\underline{z}} - \partial_\nu \hat{e}_\mu{}^{\underline{z}}) \hat{e}_{\rho \underline{z}}$$

$$= \left[\partial_\mu (e^{\alpha\phi} e_\nu{}^a) - \partial_\nu (e^{\alpha\phi} e_\mu{}^a)\right] (e^{\alpha\phi} e_{\rho a})$$

$$+ \left[\partial_\mu (e^{\beta\phi} A_\nu) - \partial_\nu (e^{\beta\phi} A_\mu)\right] (e^{\beta\phi} A_\rho)$$

$$= e^{2\alpha\phi} \left[(\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a) e_{\rho a} + \alpha (\partial_\mu \phi\, e_\nu{}^a - \partial_\nu \phi\, e_\mu{}^a) e_{\rho a} \right]$$

$$+ e^{2\beta\phi} \left[F_{\mu\nu} A_\rho + \beta (\partial_\mu \phi A_\nu - \partial_\nu \phi A_\mu) A_\rho \right]$$

$$= e^{2\alpha\phi} \left[\Omega_{[\mu\nu]\rho} + 2\alpha\, \partial_{[\mu} \phi\, e_{\nu]}{}^a e_{\rho a} \right]$$

$$+ e^{2\beta\phi} \left[F_{\mu\nu} A_\rho + 2\beta\, \partial_{[\mu} \phi\, A_{\nu]} A_\rho \right]$$

- $\hat{\Omega}_{[\mu\nu]z} = (\partial_\mu \hat{e}_\nu{}^A - \partial_\nu \hat{e}_\mu{}^A) \hat{e}_{zA} = (\partial_\mu \hat{e}_\nu{}^{\underline{z}} - \partial_\nu \hat{e}_\mu{}^{\underline{z}}) \hat{e}_{z\underline{z}}$

$$= \left[\partial_\mu (e^{\beta\phi} A_\nu) - \partial_\nu (e^{\beta\phi} A_\mu) \right] e^{\beta\phi}$$

$$= e^{2\beta\phi} \left[F_{\mu\nu} + 2\beta\, \partial_{[\mu} \phi\, A_{\nu]} \right]$$

- $\hat{\Omega}_{[\mu z]\rho} = \partial_\mu \hat{e}_z{}^A \hat{e}_{\rho A} = \partial_\mu \hat{e}_z{}^{\underline{z}} \hat{e}_{\rho \underline{z}} = \partial_\mu (e^{\beta\phi}) (e^{\beta\phi} A_\rho)$

$$= e^{2\beta\phi} \beta\, \partial_\mu \phi\, A_\rho$$

- $\hat{\Omega}_{[\mu z]z} = \partial_\mu \hat{e}_z{}^A \hat{e}_{zA} = \partial_\mu \hat{e}_z{}^{\underline{z}} \hat{e}_{z\underline{z}} = \partial_\mu (e^{\beta\phi}) e^{\beta\phi}$
$$= e^{2\beta\phi} \beta \partial_\mu \phi$$

- $\hat{\Omega}_{[z\nu]\rho} = -\partial_\nu \hat{e}_z{}^A \hat{e}_{\rho A} = -\partial_\nu \hat{e}_z{}^{\underline{z}} \hat{e}_{\rho \underline{z}} = -\partial_\nu (e^{\beta\phi})(e^{\beta\phi} A_\rho)$
$$= -e^{2\beta\phi} \beta \partial_\nu \phi A_\rho$$

- $\hat{\Omega}_{[z\nu]z} = -\partial_\nu \hat{e}_z{}^A \hat{e}_{zA} = -\partial_\nu \hat{e}_z{}^{\underline{z}} \hat{e}_{z\underline{z}} = -\partial_\nu (e^{\beta\phi})(e^{\beta\phi})$
$$= -e^{2\beta\phi} \beta \partial_\nu \phi$$

- $\hat{\Omega}_{[zz]\rho} = \hat{\Omega}_{[zz]z} = 0$

- Using $\hat{\Omega}$ we compute the spin connection with all indices curved

$$\hat{\omega}_{M[NP]}(\hat{e}) = \frac{1}{2}\left(\hat{\Omega}_{[MN]P} - \hat{\Omega}_{[NP]M} + \hat{\Omega}_{[PM]N}\right)$$
$$= \hat{\omega}_M{}^{BC}(\hat{e})\, \hat{e}_{NB}\, \hat{e}_{PC}$$

- $\hat{\omega}_{\mu[\nu\rho]} = \frac{1}{2}\left(\hat{\Omega}_{[\mu\nu]\rho} - \hat{\Omega}_{[\nu\rho]\mu} + \hat{\Omega}_{[\rho\mu]\nu}\right)$

$$= \frac{1}{2}\Big[e^{2\alpha\phi}\Big(2\,\omega_{\mu[\nu\rho]} + 2\alpha\big(\partial_{[\mu}\phi\, e_{\nu]}{}^a e_{\rho a} - \partial_{[\nu}\phi\, e_{\rho]}{}^a e_{\mu a} + \partial_{[\rho}\phi\, e_{\mu]}{}^a e_{\nu a}\big)\Big)$$
$$+ e^{2\beta\phi}\Big(F_{\mu\nu} A_\rho - F_{\nu\rho} A_\mu + F_{\rho\mu} A_\nu + 2\beta\big(\partial_{[\mu}\phi A_{\nu]} A_\rho - \partial_{[\nu}\phi A_{\rho]} A_\mu$$
$$+ \partial_{[\rho}\phi A_{\mu]} A_\nu\big)\Big)\Big]$$

- $\hat{\omega}_{z[\nu\rho]} = \frac{1}{2}\left(\hat{\Omega}_{[z\nu]\rho} - \hat{\Omega}_{[\nu\rho]z} + \hat{\Omega}_{[\rho z]\nu}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(-\beta\,\partial_\nu\phi\, A_\rho - (F_{\nu\rho} + 2\beta\,\partial_{[\nu}\phi\, A_{\rho]}) + \beta\,\partial_\rho\phi\, A_\nu\right)\right]$$

$$= \frac{1}{2}e^{2\beta\phi}\left(-F_{\nu\rho} - 4\beta\,\partial_{[\nu}\phi A_{\rho]}\right)$$

- $\hat{\omega}_{\mu[\nu z]} = \frac{1}{2}\left(\hat{\Omega}_{[\mu\nu]z} - \hat{\Omega}_{[\nu z]\mu} + \hat{\Omega}_{[z\mu]\nu}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left((F_{\mu\nu} + 2\beta\,\partial_{[\mu}\phi\, A_{\nu]}) - \beta\,\partial_\nu\phi\, A_\mu - \beta\,\partial_\mu\phi\, A_\nu\right)\right]$$

$$= \frac{1}{2}e^{2\beta\phi}\left(F_{\mu\nu} - 2\beta\,\partial_\nu\phi\, A_\mu\right)$$

- $\hat{\omega}_{z[\nu z]} = \frac{1}{2}\left(\hat{\Omega}_{[z\nu]z} - \hat{\Omega}_{[\nu z]z} + \overbrace{\hat{\Omega}_{[zz]\nu}}^{0}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(-\beta\,\partial_\nu\phi - \beta\,\partial_\nu\phi\right)\right] = -e^{2\beta\phi}\beta\,\partial_\nu\phi$$

- $\hat{\omega}_{\mu[z\rho]} = \frac{1}{2}\left(\hat{\Omega}_{[\mu z]\rho} - \hat{\Omega}_{[z\rho]\mu} + \hat{\Omega}_{[\rho\mu]z}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(\beta\,\partial_\mu\phi A_\rho + \beta\,\partial_\rho\phi A_\mu + (F_{\rho\mu} + 2\beta\,\partial_{[\rho}\phi\, A_{\mu]})\right)\right]$$

$$= \frac{1}{2}e^{2\beta\phi}\left(F_{\rho\mu} + 2\beta\,\partial_\rho\phi A_\mu\right)$$

- $\hat{\omega}_{z[z\rho]} = \frac{1}{2}\left(\underbrace{\hat{\Omega}_{[zz]\rho}}_{0} - \hat{\Omega}_{[z\rho]z} + \hat{\Omega}_{[\rho z]z}\right)$

$$= \frac{1}{2}\left[e^{2\beta\phi}\left(+\beta\,\partial_\rho\phi + \beta\,\partial_\rho\phi\right)\right] = e^{2\beta\phi}\beta\,\partial_\rho\phi$$

- $\hat{\omega}_{\mu[zz]} = \hat{\omega}_{z[zz]} = 0$

▲ Using the above resuets we compute the standard spin connection

$$\hat{\omega}_M{}^{BC} = \hat{\omega}_{M[NP]}\,\hat{e}^{BN}\,\hat{e}^{CP}$$

• $\hat{\omega}_\mu{}^{bc} = \hat{\omega}_{\mu[NP]}\,\hat{e}^{bN}\,\hat{e}^{cP}$

$$= \hat{\omega}_{\mu[\nu\rho]}\hat{e}^{b\nu}\hat{e}^{c\rho} + \hat{\omega}_{\mu[\nu z]}\hat{e}^{b\nu}\hat{e}^{cz} + \hat{\omega}_{\mu[z\rho]}\hat{e}^{bz}\hat{e}^{c\rho} + \overbrace{\hat{\omega}_{\mu[zz]}}^{0}\hat{e}^{bz}\hat{e}^{cz}$$

$$= \hat{\omega}_{\mu[\nu\rho]}\, e^{-2\alpha\phi} e^{b\nu} e^{c\rho} - \hat{\omega}_{\mu[\nu z]} e^{-2\alpha\phi} e^{b\nu} A^c - \hat{\omega}_{\mu[z\rho]} e^{-2\alpha\phi} A^b e^{c\rho}$$

$$= \frac{1}{2}\Big[2\,\omega_\mu{}^{[bc]} + 2\alpha\Big(\partial_{[\mu}\phi\, e_{\nu]}{}^a e_{\rho a} e^{b\nu} e^{c\rho} - \partial_{[\nu}\phi\, e_{\rho]}{}^a e_{\mu a} e^{b\nu} e^{c\rho}$$

$$+ \partial_{[\rho}\phi\, e_{\mu]}{}^a e_{\nu a} e^{b\nu} e^{c\rho}\Big) + e^{2(\beta-\alpha)\phi}\Big(F_{\mu\nu} A_\rho e^{b\nu} e^{c\rho} -$$

$$- F_{\nu\rho} A_\mu e^{b\nu} e^{c\rho} + F_{\rho\mu} A_\nu e^{b\nu} e^{c\rho}\Big) +$$

$$+ 2\beta e^{2(\beta-\alpha)\phi}\Big(\partial_{[\mu}\phi A_{\nu]} A_\rho e^{b\nu} e^{c\rho} - \partial_{[\nu}\phi A_{\rho]} A_\mu e^{b\nu} e^{c\rho}$$

$$+ \partial_{[\rho}\phi A_{\mu]} A_\nu e^{b\nu} e^{c\rho}\Big)\Big]$$

$$- \frac{1}{2}\Big[e^{2(\beta-\alpha)\phi}(F_{\mu\nu} - 2\beta\partial_\nu\phi A_\mu) e^{b\nu} A^c\Big] - \frac{1}{2}\Big[e^{2(\beta-\alpha)\phi}(F_{\rho\mu} + 2\beta\,\partial_\rho\phi A_\mu) A^b e^{c\rho}\Big]$$

$$= (*)$$

NOTE 1: $2\alpha\,\frac{1}{2}\big(\partial_\mu\phi\, e_\nu{}^a e_{\rho a} e^{b\nu} e^{c\rho} - \partial_\nu\phi\, e_\mu{}^a e_{\rho a} e^{b\nu} e^{c\rho}$

$- \partial_\nu\phi\, e_\rho{}^a e_{\mu a} e^{b\nu} e^{c\rho} + \partial_\rho\phi\, e_\nu{}^a e_{\mu a} e^{b\nu} e^{c\rho}$

$+ \partial_\rho\phi\, e_\mu{}^a e_{\nu a} e^{b\nu} e^{c\rho} - \partial_\mu\phi\, e_\rho{}^a e_{\nu a} e^{b\nu} e^{c\rho}\big)$

$$= \alpha\big(\underline{\partial_\mu\phi\,\eta^{bc}} - \partial_\nu\phi\, e^{b\nu} e_\mu{}^c - \partial_\nu\phi\, e_\mu{}^c e^{b\nu} + \partial_\rho\phi\, e_\mu{}^b e^{c\rho}$$

$$+ \partial_\rho\phi\, e_\mu{}^b e^{c\rho} - \underline{\partial_\mu\phi\,\eta^{bc}}\big)$$

$$= \alpha\big(2\,\partial_\rho\phi\, e_\mu{}^b e^{c\rho} - 2\,\partial_\nu\phi\, e_\mu{}^c e^{b\nu}\big) = \quad [\partial^a \equiv e^{a\rho}\partial_\rho]$$

$$= 2\alpha\big(e_\mu{}^b\partial^c\phi - e_\mu{}^c\partial^b\phi\big) = 4\alpha\, e_\mu{}^{[b}\partial^{c]}\phi$$

$$= 4\alpha\,\partial^{[c}\phi\, e_\mu{}^{b]}$$

NOTE 2: $e^{2(\beta-\alpha)\phi}\left(F_{\mu\nu} A_\rho e^{b\nu} e^{c\rho} - F_{\nu\rho} A_\mu e^{b\nu} e^{c\rho} + F_{\rho\mu} A_\nu e^{b\nu} e^{c\rho}\right)$

$$= e^{2(\beta-\alpha)\phi}\left(F_\mu{}^b A^c - F^{bc} A_\mu + F^c{}_\mu A^b\right)$$

$$= e^{2(\beta-\alpha)\phi}\left(2 F_\mu{}^{[b} A^{c]} - F^{bc} A_\mu\right)$$

NOTE 3: $2\beta e^{2(\beta-\alpha)\phi}\frac{1}{2}\Big(\partial_\mu\phi A_\nu A_\rho e^{b\nu} e^{c\rho} - \partial_\nu\phi A_\mu A_\rho e^{b\nu} e^{c\rho} - \partial_\nu\phi A_\rho A_\mu e^{b\nu} e^{c\rho} + \partial_\rho\phi A_\nu A_\mu e^{b\nu} e^{c\rho} + \partial_\rho\phi A_\mu A_\nu e^{b\nu} e^{c\rho} - \partial_\mu\phi A_\rho A_\nu e^{b\nu} e^{c\rho}\Big)$

$$= \beta e^{2(\beta-\alpha)\phi}\Big(\partial_\mu\phi A^b A^c - \partial^b\phi A_\mu A^c - \partial^b\phi A^c A_\mu + \partial^c\phi A^b A_\mu + \partial^c\phi A_\mu A^b - \partial_\mu\phi A^c A^b\Big)$$

$$= \beta e^{2(\beta-\alpha)\phi}\left(-2 A_\mu \partial^b\phi A^c + 2 A_\mu \partial^c\phi A^b\right)$$

$$= -4\beta e^{2(\beta-\alpha)\phi} A_\mu \partial^{[b}\phi A^{c]}$$

$$(*) = \frac{1}{2}\Big[2\omega_\mu{}^{[bc]} - 4\alpha\, e_\mu{}^{[c}\partial^{b]}\phi + e^{2(\beta-\alpha)\phi}\left(2F_\mu{}^{[b}A^{c]} - F^{bc}A_\mu\right) - 4\beta e^{2(\beta-\alpha)\phi} A_\mu \partial^{[b}\phi A^{c]}\Big]$$
$$-\frac{1}{2}e^{2(\beta-\alpha)\phi}\Big[\underbrace{F_\mu{}^b A^c - F_\mu{}^c A^b}_{2F_\mu{}^{[b}A^{c]}} - 2\beta\underbrace{\left(\partial^b\phi A^c A_\mu - \partial^c\phi A^b\right)}_{2\partial^{[b}\phi A^{c]}} A_\mu\Big]$$

$$= \frac{1}{2}\Big[2\omega_\mu{}^{[bc]} - 4\alpha\, e_\mu{}^{[c}\partial^{b]}\phi - e^{2(\beta-\alpha)\phi}F^{bc}A_\mu + e^{2(\beta-\alpha)\phi}\Big(\underline{2F_\mu{}^{[b}A^{c]}} - \underline{4\beta A_\mu\partial^{[b}\phi A^{c]}} - \underline{2F_\mu{}^{[b}A^{c]}} + \underline{4\beta\partial^{[b}\phi A^{c]}}\Big)\Big]$$

$$= \frac{1}{2}\Big[2\omega_\mu{}^{[bc]} + 4\alpha\,\partial^{[c}\phi\, e_\mu{}^{b]} - e^{2(\beta-\alpha)\phi}F^{bc}A_\mu\Big] =$$

$$= \omega_\mu{}^{[bc]} + \alpha\,(\partial^c\phi\, e_\mu{}^b - \partial^b\phi\, e_\mu{}^c) - \frac{1}{2}\, e^{2(\beta-\alpha)\phi} F^{bc} A_\mu .$$

- $$\hat\omega_z{}^{bc} = \hat\omega_{z[NP]}\,\hat e^{bN}\,\hat e^{cP}$$

$$= \hat\omega_{z[\nu\rho]}\hat e^{b\nu}\hat e^{c\rho} + \hat\omega_{z[\nu z]}\hat e^{b\nu}\hat e^{cz} + \hat\omega_{z[z\rho]}\hat e^{bz}\hat e^{c\rho} + \overbrace{\hat\omega_{z[zz]}}^{0}\hat e^{bz}\hat e^{cz}$$

$$= \hat\omega_{z[\nu\rho]}\, e^{-2\alpha\phi} e^{b\nu} e^{c\rho} - \hat\omega_{z[\nu z]}\, e^{-2\alpha\phi} e^{b\nu} A^c - \hat\omega_{z[z\rho]}\, e^{-2\alpha\phi} A^b e^{c\rho}$$

$$= \frac{1}{2} e^{2(\beta-\alpha)\phi}\left[-F_{\nu\rho} - 4\beta\,\partial_{[\nu}\phi A_{\rho]}\right] e^{b\nu} e^{c\rho} + e^{2(\beta-\alpha)\phi}\beta\,\partial_\nu\phi\, e^{b\nu} A^c$$

$$- e^{2(\beta-\alpha)\phi}\beta\,\partial_\rho\phi\, A^b e^{c\rho} =$$

$$= -\frac{1}{2} e^{2(\beta-\alpha)\phi} F^{bc} + e^{2(\beta-\alpha)\phi}\beta\left[\underline{-\partial^b\phi A^c} + \underline{\partial^c\phi A^b} + \underline{\partial^b\phi A^c} - \underline{\partial^c\phi A^b}\right]$$

$$= -\frac{1}{2} e^{2(\beta-\alpha)\phi} F^{bc}.$$

Therefore, using compact notation, we find that

$$\boxed{\hat\omega^{bc} = \omega^{[bc]} + \alpha e^{-\alpha\phi}\,(\partial^c\phi\,\hat e^b - \partial^b\phi\,\hat e^c) - \frac{1}{2} F^{bc} e^{(\beta-2\alpha)\phi}\,\hat e^{\underline{z}}}$$

- $$\hat\omega_\mu{}^{b\underline{z}} = \hat\omega_{\mu[NP]}\,\hat e^{bN}\,\hat e^{\underline{z}P}$$

$$= \hat\omega_{\mu[\nu\rho]}\hat e^{b\nu}\underbrace{\hat e^{\underline{z}\rho}}_{0} + \hat\omega_{\mu[\nu z]}\hat e^{b\nu}\hat e^{\underline{z}z} + \hat\omega_{\mu[z\rho]}\overbrace{\hat e^{bz}}^{0}\hat e^{\underline{z}\rho} + \overbrace{\hat\omega_{\mu[zz]}}^{0}\hat e^{bz}\hat e^{\underline{z}z}$$

$$= \hat\omega_{\mu[\nu z]}\, e^{-(\alpha+\beta)\phi} e^{b\nu} = \frac{1}{2} e^{2\beta\phi}(F_{\mu\nu} - 2\beta\,\partial_\nu\phi A_\mu)\, e^{-(\alpha+\beta)\phi} e^{b\nu}$$

$$= \frac{1}{2} e^{(\beta-\alpha)\phi}(F_\mu{}^b - 2\beta\,\partial^b\phi A_\mu) \quad \to F^b{}_c\, e_\mu{}^c$$

$$= -\beta e^{(\beta-\alpha)\phi}\partial^b\phi A_\mu - \frac{1}{2} e^{(\beta-\alpha)\phi} F^b{}_\mu$$

$$= -e^{(\beta-\alpha)\phi}\left[\beta\,\partial^b\phi A_\mu + \frac{1}{2}F^b{}_\mu\right] = -\hat\omega_\mu{}^{\underline{z}b}$$

- $\hat{\omega}_z{}^{b\underline{z}} = \hat{\omega}_{z[NP]}\, \hat{e}^{bN}\, \hat{e}^{\underline{z}P}$

$$= \hat{\omega}_{z[\nu\rho]}\hat{e}^{b\nu}\underbrace{\hat{e}^{\underline{z}\rho}}_{0} + \hat{\omega}_{z[\nu z]}\hat{e}^{b\nu}\hat{e}^{\underline{z}z} + \hat{\omega}_{z[z\rho]}\overbrace{\hat{e}^{bz}\hat{e}^{\underline{z}\rho}}^{0} + \overbrace{\hat{\omega}_{z[zz]}}^{0}\hat{e}^{bz}\hat{e}^{\underline{z}z}$$

$$= \hat{\omega}_{z[\nu z]}\, e^{-(\alpha+\beta)\phi}\, e^{b\nu} = -e^{2\beta\phi}\beta\partial_\nu\phi\, e^{-(\alpha+\beta)\phi} e^{b\nu}$$

$$= -\beta\, e^{(\beta-\alpha)\phi}\,\partial^b\phi \;= -\hat{\omega}_z{}^{\underline{z}b}$$

Using again compact notation we find:

$$\boxed{\hat{\omega}^{b\underline{z}} = -\omega^{\underline{z}b} = -\beta e^{-\alpha\phi}\partial^b\phi\, \hat{e}^z - \frac{1}{2}{}^{(\beta-2\alpha)\phi} F^b{}_c\, \hat{e}^c}$$

- $\hat{\omega}_\mu{}^{\underline{z}\underline{z}} = \hat{\omega}_z{}^{\underline{z}\underline{z}} = 0$

▲ Once we have the spin connection we compute the Riemann tensor

$$\hat{R}_{MN}{}^{BC} = \partial_M\hat{\omega}_N{}^{BC} - \partial_N\hat{\omega}_M{}^{BC} + \hat{\omega}_M{}^B{}_D\,\hat{\omega}_N{}^{DC} - \hat{\omega}_N{}^B{}_D\,\hat{\omega}_M{}^{DC}$$

- $\hat{R}_{\mu\nu}{}^{bc} = \partial_\mu\hat{\omega}_\nu{}^{bc} + \hat{\omega}_\mu{}^b{}_D\,\hat{\omega}_\nu{}^{Dc} - \partial_\nu\hat{\omega}_\mu{}^{bc} - \hat{\omega}_\nu{}^b{}_D\,\hat{\omega}_\mu{}^{Dc}$

$$= \partial_\mu\omega_\nu{}^{bc} + \alpha\,\partial_\mu(\partial^c\phi\, e_\nu{}^b - \partial^b\phi\, e_\nu{}^c) - \frac{1}{2}\partial_\mu\left(e^{2(\beta-\alpha)\phi} F_{bc} A_\nu\right)$$

$$+\hat{\omega}_\mu{}^b{}_d\,\hat{\omega}_\nu{}^{dc} + \hat{\omega}_\mu{}^b{}_{\underline{z}}\,\hat{\omega}_\nu{}^{\underline{z}c} - (\mu\leftrightarrow\nu)$$

$$= \partial_\mu\omega_\nu{}^{bc} + \alpha\,\partial_\mu(\partial^c\phi\, e_\nu{}^b - \partial^b\phi\, e_\nu{}^c) - \frac{1}{2}e^{2(\beta-\alpha)\phi}\Big[2(\beta-\alpha)\partial_\mu\phi F^{bc} A_\nu$$

$$+ \partial_\mu F^{bc} A_\nu + F^{bc}\partial_\mu A_\nu\Big]$$

$$+\Big[\omega_\mu{}^b{}_d + \alpha(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d}) - \frac{1}{2}e^{2(\beta-\alpha)\phi}F^b{}_d A_\mu\Big]$$

$$\Big[\omega_\nu{}^{dc} + \alpha(\partial^c\phi\, e_\nu{}^d - \partial^d\phi\, e_\nu{}^c) - \frac{1}{2}e^{2(\beta-\alpha)\phi}F^{dc} A_\nu\Big]$$

$$- e^{2(\beta-\alpha)\phi}\left[\beta\,\partial^b\phi\, A_\mu + \tfrac12 F^b{}_\mu\right]\left[\beta\,\partial^c\phi\, A_\nu + \tfrac12 F^c{}_\nu\right] - (\mu\leftrightarrow\nu)$$

$$= \overbrace{\partial_\mu \omega_\nu{}^{bc} + \omega_\mu{}^b{}_d\, \omega_\nu{}^{dc}}^{R_{\mu\nu}{}^{bc}} + \alpha\,\partial_\mu\left(\partial^c\phi\, e_\nu{}^b - \partial^b\phi\, e_\nu{}^c\right)$$

$$- e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\, F^{bc}A_\nu + \tfrac12\partial_\mu F^{bc}A_\nu + \tfrac12 F^{bc}\partial_\mu A_\nu$$

$$+\beta^2\, \underline{\partial^b\phi\,\partial^c\phi\, A_\mu A_\nu} + \tfrac12\beta\,\partial^b\phi\, A_\mu F^c{}_\nu + \tfrac12\beta\,\partial^c\phi\, A_\nu F^b{}_\mu + \tfrac14 F^b{}_\mu F^c{}_\nu\Big]$$

$$+\alpha\,\omega_\mu{}^b{}_d\left(\partial^c\phi\, e_\nu{}^d - \partial^d\phi\, e_\nu{}^c\right) - \tfrac12 e^{2(\beta-\alpha)\phi}\,\omega_\mu{}^b{}_d F^{dc}A_\nu$$

$$+\alpha\,\omega_\nu{}^{dc}\left(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d}\right) - \tfrac12\alpha\, e^{2(\beta-\alpha)\phi}\left(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d}\right)F^{dc}A_\nu$$

$$+\alpha^2\left(\partial_\nu\phi\,\partial^c\phi\, e_\mu{}^b - \partial_d\phi\,\partial^d\phi\, e_\mu{}^b e_\nu{}^c - \underline{\partial^b\phi\,\partial^c\phi\, g_{\mu\nu}} + \partial^b\phi\,\partial_\mu\phi\, e_\nu{}^c\right)$$

$$-\tfrac12 e^{2(\beta-\alpha)\phi}\,\omega_\nu{}^{dc} F^b{}_d A_\mu + \underline{\tfrac14 e^{4(\beta-\alpha)\phi} F^b{}_d F^{dc} A_\mu A_\nu}$$

$$-\tfrac12\alpha\, e^{2(\beta-\alpha)\phi}\left(\partial^c\phi\, e_\nu{}^d - \partial^d\phi\, e_\nu{}^c\right)F^b{}_d A_\mu - (\mu\leftrightarrow\nu)$$

NOTE: Underlined terms vanish because they are $\mu\leftrightarrow\nu$ symmetric

$$= \tfrac12 R_{\mu\nu}{}^{bc} + \alpha\left(\partial_\mu\partial^c\phi\, e_\nu{}^b + \partial^c\phi\,\partial_\mu e_\nu{}^b - \partial_\mu\partial^b\phi\, e_\nu{}^c - \partial^b\phi\,\partial_\mu e_\nu{}^c\right)$$

$$- e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\, F^{bc}A_\nu + \tfrac12\beta\,\partial^b\phi\, F^c{}_\nu A_\mu + \tfrac12\beta\,\partial^c\phi\, F^b{}_\mu A_\nu$$

$$+\tfrac12\alpha\,\partial_d\phi\, F^{dc}A_\nu e_\mu{}^b - \tfrac12\alpha\,\partial^b\phi\, F_\mu{}^c A_\nu$$

$$+\tfrac12\alpha\,\partial^c\phi\, F^b{}_\nu A_\mu - \tfrac12\alpha\,\partial^d\phi\, F^b{}_d A_\mu e_\nu{}^c$$

$$+\tfrac12\partial_\mu F^{bc}A_\nu + \tfrac12 F^{bc}\partial_\mu A_\nu + \tfrac14 F^b{}_\mu F^c{}_\nu$$

$$+\tfrac12\omega_\mu{}^b{}_d F^{dc}A_\nu + \tfrac12\omega_\nu{}^{dc}F^b{}_d A_\mu\Big]$$

$$+\alpha\,\omega_\mu{}^b{}_d\left(\partial^c\phi\, e_\nu{}^d - \partial^d\phi\, e_\nu{}^c\right) + \alpha\,\omega_\nu{}^{dc}\left(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d}\right)$$

$$+\alpha^2\left(\partial_\nu\phi\,\partial^c\phi\, e_\mu{}^b + \partial^b\phi\,\partial_\mu\phi\, e_\nu{}^c - \partial_d\phi\,\partial^d\phi\, e_\mu{}^b e_\nu{}^c - (\mu\leftrightarrow\nu)\right.$$

$$\bullet\quad \hat{R}_{\mu\nu}{}^{b\underline{z}} = \partial_\mu \hat{\omega}_\nu{}^{b\underline{z}} + \hat{\omega}_\mu{}^{b}{}_{D}\, \hat{\omega}_\nu{}^{D\underline{z}} - (\mu \leftrightarrow \nu)$$

$$= \partial_\mu \hat{\omega}_\nu{}^{b\underline{z}} + \hat{\omega}_\mu{}^{b}{}_{c}\, \hat{\omega}_\nu{}^{c\underline{z}} + \hat{\omega}_\mu{}^{b}{}_{\underline{z}}\, \overbrace{\hat{\omega}_\nu{}^{\underline{z}\underline{z}}}^{0} - (\mu \leftrightarrow \nu)$$

$$= -\beta\, e^{(\beta-\alpha)\phi}\left[(\beta-\alpha)\partial_\mu\phi\, \partial^b\phi\, A_\nu + \partial_\mu\partial^b\phi\, A_\nu + \partial^b\phi\, \partial_\mu A_\nu\right]$$

$$-\frac{1}{2} e^{(\beta-\alpha)\phi}\left[(\beta-\alpha)\partial_\mu\phi\, F^b{}_\nu + \partial_\mu F^b{}_\nu\right]$$

$$-\left[\omega_\mu{}^b{}_c + \alpha\left(\partial_c\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu c}\right) - \frac{1}{2} e^{2(\beta-\alpha)\phi} F^b{}_c A_\mu\right]$$

$$e^{(\beta-\alpha)\phi}\left[\beta\, \partial^c\phi\, A_\nu + \frac{1}{2} F^c{}_\nu\right] - (\mu \leftrightarrow \nu)$$

$$= -e^{(\beta-\alpha)\phi}\Big[\beta(\beta-\alpha)\partial_\mu\phi\, \partial^b\phi\, A_\nu + \beta\, \partial_\mu\partial^b\phi\, A_\nu + \beta\, \partial^b\phi\, \partial_\mu A_\nu$$

$$+ \frac{1}{2}(\beta-\alpha)\partial_\mu\phi\, F^b{}_\nu + \frac{1}{2}\partial_\mu F^b{}_\nu + \beta\, \omega_\mu{}^b{}_c\, \partial^c\phi\, A_\nu + \frac{1}{2}\omega_\mu{}^b{}_c F^c{}_\nu$$

$$+ \alpha\beta\, \partial_c\phi\, \partial^c\phi\, e_\mu{}^b A_\nu + \frac{1}{2}\alpha\, \partial_c\phi\, e_\mu{}^b F^c{}_\nu$$

$$- \alpha\beta\, \partial_\mu\phi\, \partial^b\phi\, A_\nu - \frac{1}{2}\alpha\, \partial^b\phi\, F_{\mu\nu}\Big]$$

$$+ e^{3(\beta-\alpha)\phi}\left[\underline{\frac{1}{2}\beta\, F^b{}_c\, \partial^c\phi\, A_\mu A_\nu} + \frac{1}{4} F^b{}_c F^c{}_\nu A_\mu\right] - \underline{(\mu \leftrightarrow \nu)}$$

$$= -e^{(\beta-\alpha)\phi}\Big[\beta^2 \partial_\mu\phi\, \partial^b\phi\, A_\nu - 2\alpha\beta\, \partial_\mu\phi\, \partial^b\phi\, A_\nu + \beta\, \partial_\mu\partial^b\phi\, A_\nu$$

$$+ \alpha\beta\, \partial_c\phi\, \partial^c\phi\, e_\mu{}^b A_\nu + \frac{1}{2}\alpha\, \partial_c\phi\, F^c{}_\nu\, e_\mu{}^b + \frac{1}{2}(\beta-\alpha)\partial_\mu\phi\, F^b{}_\nu$$

$$+ \beta\, \partial^b\phi\, \partial_\mu A_\nu - \frac{1}{2}\alpha\, \partial^b\phi\, F_{\mu\nu} + \frac{1}{2}\partial_\mu F^b{}_\nu$$

$$+ \beta\, \omega_\mu{}^b{}_c\, \partial^c\phi\, A_\nu + \frac{1}{2}\omega_\mu{}^b{}_c F^c{}_\nu\Big]$$

$$+ e^{3(\beta-\alpha)\phi}\, \frac{1}{4} F^b{}_c F^c{}_\nu A_\mu - (\mu \leftrightarrow \nu) = -\hat{R}_{\mu\nu}{}^{\underline{z} b}$$

- $\hat{R}_{\mu z}{}^{bc} = \partial_\mu \hat{\omega}_z{}^{bc} + \hat{\omega}_\mu{}^b{}_D\, \hat{\omega}_z{}^{Dc} - (\mu \leftrightarrow z)$

$$= \partial_\mu \hat{\omega}_z{}^{bc} + \hat{\omega}_\mu{}^b{}_d\, \hat{\omega}_z{}^{dc} + \hat{\omega}_\mu{}^b{}_{\underline{z}}\, \hat{\omega}_z{}^{\underline{z}c}$$

$$- \underbrace{\partial_z \hat{\omega}_\mu{}^{bc}}_{0} - \hat{\omega}_z{}^b{}_d\, \hat{\omega}_\mu{}^{dc} - \hat{\omega}_z{}^b{}_{\underline{z}}\, \hat{\omega}_\mu{}^{\underline{z}c}$$

$$= -\frac{1}{2} e^{2(\beta-\alpha)\phi}\left[2(\beta-\alpha)\partial_\mu\phi\, F^{bc} + \partial_\mu F^{bc}\right]$$

$$-\left[\omega_\mu{}^b{}_d + \alpha(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d}) - \frac{1}{2}e^{2(\beta-\alpha)\phi} F^b{}_d A_\mu\right]\frac{1}{2}e^{2(\beta-\alpha)\phi} F^{dc}$$

$$- e^{(\beta-\alpha)\phi}\left[\beta\partial^b\phi\, A_\mu + \frac{1}{2}F^b{}_\mu\right]\beta e^{(\beta-\alpha)\phi}\partial^c\phi$$

$$+ \frac{1}{2}e^{2(\beta-\alpha)\phi} F^b{}_d\left[\omega_\mu{}^{dc} + \alpha(\partial^c\phi\, e_\mu{}^d - \partial^d\phi\, e_\mu{}^c) - \frac{1}{2}e^{2(\beta-\alpha)\phi}F^{dc}A_\mu\right]$$

$$+ \beta e^{(\beta-\alpha)\phi}\partial^b\phi\; e^{(\beta-\alpha)\phi}\left[\beta\partial^c\phi\, A_\mu + \frac{1}{2}F^c{}_\mu\right]$$

$$= -e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\, F^{bc} + \frac{1}{2}\partial_\mu F^{bc} + \frac{1}{2}\omega_\mu{}^b{}_d F^{dc}$$

$$+ \frac{1}{2}\alpha(\partial_d\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu d})F^{dc} + \beta^2\partial^b\phi\,\partial^c\phi\, A_\mu + \frac{1}{2}\beta F^b{}_\mu\partial^c\phi$$

$$- \frac{1}{2}\omega_\mu{}^{dc}F^b{}_d - \frac{1}{2}\alpha(\partial^c\phi\, e_\mu{}^d - \partial^d\phi\, e_\mu{}^c)F^b{}_d - \beta^2\partial^b\phi\,\partial^c\phi\, A_\mu$$

$$- \frac{1}{2}\beta\partial^b\phi\, F^c{}_\mu\Big]$$

$$+ e^{4(\beta-\alpha)\phi}\left[\frac{1}{4}F^b{}_d F^{dc} A_\mu - \frac{1}{4}F^b{}_d F^{dc}A_\mu\right]$$

$$= -e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\, F^{bc} - \frac{1}{2}\alpha\partial^d\phi\, F^c{}_d e_\mu{}^b + \frac{1}{2}\alpha\partial^b\phi\, F^c{}_\mu$$

$$- \frac{1}{2}\alpha\partial^c\phi\, F^b{}_\mu + \frac{1}{2}\alpha\partial^d\phi\, F^b{}_d e_\mu{}^c + \frac{1}{2}\beta\partial^c\phi\, F^b{}_\mu - \frac{1}{2}\beta\partial^b\phi\, F^c{}_\mu$$

$$- \frac{1}{2}\omega_\mu{}^b{}_d F^{cd} + \frac{1}{2}\omega_\mu{}^c{}_d F^{bd} + \frac{1}{2}\partial_\mu F^{bc}\Big]$$

$$= -e^{2(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\, F^{bc} + \alpha\,\partial^{[b}\phi\, F^{c]}{}_\mu + \alpha\,\partial^d\phi\, F^{[b}{}_d\, e_\mu{}^{c]}$$
$$+ \beta\, F^{[b}{}_\mu\, \partial^{c]}\phi - \omega_\mu{}^{[b}{}_d\, F^{c]d} + \tfrac{1}{2}\partial_\mu F^{bc}\Big]$$
$$= -\hat{R}_{z\mu}{}^{bc}$$

- $\hat{R}_{\mu z}{}^{b\underline{z}} = \partial_\mu \hat{\omega}_z{}^{b\underline{z}} + \hat{\omega}_\mu{}^b{}_D\, \hat{\omega}_z{}^{D\underline{z}} - (\mu \leftrightarrow z)$

$$= \partial_\mu \hat{\omega}_z{}^{b\underline{z}} + \hat{\omega}_\mu{}^b{}_c\, \hat{\omega}_z{}^{c\underline{z}} + \hat{\omega}_\mu{}^b{}_{\underline{z}}\, \overbrace{\hat{\omega}_z{}^{\underline{z}\underline{z}}}^{0}$$
$$- \underbrace{\partial_z \hat{\omega}_\mu{}^{b\underline{z}}}_{0} - \hat{\omega}_z{}^b{}_c\, \hat{\omega}_\mu{}^{c\underline{z}} - \omega_z{}^b{}_{\underline{z}}\, \underbrace{\hat{\omega}_\mu{}^{\underline{z}\underline{z}}}_{0}$$

$$= \partial_\mu \hat{\omega}_z{}^{b\underline{z}} + \hat{\omega}_\mu{}^b{}_c\, \hat{\omega}_z{}^{c\underline{z}} - \hat{\omega}_z{}^b{}_c\, \hat{\omega}_\mu{}^{c\underline{z}}$$

$$= -\beta\, e^{(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\, \partial^b\phi + \partial_\mu\partial^b\phi\Big]$$
$$- \Big[\omega_\mu{}^b{}_c + \alpha(\partial_c\phi\, e_\mu{}^b - \partial^b\phi\, e_{\mu c}) - \tfrac{1}{2} e^{2(\beta-\alpha)\phi} F^b{}_c A_\mu\Big]\beta\, e^{(\beta-\alpha)\phi}\partial^c\phi$$
$$- \tfrac{1}{2} e^{2(\beta-\alpha)\phi} F^b{}_c \cdot e^{(\beta-\alpha)\phi}\Big[\beta\,\partial^c\phi\, A_\mu + \tfrac{1}{2} F^c{}_\mu\Big]$$

$$= -\beta\, e^{(\beta-\alpha)\phi}\Big[(\beta-\alpha)\partial_\mu\phi\, \partial^b\phi + \partial_\mu\partial^b\phi + \omega_\mu{}^b{}_c\, \partial^c\phi$$
$$+ \alpha\, \partial_c\phi\, \partial^c\phi\, e_\mu{}^b - \alpha\, \partial^b\phi\, \partial_\mu\phi\Big]$$

$$+ e^{3(\beta-\alpha)\phi}\Big[\underline{\tfrac{1}{2}\beta F^b{}_c\, \partial^c\phi\, A_\mu} - \underline{\tfrac{1}{2}\beta F^b{}_c\, \partial^c\phi\, A_\mu}$$
$$- \tfrac{1}{4} F^b{}_c\, F^c{}_\mu\Big]$$

$$= -\beta\, e^{(\beta-\alpha)\phi}\Big[(\beta-2\alpha)\partial_\mu\phi\, \partial^b\phi + \partial_\mu\partial^b\phi + \alpha\,\partial_c\phi\, \partial^c\phi\, e_\mu{}^b + \omega_\mu{}^b{}_c\, \partial^c\phi\Big]$$
$$- \tfrac{1}{4} e^{3(\beta-\alpha)\phi} F^b{}_c\, F^c{}_\mu = -\hat{R}_{\mu z}{}^{\underline{z}b} = -\hat{R}_{z\mu}{}^{b\underline{z}} = \hat{R}_{z\mu}{}^{\underline{z}b}$$

▲ With the Riemann tensor we compute now the curved/flat Ricci tensor

$$\hat R_{NC} = \hat R_{MN}{}^{B}{}_{C}\, \hat e_B{}^M$$

• $\hat R_{\nu c} = \hat R_{M\nu}{}^{B}{}_{c}\, \hat e_B{}^M$

$$= \hat R_{\mu\nu}{}^{b}{}_{c}\, \hat e_b{}^\mu + \hat R_{\mu\nu}{}^{\underline z}{}_{c}\, \overbrace{\hat e_{\underline z}{}^\mu}^{0} + \hat R_{z\nu}{}^{b}{}_{c}\, e_b{}^z + \hat R_{z\nu}{}^{\underline z}{}_{c}\, \hat e_{\underline z}{}^z$$

$$= e^{-\alpha\phi} e_b{}^\mu \hat R_{\mu\nu}{}^{b}{}_{c} - e^{-\alpha\phi} A_b \hat R_{z\nu}{}^{b}{}_{c} + e^{-\beta\phi} R_{z\nu}{}^{\underline z}{}_{c}$$

$$= e^{-\alpha\phi} R_{\nu c} + e^{-\alpha\phi}\alpha\big(\partial_b\partial_c\phi\, e_\nu{}^b - \partial_\nu\partial_c\phi\, D + \partial_c\phi\,\partial_b e_\nu{}^b - \partial_c\phi\,\partial_\nu e_\mu{}^b e_b{}^\mu$$
$$- \partial^2\phi\, e_{\nu c} + \partial_\nu\partial^b\phi\, \eta_{bc} - \partial^b\phi\,\partial_b e_{\nu c}$$
$$+ \partial^b\phi\,\partial_\nu e_{\mu c}\, e_b{}^\mu\big)$$

$$- e^{(2\beta-3\alpha)\phi}\Big[(\beta-\alpha)\big(\partial_b\phi\, F^b{}_c A_\nu - \partial_\nu\phi\, F^b{}_c A_b\big)$$
$$+ \tfrac12\beta\big(\partial^b\phi\, F_{c\nu} A_b + \partial_b\phi\, F^b{}_c A_\nu\big)$$
$$+ \tfrac12\beta\big(\cancelto{0}{\partial_c\phi\, F^b{}_b A_\nu} - \partial_c\phi\, F^b{}_\nu A_b\big)$$
$$+ \tfrac12\alpha\,\partial_b\phi\, F^b{}_c\big(A_\nu \underbrace{\delta_d{}^d}_{D} - \underbrace{A_d\, e_\nu{}^d}_{A_\nu}\big)$$
$$- \tfrac12\alpha\,\partial_b\phi\big(F^b{}_c A_\nu - F_{\nu c} A^b\big)$$
$$+ \tfrac12\alpha\,\partial_c\phi\big(F^b{}_\nu A_b - \cancelto{0}{F^b{}_b A_\nu}\big)$$
$$+ \tfrac12\alpha\,\partial_b\phi\, F^{bd}\big(A_d\, e_{\nu c} - A_\nu\, \eta_{dc}\big)$$
$$+ \tfrac12\big(\partial_b F^b{}_c A_\nu - \partial_\nu F^b{}_c A_b\big) + \tfrac12 F^b{}_c F_{b\nu}$$
$$+ \tfrac14\big(\cancelto{0}{F^b{}_b F_{c\nu}} - F^b{}_\nu F_{cb}\big)$$
$$+ \tfrac12 F^d{}_c\big(\omega_b{}^b{}_d A_\nu - \omega_\nu{}^b{}_d A_b\big)$$
$$+ \tfrac12 F^b{}_d\big(\omega_\nu{}^d{}_c A_b - \omega_b{}^d{}_c A_\nu\big)\Big]$$

SO(1,D-1) generators are antisymmetric

$$
\begin{aligned}
&+ e^{-\alpha\phi}\Big[\alpha\, \omega_b{}^b{}_d\left(\partial_c\phi\, e_\nu{}^d - \partial^d\phi\, e_{\nu c}\right) - \alpha\, \omega_\nu{}^b{}_d\left(\partial_c\phi\, \delta_b{}^d - \partial^d\phi\, \eta_{bc}\right) \\
&\quad + \alpha\, \omega_\nu{}^d{}_c\left(\partial_d\phi\, \overset{D}{\delta_b{}^b} - \partial^b\phi\, \eta_{bd}\right) - \alpha\, \omega_b{}^d{}_c\left(\partial_d\phi\, e_\nu{}^b - \partial^b\phi\, e_{\nu d}\right) \\
&\quad + \alpha^2\Big(\partial_\nu\phi\,\partial_c\phi\, \overset{D}{\delta_b{}^b} - \partial_b\phi\,\partial_c\phi\, e_\nu{}^b + \partial^b\phi\,\partial_b\phi\, e_{\nu c} - \partial^b\phi\,\partial_\nu\phi\,\eta_{bc} \\
&\quad\quad - \partial_d\phi\,\partial^d\phi\, \underset{D}{\delta_b{}^b}\, e_{\nu c} + \partial_d\phi\,\partial^d\phi\, e_\nu{}^b\,\eta_{bc}\Big)\Big] \\
&- A_b\, e^{(2\beta-3\alpha)\phi}\Big[(\beta-\alpha)\,\partial_\nu\phi\, F^b{}_c - \tfrac{1}{2}\alpha\,\partial^d\phi\, F_{cd}\, e_\nu{}^b + \tfrac{1}{2}\alpha\,\partial^b\phi\, F_{c\nu} \\
&\quad - \tfrac{1}{2}\alpha\,\partial_c\phi\, F^b{}_\nu + \tfrac{1}{2}\alpha\,\partial^d\phi\, F^b{}_d\, e_{\nu c} + \tfrac{1}{2}\beta\,\partial_c\phi\, F^b{}_\nu \\
&\quad - \tfrac{1}{2}\beta\,\partial^b\phi\, F_{c\nu} - \tfrac{1}{2}\omega_\nu{}^b{}_d\, F_c{}^d + \tfrac{1}{2}\omega_{\nu cd}\, F^{bd} \\
&\quad + \tfrac{1}{2}\partial_\nu F^b{}_c\Big] \\
&- \beta\, e^{-\alpha\phi}\Big[(\beta-2\alpha)\,\partial_\nu\phi\,\partial_c\phi + \partial_\nu\partial_c\phi + \alpha\,\partial_d\phi\,\partial^d\phi\, e_{\nu c} + \omega_{\nu cd}\,\partial^d\phi\Big] \\
&- \tfrac{1}{4} e^{(2\beta-3\alpha)\phi}\, F_{cd}\, F^d{}_\nu
\end{aligned}
$$

$$
\begin{aligned}
&= e^{-\alpha\phi}\Big[R_{\nu c} + \alpha\Big(\partial_b\partial_c\phi\, e_\nu{}^b - \partial^2\phi\, e_{\nu c} - (D-1)\,\partial_\nu\partial_c\phi \\
&\quad + \partial_c\phi\,\partial_b e_\nu{}^b - \partial^b\phi\,\partial_b e_{\nu c} - \partial_c\phi\,\partial_\nu e_\mu{}^b\, e_b{}^\mu + \partial^b\phi\,\partial_\nu e_{\mu c}\, e_b{}^\mu \\
&\quad + \omega_b{}^b{}_\nu\,\partial_c\phi - \omega_b{}^{bd}\,\partial_d\phi\, e_{\nu c} + (3-D)\,\omega_{\nu c}{}^d\,\partial_d\phi \\
&\quad + \omega^d{}_{\nu c}\,\partial_d\phi\Big) + \alpha^2(D-2)\left(\partial_\nu\phi\,\partial_c\phi - \partial^d\phi\,\partial_d\phi\, e_{\nu c}\right) \\
&\quad - \beta^2\,\partial_\nu\phi\,\partial_c\phi + \alpha\beta\left(2\,\partial_\nu\phi\,\partial_c\phi - \partial_d\phi\,\partial^d\phi\, e_{\nu c}\right) \\
&\quad - \beta\left(\partial_\nu\partial_c\phi + \omega_{\nu c}{}^d\,\partial_d\phi\right)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&- e^{(2\beta-3\alpha)\phi}\Big[\alpha\,\frac{D-4}{2}\,\partial_b\phi\, F^b{}_c\, A_\nu + \beta\,\frac{3}{2}\,\partial_b\phi\, F^b{}_c\, A_\nu + \tfrac{1}{2}\partial_b F^b{}_c\, A_\nu + \tfrac{1}{2} F_{b\nu}\, F^b{}_c \\
&\quad + \tfrac{1}{2}\omega_b{}^b{}_d\, F^d{}_c\, A_\nu - \tfrac{1}{2}\omega_b{}^d{}_c\, F^b{}_d\, A_\nu\Big]
\end{aligned}
$$

$$
\bullet\ \hat R_{zc} = \hat R_{Mz}{}^{B}{}_{c}\, \hat e_{B}{}^{M}
$$

$$
= \hat R_{\mu z}{}^{b}{}_{c}\, \hat e_{b}{}^{\mu} + \hat R_{\mu z}{}^{\underline z}{}_{c}\, \overbrace{\hat e_{\underline z}{}^{\mu}}^{0} + \overbrace{\hat R_{zz}{}^{b}{}_{c}}^{0}\, \hat e_{b}{}^{z} + \overbrace{\hat R_{zz}{}^{\underline z}{}_{c}}^{0}\, \hat e_{\underline z}{}^{z}
$$

$$
= e^{-\alpha\phi} e_{b}{}^{\mu}\, \hat R_{\mu z}{}^{b}{}_{c}
$$

$$
= -e^{(2\beta-3\alpha)\phi}\Big[\alpha\Big(-\partial_b\phi\, F^{b}{}_{c} + \frac{D}{2}\partial^d\phi\, F_{dc} - \partial^b\phi\, F_{bc}\Big) + \frac12 \partial_b F^{b}{}_{c}
$$

$$
+ \frac32 \partial_b\phi\, F^{b}{}_{c} + \frac12 \omega_{b}{}^{b}{}_{d} F^{d}{}_{c} - \frac12 \omega_{bcd} F^{db}\Big]
$$

$$
\bullet\ \hat R_{\nu\underline z} = \hat R_{M\nu}{}^{B}{}_{\underline z}\, \hat e_{B}{}^{M}
$$

$$
= \hat R_{\mu\nu}{}^{b}{}_{\underline z}\, \hat e_{b}{}^{\mu} + \hat R_{\mu\nu}{}^{\underline z}{}_{\underline z}\, \overbrace{\hat e_{\underline z}{}^{\mu}}^{0} + \hat R_{z\nu}{}^{b}{}_{\underline z}\, \hat e_{b}{}^{z} + \overbrace{\hat R_{z\nu}{}^{\underline z}{}_{\underline z}}^{0}\, \hat e_{\underline z}{}^{z}
$$

$$
= e^{-\alpha\phi} e_{b}{}^{\mu}\, \hat R_{\mu\nu}{}^{b}{}_{\underline z} - e^{-\alpha\phi} A_b\, \hat R_{z\nu}{}^{b}{}_{\underline z}
$$

$$
= -e^{(\beta-2\alpha)\phi}\Big[\beta^2\big(\partial_b\phi\,\partial^b\phi\, A_\nu - \underline{\partial_\nu\phi\,\partial^b\phi\, A_b}\big)
$$

$$
+ \alpha\beta\big(-2\partial_b\phi\,\partial^b\phi\, A_\nu + 2\partial_\nu\phi\,\partial^b\phi\, A_b + (D-1)\partial_c\phi\,\partial^c\phi\, A_\nu\big)
$$

$$
+ \alpha\Big(\frac12 \partial_c\phi\,(D-1)F^{c}{}_{\nu} - \frac12 \partial_b\phi\, F^{b}{}_{\nu} - \partial^b\phi\, F_{b\nu}\Big)
$$

$$
+ \beta\Big(\partial^2\phi\, A_\nu - \partial_\nu\partial^b\phi\, A_b + \frac12 \partial_b\phi\, F^{b}{}_{\nu} + \partial^b\phi\, F_{b\nu}
$$

$$
+ \omega_{b}{}^{b}{}_{c}\,\partial^c\phi\, A_\nu - \omega_{\nu}{}^{b}{}_{c}\,\partial^c\phi\, A_b\Big) + \frac12 \partial_b F^{b}{}_{\nu}
$$

$$
- \frac12 \partial_\nu F^{b}{}_{\mu} e_{b}{}^{\mu} + \frac12 \omega_{b}{}^{b}{}_{c} F^{c}{}_{\nu} - \frac12 \omega_{\nu}{}^{b}{}_{c} F^{c}{}_{b}\Big]
$$

$$
+ e^{(3\beta-4\alpha)\phi}\Big[\underline{\frac14 F^{b}{}_{c} F^{c}{}_{\nu} A_b} - \frac14 F^{b}{}_{c} F^{c}{}_{b} A_\nu\Big]
$$

$$
-\beta e^{(\beta-2\alpha)\phi} A_b\Big[\alpha\big(-2\partial_\nu\phi\,\partial^b\phi + \partial_c\phi\,\partial^c\phi\, e_{\nu}{}^{b}\big) + \underline{\beta\,\partial_\nu\phi\,\partial^b\phi}
$$

$$
+ \partial_\nu\partial^b\phi + \omega_{\nu}{}^{b}{}_{c}\,\partial^c\phi\Big] - \underline{\frac14 e^{(3\beta-4\alpha)\phi} A_b F^{b}{}_{c} F^{c}{}_{\nu}}
$$

$$= -e^{(\beta-2\alpha)\phi}\Big[\beta^2\,\partial_b\phi\,\partial^b\phi\,A_\nu + \alpha\beta\,(D-2)\,\partial_b\phi\,\partial^b\phi\,A_\nu$$

$$+\alpha\,\frac{D-4}{2}\,\partial_b\phi\,F^b{}_\nu + \beta\left(\partial^2\phi\,A_\nu + \frac{3}{2}\,\partial_b\phi\,F^b{}_\nu + \omega_b{}^b{}_c\,\partial^c\phi\,A_\nu\right)$$

$$+\frac{1}{2}\,\partial_b F^b{}_\nu - \frac{1}{2}\,\partial_\nu F^b{}_\mu\, e_b{}^\mu + \frac{1}{2}\,\omega_b{}^b{}_c\,F^c{}_\nu - \frac{1}{2}\,\omega_\nu{}^b{}_c\,F^c{}_b\Big]$$

$$-\frac{1}{4}\,e^{(3\beta-4\alpha)\phi}\,F^b{}_c\,F^c{}_b\,A_\nu$$

- $\hat{R}_{z\underline{z}} = \hat{R}_{Mz}{}^B{}_{\underline{z}}\,\hat{e}_B{}^M$

$$= \hat{R}_{\mu z}{}^b{}_{\underline{z}}\,\hat{e}_b{}^\mu + \hat{R}_{\mu z}{}^{\underline{z}}{}_{\underline{z}}\,\overbrace{\hat{e}_{\underline{z}}{}^\mu}^{0} + \overbrace{\hat{R}_{zz}{}^b{}_{\underline{z}}}^{0}\,\hat{e}_b{}^z + \overbrace{\hat{R}_{zz}{}^{\underline{z}}{}_{\underline{z}}}^{0}\,e_{\underline{z}}{}^z$$

$$= e^{-\alpha\phi}\,e_b{}^\mu\,\hat{R}_{\mu z}{}^b{}_{\underline{z}}$$

$$= -\beta\,e^{(\beta-2\alpha)\phi}\Big[(\beta-2\alpha)\,\partial_b\phi\,\partial^b\phi + \partial^2\phi + \alpha D\,\partial_c\phi\,\partial^c\phi + \omega_b{}^b{}_c\,\partial^c\phi\Big]$$

$$-\frac{1}{4}\,e^{(3\beta-4\alpha)\phi}\,F^b{}_c\,F^c{}_b$$

$$= -e^{(\beta-2\alpha)\phi}\Big[\beta^2\,\partial_b\phi\,\partial^b\phi + \alpha\beta\,(D-2)\,\partial_b\phi\,\partial^b\phi + \beta\left(\partial^2\phi + \omega_b{}^b{}_c\,\partial^c\phi\right)\Big]$$

$$-\frac{1}{4}\,e^{(3\beta-4\alpha)\phi}\,F^b{}_c\,F^c{}_b$$

▲ Now we compute the vielbein components of the Ricci tensor

$$\hat{R}_{AC} = \hat{e}_A{}^N\,\hat{R}_{NC}$$

- $\hat{R}_{ac} = \hat{e}_a{}^N\,\hat{R}_{Nc} = \hat{e}_a{}^\nu\,\hat{R}_{\nu c} + \hat{e}_a{}^z\,\hat{R}_{zc}$

$$= e^{-\alpha\phi}\,e_a{}^\nu\,\hat{R}_{\nu c} - e^{-\alpha\phi}\,A_a\,\hat{R}_{zc}$$

$$
\begin{aligned}
= e^{-2\alpha\phi}\Big[R_{ac} &+ \alpha\Big(\partial_a \partial_c \phi - \partial^2 \phi\, \eta_{ac} - (D-1)\, \partial_a \partial_c \phi \\
&+ \partial_c \phi\, \partial_b e_\nu{}^b e_a{}^\nu - \partial^b \phi\, \partial_b e_{\nu c} e_a{}^\nu - \partial_c \phi\, \partial_a e_\mu{}^b e_b{}^\mu + \partial^b \phi\, \partial_a e_{\mu c} e_b{}^\mu \\
&+ \omega_b{}^b{}_a\, \partial_c \phi - \omega_b{}^{bd}\, \partial_d \phi\, \eta_{ca} + (3-D)\, \omega_{ac}{}^d\, \partial_d \phi + \omega^d{}_{ac}\, \partial_d \phi \Big) \\
&+ \alpha^2 (D-2) \left(\partial_a \phi\, \partial_c \phi - \partial^d \phi\, \partial_d \phi\, \eta_{ac} \right) - \beta^2 \partial_a \phi\, \partial_c \phi \\
&+ \alpha\beta \left(2\, \partial_a \phi\, \partial_c \phi - \partial_d \phi\, \partial^d \phi\, \eta_{ac} \right) - \beta \left(\partial_a \partial_c \phi + \omega_{ac}{}^d\, \partial_d \phi \right) \Big]
\end{aligned}
$$

$$
\begin{aligned}
- e^{(2\beta-4\alpha)\phi} \Big[&\alpha\, \frac{D-4}{2}\, \partial_b \phi\, F^b{}_c A_a + \beta\, \frac{3}{2}\, \partial_b \phi\, F^b{}_c A_a + \frac{1}{2}\, \partial_b F^b{}_c A_a \\
&+ \frac{1}{2} F_{ba} F^b{}_c + \frac{1}{2}\, \omega_b{}^b{}_d F^d{}_c A_a - \frac{1}{2}\, \omega_b{}^d{}_c F^b{}_d A_a \Big]
\end{aligned}
$$

$$
\begin{aligned}
+ e^{(2\beta-4\alpha)\phi} \Big[&\alpha \left(-\partial_b \phi\, F^b{}_c A_a + \frac{D}{2}\, \partial^d \phi\, F_{dc} A_a - \partial^b \phi\, F_{bc} A_a \right) \\
&+ \frac{1}{2}\, \partial_b F^b{}_c A_a + \beta\, \frac{3}{2}\, \partial_b \phi\, F^b{}_c A_a \\
&+ \frac{1}{2}\, \omega_b{}^b{}_d F^d{}_c A_a - \frac{1}{2}\, \omega_{bcd} F^{db} A_a \Big]
\end{aligned}
$$

$$
\begin{aligned}
= e^{-2\alpha\phi}\Big[R_{ac} &+ \alpha\Big(-(D-2)\, \partial_a \partial_c \phi - \partial^2 \phi\, \eta_{ac} \\
&+ \partial_c \phi\, \partial_b e_\nu{}^b e_a{}^\nu - \partial^b \phi\, \partial_b e_{\nu c} e_a{}^\nu - \partial_c \phi\, \partial_a e_\mu{}^b e_b{}^\mu + \partial^b \phi\, \partial_a e_{\mu c} e_b{}^\mu \\
&+ \omega_b{}^b{}_a\, \partial_c \phi - \omega_b{}^{bd}\, \partial_d \phi\, \eta_{ca} + (3-D)\, \omega_{ac}{}^d\, \partial_d \phi + \omega^d{}_{ac}\, \partial_d \phi \Big) \\
&+ \alpha^2 (D-2) \left(\partial_a \phi\, \partial_c \phi - \partial^d \phi\, \partial_d \phi\, \eta_{ac} \right) - \beta^2 \partial_a \phi\, \partial_c \phi \\
&+ \alpha\beta \left(2\, \partial_a \phi\, \partial_c \phi - \partial_d \phi\, \partial^d \phi\, \eta_{ac} \right) - \beta \left(\partial_a \partial_c \phi + \omega_{ac}{}^d\, \partial_d \phi \right) \Big] \\
- e^{(2\beta-4\alpha)\phi} \Big[&\partial_b \phi\, F^b{}_c A_a\, \frac{(D-4)\alpha + 3\beta}{2} + \frac{1}{2}\, \partial_b F^b{}_c A_a + \frac{1}{2} F_{ba} F^b{}_c \\
&+ \frac{1}{2}\, \omega_b{}^b{}_d F^d{}_c A_a - \frac{1}{2}\, \omega_b{}^d{}_c F^b{}_d A_a \\
&- \alpha\, \frac{D-4}{2}\, \partial_b \phi\, F^b{}_c A_a - \frac{1}{2}\, \partial_b F^b{}_c A_a
\end{aligned}
$$

$$-\beta\,\frac{3}{2}\,\partial_b\phi\,F^b{}_c\,A_a$$

$$-\frac{1}{2}\,\omega_b{}^b{}_d\,F^d{}_c\,A_a+\frac{1}{2}\,\omega_b{}^d{}_c\,F^b{}_d\,A_a\Big]$$

$$=e^{-2\alpha\phi}\Big[R_{ac}+\alpha\Big(-(D-2)\,\partial_a\partial_c\phi\;\underbrace{-\,\partial^2\phi}_{\to\,\Box\phi}\,\eta_{ac}$$

$$+\partial_c\phi\,\partial_b e_\nu{}^b e_a{}^\nu-\partial^b\phi\,\partial_b e_{\nu c}e_a{}^\nu-\partial_c\phi\,\partial_a e_\mu{}^b e_b{}^\mu+\partial^b\phi\,\partial_a e_{\mu c}e_b{}^\mu$$

$$+\omega_b{}^b{}_a\,\partial_c\phi-\omega_b{}^{bd}\,\partial_d\phi\,\eta_{ac}\underbrace{-(D-3)}_{-(D-2)+1}\omega_{ac}{}^d\,\partial_d\phi+\omega^d{}_{ac}\,\partial_d\phi\Big)$$

$$+\alpha^2(D-2)\Big(\partial_a\phi\,\partial_c\phi-\partial^d\phi\,\partial_d\phi\,\eta_{ac}\Big)-\beta^2\,\partial_a\phi\,\partial_c\phi$$

$$+\alpha\beta\Big(2\,\partial_a\phi\,\partial_c\phi-\partial_d\phi\,\partial^d\phi\,\eta_{ac}\Big)-\beta\Big(\underbrace{\partial_a\partial_c\phi+\omega_{ac}{}^d\,\partial_d\phi}_{\nabla_a\nabla_c\phi}\Big)\Big]$$

$$-\frac{1}{2}\,e^{(2\beta-4\alpha)\phi}\,F_a{}^b\,F_{cb}=(*)$$

NOTE 4: We will see later that one must set $\beta=-(D-2)\alpha$

$$(*)=-\frac{1}{2}\,e^{(2\beta-4\alpha)\phi}\,F_a{}^b\,F_{cb}+e^{-2\alpha\phi}\Big[R_{ac}+\overbrace{(D-2)\alpha}^{-\beta}\,\nabla_a\nabla_c\phi$$

$$+\partial_a\phi\,\partial_c\phi\Big(\underbrace{\alpha^2(D-2)-\beta^2+2\alpha\beta}_{\alpha\beta-\beta^2=-(D-2)(D-1)\alpha^2}\Big)-\partial^b\phi\,\partial_b\phi\,\eta_{ac}\Big(\underbrace{\alpha^2(D-2)+\alpha\beta}_{0}\Big)$$

$$+\alpha\Big(-\Box\phi\,\eta_{ac}-(D-2)\,\nabla_a\nabla_c\phi+\omega_{ac}{}^d\,\partial_d\phi+\omega^d{}_{ac}\,\partial_d\phi$$

$$+\omega_b{}^b{}_a\,\partial_c\phi+\partial_c\phi\,\partial_b e_\nu{}^b e_a{}^\nu-\partial_d\phi\,\partial^d e_{\nu c}e_a{}^\nu$$

$$-\partial_c\phi\,\partial_a e_\nu{}^b e_b{}^\nu+\partial_d\phi\,\partial_a e_{\nu c}e^{d\nu}\Big)\Big]$$

NOTE 5: $\alpha^2=\frac{1}{2(D-2)(D-1)}$ [We will see later]

$$=-\frac{1}{2}\,e^{(2\beta-4\alpha)\phi}\,F_a{}^b\,F_{cb}+e^{-2\alpha\phi}\Big[R_{ac}-\frac{1}{2}\,\partial_a\phi\,\partial_c\phi-\alpha\,\Box\phi\,\eta_{ac}$$

$$+\partial_d\phi\Big(\underbrace{\omega_{ac}{}^d+\omega^d{}_{ac}-\partial^d e_{\nu c}e_a{}^\nu+\partial_a e_{\nu c}e^{d\nu}}_{0}\Big)+\partial_c\phi\Big(\underbrace{\omega_b{}^b{}_a+\partial_b e_\nu{}^b e_a{}^\nu-\partial_a e_\nu{}^b e_b{}^\nu}_{0}\Big)=(*)$$

Remark 1

$$\omega_b{}^{ba} = e_b{}^\mu \omega_\mu{}^{ba} = -e_b{}^\mu \omega_\mu{}^{ab}(e)$$

$$= -e_b{}^\mu \frac{1}{2}\Big[e^{\nu a}\partial_\mu e_\nu{}^b - e^{\nu b}\partial_\mu e_\nu{}^a - e^{\nu a}\partial_\nu e_\mu{}^b + e^{\nu b}\partial_\nu e_\mu{}^a$$
$$- e^{\nu a} e^{b\sigma} e_{\mu c}\partial_\nu e_\sigma{}^c + e^{\nu b} e^{a\sigma} e_{\mu c}\partial_\nu e_\sigma{}^c\Big]$$

$$= -\frac{1}{2}\Big[\partial_b e_\nu{}^b e^{\nu a} - \partial_b e_\nu{}^a e^{\nu b} - \partial^a e_\mu{}^b e_b{}^\mu + \partial^b e_\mu{}^a e_b{}^\mu$$
$$- e^{\nu a} e^{b\sigma}\partial_\nu e_{\sigma b} + e^{\nu b} e^{a\sigma}\partial_\nu e_{\sigma b}\Big]$$

$$= -\frac{1}{2}\Big[\partial_b e_\nu{}^b e^{\nu a} - \partial^\nu e_\nu{}^a - \partial^a e_\mu{}^b e_b{}^\mu + \partial^\mu e_\mu{}^a$$
$$- \partial^a e_{\sigma b} e^{b\sigma} + \partial^b e_{\sigma b} e^{a\sigma}\Big]$$

$$= -\frac{1}{2}\Big[2\partial_b e_\nu{}^b e^{\nu a} - \partial^a e_{\sigma b} e^{b\sigma}\Big] = \partial^a e_\nu{}^b e_b{}^\nu - \partial_b e_\nu{}^b e^{\nu a}$$

$$\Rightarrow \omega_b{}^b{}_a = \partial_a e_\nu{}^b e_b{}^\nu - \partial_b e_\nu{}^b e_a{}^\nu$$

Remark 2 [We can compute faster by using anholonomy coefficients]

$$\omega_{acd} + \omega_{dac} = \frac{1}{2}\Big[\underline{\Omega_{[ac]d}} - \underline{\Omega_{[cd]a}} + \Omega_{[da]c}$$
$$+ \Omega_{[da]c} - \underline{\Omega_{[ac]d}} + \underline{\Omega_{[cd]a}}\Big]$$
$$= \Omega_{[da]c}$$

see note below

$$\Rightarrow \omega_{ac}{}^d + \omega^d{}_{ac} = \Omega_{[ba]c}\,\eta^{bd} = -(\partial_b e_a{}^\rho - \partial_a e_b{}^\rho)e_{\rho c}\,\eta^{bd}$$
$$= -\partial^d e_a{}^\nu e_{\nu c} + \partial_a e^{d\nu} e_{\nu c}$$
$$= \partial^d e_{\nu c} e_a{}^\nu - \partial_a e_{\nu c} e^{d\nu}$$

NOTE: $\Omega_{[\mu\nu]\rho} = (\partial_\mu e_\nu{}^d - \partial_\nu e_\mu{}^d)e_{d\rho}$

$$\Omega_{[ab]c} = e_a{}^\mu e_b{}^\nu e_c{}^\rho\,\Omega_{[\mu\nu]\rho} = (\partial_a e_\nu{}^d e_b{}^\nu - \partial_b e_\mu{}^d e_a{}^\mu)\eta_{cd}$$

Important $\leftarrow$ $= -(\partial_a e_b{}^\rho - \partial_b e_a{}^\rho)e_{\rho c}$

$$(*) = e^{-2\alpha\phi}\left[R_{ac} - \frac{1}{2}\partial_a\phi\,\partial_c\phi - \alpha\,\Box\phi\,\eta_{ac}\right] - \frac{1}{2}e^{-2D\alpha\phi}F_a{}^b F_{cb}$$

- $$\hat{R}_{\underline{z}\underline{z}} = \hat{e}_{\underline{z}}{}^N \hat{R}_{N\underline{z}} = \overbrace{\hat{e}_{\underline{z}}{}^{\mu}}^{0}\hat{R}_{\mu\underline{z}} + \hat{e}_{\underline{z}}{}^z \hat{R}_{z\underline{z}}$$

$$= e^{-\beta\phi}\hat{R}_{z\underline{z}}$$

$$= -e^{-2\alpha\phi}\left[\partial_b\phi\,\partial^b\phi\left(\beta^2 + (D-2)\alpha\beta\right) + \overbrace{\beta\left(\partial^2\phi + \omega_b{}^b{}_c\,\partial^c\phi\right)}^{\eta^{ab}\nabla_a\nabla_b\phi\,=\,\Box\phi}\right]$$

$$+ \frac{1}{4}e^{(2\beta-4\alpha)\phi}F_c{}^b F^c{}_b$$

$$= e^{-2\alpha\phi}\left[-\underbrace{\left(\beta^2 + (D-2)\alpha\beta\right)}_{0\ \text{(see note 4)}}\partial_b\phi\,\partial^b\phi - \beta\,\Box\phi\right] + \frac{1}{4}e^{(2\beta-4\alpha)\phi}F^2$$

▲ Using these two vielbein components of the Ricci tensor it's finally straightforward to compute the Ricci scalar

$$\hat{R} = \hat{R}_{AB}\,\eta^{AB}$$

We find:

$$\hat{R} = \hat{R}_{ac}\eta^{ac} + \hat{R}_{\underline{z}\underline{z}} = e^{-2\alpha\phi}\left[R - \frac{1}{2}(\partial\phi)^2 - \overbrace{(D\alpha+\beta)}^{D\alpha-(D-2)\alpha\,=\,2\alpha}\Box\phi\right]$$

$$-\frac{1}{2}e^{-2D\alpha\phi}\left(F^2 - \frac{1}{2}F^2\right) =$$

$$= e^{-2\alpha\phi}\left[R - \frac{1}{2}(\partial\phi)^2 - \underbrace{2\alpha\,\Box\phi}_{\text{boundary term}} - \frac{1}{4}e^{-2(D-1)\alpha\phi}F^2\right]$$

▶ The full $(D+1)$-dimensional action then reduces to

$$S_{D+1} = \frac{1}{2\kappa_{D+1}^2}\int d^{D+1}x \; \hat{e} \; \hat{R}$$

$$= \frac{1}{2\kappa_{D+1}^2}\int_0^{2\pi L} dz \int d^D x \; e^{(\alpha D+\beta)\phi} e \; \hat{R}$$

$$= \frac{1}{2\underbrace{\frac{\kappa_{D+1}^2}{2\pi L}}_{\kappa_D^2 = \frac{\kappa_{D+1}^2}{2\pi L}}}\int d^D x \; \underbrace{e^{[(D-2)\alpha+\beta]\phi}}_{\substack{\text{Canonical E-H if} \\ \beta = -(D-2)\alpha \\ \text{(see note 4)}}} e \left[R - \frac{1}{2}(\partial\phi)^2 - \underbrace{\frac{1}{4} e^{-2(D-1)\alpha\phi}}_{\substack{\text{Proper normalisation if} \\ \alpha^2 = \frac{1}{2(D-2)(D-1)}}} F^2\right]$$

(see note 5)

$$= \frac{1}{2\kappa_D^2}\int d^D x \, e \left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4} e^{-2(D-1)\alpha\phi} F^2\right] = S_D$$

Therefore we recover an Einstein – Maxwell – Dilaton theory !!

$$\boxed{S_{D+1} = \frac{1}{2\kappa_D^2}\int d^D x \; e \left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4} e^{-2(D-1)\alpha\phi} F^2\right]}$$

with $\kappa_D^2 = \frac{\kappa_{D+1}^2}{2\pi L}$

<u>Example</u> : If $D=4 \Rightarrow e^{-2(D-1)\alpha\phi} = e^{-\sqrt{3}\phi}$

Exercise: Compute the $\hat{R}_{b\underline{z}}$ component of the Ricci tensor

$$\hat{R}_{b\underline{z}} = \hat{e}_b{}^N \hat{R}_{N\underline{z}} = \hat{e}_b{}^0 \hat{R}_{0\underline{z}} + \hat{e}_b{}^z \hat{R}_{\underline{z}\underline{z}}$$

$$= e^{-\alpha\phi} e_b{}^0 \hat{R}_{0\underline{z}} - e^{-\alpha\phi} A_b \hat{R}_{\underline{z}\underline{z}}$$

$$= -e^{(\beta-3\alpha)\phi}\Big[\beta^2\partial_c\phi\partial^c\phi A_b + \alpha\beta(D-2)\partial_c\phi\partial^c\phi A_b + \alpha\frac{D-4}{2}\partial_c\phi F^c{}_b$$
$$+\beta\left(\partial^2\phi A_b + \tfrac{3}{2}\partial_c\phi F^c{}_b + \omega_c{}^c{}_d\partial^d\phi A_b\right) + \tfrac{1}{2}\partial_c F^c{}_0 e_b{}^0$$
$$-\tfrac{1}{2}\partial_b F^c{}_0 e_c{}^0 + \tfrac{1}{2}\omega_c{}^c{}_d F^d{}_b - \tfrac{1}{2}\omega_b{}^c{}_d F^d{}_c\Big]$$
$$+\frac{1}{4}e^{(3\beta-5\alpha)\phi}F^2 A_b$$
$$+e^{(\beta-3\alpha)\phi}\Big[\beta^2\partial_c\phi\partial^c\phi A_b + \alpha\beta(D-2)\partial_c\phi\partial^c\phi A_b + \beta\,\Box\phi\, A_b\Big]$$
$$-\frac{1}{4}e^{(3\beta-5\alpha)\phi}F^2 A_b$$

$$= -\frac{1}{2}\underbrace{e^{(\beta-3\alpha)\phi}}_{e^{-(D+1)\alpha}}\Big[-\overbrace{\big((D-4)\alpha+3\beta\big)}^{-2(D-1)\alpha}\partial_c\phi F_b{}^c$$
$$+\partial_c F^c{}_0 e_b{}^0 - \partial_b F^c{}_0 e_c{}^0$$
$$+\omega_c{}^c{}_d F^d{}_b + \omega_b{}^c{}_d F_c{}^d\Big]$$

NOTE: $-(D+1)\alpha = (D-3)\alpha - 2(D-1)$

$$= \frac{1}{2}e^{(D-3)\alpha\phi} \times e^{-2(D-1)\alpha\phi}\Big[-2(D-1)\alpha\,\partial_c\phi F_b{}^c$$
$$-\underbrace{\partial_c F^c{}_0 e_b{}^0 + \partial_b F^c{}_0 e_c{}^0}_{\partial_c F_b{}^c - F_0{}^c\partial_c e_b{}^0 + F_0{}^c\partial_b e_c{}^0} + \omega_c{}^c{}_d F_b{}^d + \omega_b{}^c{}_d F^d{}_c\Big]$$

NOTE: $\nabla_c F_b{}^c = \partial_c F_b{}^c + \omega_c{}^c{}_d F_b{}^d - \omega_c{}^d{}_b F_d{}^c$

$$= \frac{1}{2} e^{(D-3)\alpha\phi} \times e^{-2(D-1)\alpha\phi} \Big[-2(D-1)\alpha \overbrace{\partial_c \phi}^{\nabla_c \phi} F_b{}^c$$

$$+ \underbrace{\partial_c F_b{}^c + \omega_c{}^c{}_d F_b{}^d - \omega_c{}^d{}_b F_d{}^c}_{\nabla_c F_b{}^c} + \omega_{cdb} F^{dc} - F_a{}^c \partial_c e_b{}^a + F_a{}^c \partial_b e_c{}^a$$

$$+ \omega_{bcd} F^{dc} \Big] = (*)$$

Remark 3

$$\omega_{cdb} + \omega_{bcd} = \frac{1}{2} \Big[\Omega_{[cd]b} - \Omega_{[db]c} + \Omega_{[cb]d} + \Omega_{[cb]d} - \Omega_{[cd]b} + \Omega_{[db]c} \Big]$$

$$= \Omega_{[cb]d} = -(\partial_b e_c{}^a - \partial_c e_b{}^a) e_{ad}$$

$$\Rightarrow (\omega_{cdb} + \omega_{bcd}) F^{dc} = -\partial_b e_c{}^a e_{ad} F^{dc} + \partial_c e_b{}^a e_{ad} F^{dc}$$

$$= F_a{}^c \partial_c e_b{}^a - F_a{}^c \partial_b e_c{}^a$$

$$(*) = \frac{1}{2} e^{(D-3)\alpha\phi} \nabla_c \Big[e^{-2(D-1)\alpha\phi} F_b{}^c \Big] = \hat{R}_{\underline{z} b}$$

II. (D+1)-dimensional vs D-dimensional EOMs and symmetries

In this section we discuss the equations of motion (EOMs) that result from S_{D+1} and S_D.

i) (D+1)-dimensional EOMs

$$S_{D+1} = \frac{1}{2K_{D+1}^2} \int d\overset{D+1}{x}\; \hat{e}\; \hat{R} \implies \hat{G}_{MN} = \hat{R}_{MN} - \frac{1}{2} \hat{g}_{MN} \hat{R} = 0$$

NOTE: $\hat{g}^{MN} \hat{G}_{MN} = \hat{R} - \frac{1}{2}(D+1)\hat{R} = \left(1 - \frac{1}{2}(D+1)\right)\hat{R} = 0$

$\implies \hat{R} = 0 \implies \hat{R}_{MN} = 0$

$\hookrightarrow D \neq 1 \qquad \implies \hat{R}_{AB} = 0$

$$\hat{R}_{AB} = 0 \quad \begin{cases} \hat{R}_{ab} = e^{-2\alpha\phi}\left[R_{ab} - \frac{1}{2}\partial_a\phi\partial_b\phi - \alpha\,\Box\phi\,\eta_{ab}\right] - \frac{1}{2} e^{-2D\alpha\phi} F_a{}^c F_{bc} = 0 \\[2ex] \hat{R}_{a\underline{z}} = \frac{1}{2} e^{(D-3)\alpha\phi} \nabla_c \left[e^{-2(D-1)\alpha\phi} F_b{}^c \right] = 0 \\[2ex] \hat{R}_{\underline{z}\underline{z}} = (D-2)\alpha\, e^{-2\alpha\phi}\,\Box\phi + \frac{1}{4} e^{-2D\alpha\phi} F^2 = 0 \end{cases}$$

- It is important to notice that

$$\phi = 0 \implies \hat{R}_{\underline{z}\underline{z}} = \frac{1}{4} F^2 = 0 \implies \underline{F = 0}$$

$\hookrightarrow$ Trivial Maxwell !!

ii) D-dimensional EOMs

$$S_D = \frac{1}{2\kappa_D^2} \int d^D x \; e \left[R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{-2(D-1)\alpha\phi} F_{\mu\nu} F^{\mu\nu} \right]$$

The EOMs that follow from the above action are:

- $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (\partial\phi)^2 g_{\mu\nu} \right) + \frac{1}{2} e^{-2(D-1)\alpha\phi} \left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} F^2 g_{\mu\nu} \right)$$

- $$\nabla^\mu \left(e^{-2(D-1)\alpha\phi} F_{\mu\nu} \right) = 0$$

- $$\Box \phi = -\frac{1}{2} (D-1)\alpha \; e^{-2(D-1)\alpha\phi} F^2$$

▲ It is important to notice again that

$$\phi = 0 \implies F^2 = 0 \implies \underline{F = 0}$$

↳ Trivial Maxwell !!

IMPORTANT: Having set $\phi = 0$ in the Ansatz for the (D+1)-dimensional metric would have been inconsistent !! [common mistake]
[Einstein - Maxwell - DILATON]

iii) (D+1)-dimensional symmetries

The symmetry group is (D+1)-dimensional general coordinate transformations. At the infinitesimal level we have

$$\delta_{\hat{\xi}}\, \hat{g}_{MN} = \hat{\xi}^P \partial_P \hat{g}_{MN} + \hat{g}_{PN} \partial_M \hat{\xi}^P + \hat{g}_{MP} \partial_N \hat{\xi}^P$$

with

$$\hat{\xi}^M(x,z) = \left(\hat{\xi}^\mu(x,z) \,,\; \hat{\xi}^z(x,z) \right)$$

▲ However, in order to preserve the KK Ansatz of the (D+1)-dimensional metric, there are the restrictions:

Diffeom: $\hat{\xi}^\mu = \xi^\mu(x) \quad , \quad \hat{\xi}^z = \lambda(x) + \underbrace{c\, z}_{\text{linear dependence on } S^1}$

▲ On the other hand, the (D+1)-dimensional EOMs have a scaling symmetry [not the E-H action!!] of the form:

$$\left.\begin{array}{l} \hat{g}_{MN} \to a^2\, \hat{g}_{MN} \\ \hat{e} \to a^{(D+1)}\, \hat{e} \\ \hat{R} \to a^{-2}\, \hat{R} \end{array}\right\} \hat{e}\,\hat{R} \to \underbrace{a}_{a \in \mathbb{R}}{}^{(D-1)}\, \hat{e}\,\hat{R} \implies \underbrace{\delta_a \hat{g}_{MN} = 2a\, \hat{g}_{MN}}_{\text{infinitesim.}}$$

iv) D-dimensional symmetries

Starting from (D+1)-dimensional diffeomorphisms we will obtain D-dimensional diff + U(1) gauge symmetry + Global symmetries.

Ex : Using $\left\{ \begin{aligned} \hat g_{\mu\nu} &= e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu \\ \hat g_{\mu z} = \hat g_{z\mu} &= e^{2\beta\phi} A_\mu \\ \hat g_{zz} &= e^{2\beta\phi} \end{aligned} \right\}$ with $\beta = -(D-2)\alpha$

show that $\delta \hat g_{MN} = (\delta_\xi + \delta_a)\, \hat g_{MN}$ gives rise to :

$$\delta\phi = \xi^\rho \partial_\rho \phi - \frac{1}{(D-2)\alpha}(c+a)$$

$$\delta A_\mu = \xi^\rho \partial_\rho A_\mu + A_\rho \partial_\mu \xi^\rho + \partial_\mu \lambda - c A_\mu$$

$$\delta g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu} \partial_\mu \xi^\rho + g_{\mu\rho} \partial_\nu \xi^\rho + \frac{2}{(D-2)} \left[c + a(D-1) \right] g_{\mu\nu}$$

- Setting $a = -\frac{c}{(D-1)}$ one finds :

$$\delta\phi = \underbrace{\delta_\xi \phi}_{} - \overbrace{\frac{c}{(D-1)\alpha}}^{\text{shift}}$$

$$\delta A_\mu = \underbrace{\delta_\xi A_\mu}_{} + \underbrace{\partial_\mu \lambda}_{} - \overbrace{c A_\mu}^{\text{scaling}}$$

$$\delta g_{\mu\nu} = \underbrace{\delta_\xi g_{\mu\nu}}_{}$$

→ Global symmetry $\equiv \mathbb{R}$ (real parameter)

→ $U(1)$ gauge symmetry

→ D-dimensional diffeomorphisms

- Setting $a = -c$ one finds : $\underbrace{n\text{-legs} \Rightarrow n c}$

$$\delta\phi = \delta_\xi \phi \qquad (0\text{-legs})$$

$$\delta A_\mu = \delta_\xi A_\mu + \partial_\mu \lambda - \underbrace{c A_\mu} \qquad (1\text{-leg})$$

$$\delta g_{\mu\nu} = \delta_\xi g_{\mu\nu} - \underbrace{2 c \, g_{\mu\nu}} \qquad (2\text{-legs})$$

⟶ Real scaling $\mathbb{R}$ symmetry of the D-dimensional <u>EOMs</u> known as "trombone" scaling symmetry.

<u>Important</u> : There are two inequivalent $\mathbb{R}$ global symmetries. One is an actual symmetry of the D-dimens action whereas the other is only of the EOMs.

<u>Important</u> : In modern language, the global symmetries of the action upon dimensional reduction are referred to as "dualities". In this case the duality group is just $G_{\text{global}} = \mathbb{R}$ symmetry and affects scalar and vector fields in the reduced theory.

III. Kaluza – Klein reduction of Maxwell and scalar on S^1

In this section we look at other reductions on S^1. The starting point is a $(D+1)$-dimensional Maxwell field $\hat{B}_M$ with field strength $\hat{F}_{MN} = \partial_M \hat{B}_N - \partial_N \hat{B}_M$.

- The K-K Ansatz for $\hat{B}_M$ reads:

$$\boxed{\hat{B}_M = \left(\hat{B}_\mu \, , \hat{B}_z \right) = \left(B_\mu(x) \, , \chi(x) \right)}$$

which yields a field strength of the form:

$$\hat{F}_{MN} = \begin{bmatrix} \hat{F}_{\mu\nu} & \hat{F}_{\mu z} \\ \hat{F}_{z\nu} & 0 \end{bmatrix} = \begin{bmatrix} F_{\mu\nu} & \partial_\mu \chi \\ -\partial_\nu \chi & 0 \end{bmatrix}$$

with:

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$F_{\mu z} = \partial_\mu \chi$$

$$F_{z\nu} = -\partial_\nu \chi$$

- The Maxwell's action in $(D+1)$-dimensions then reduces to:

$$S_{\hat{B}} = -\frac{1}{4}\int d^{D+1}x \, \underbrace{\sqrt{|\hat{g}|}}_{\hat{e}} \, \hat{F}_{MN} \, \hat{F}^{MN} = -\frac{1}{4}\int d^{D+1}x \, \hat{e} \, \hat{F}_{AB} \, \hat{F}^{AB} = (*)$$

NOTE 1: $\hat{F}_{AB} = \hat{e}_A{}^M \hat{e}_B{}^N \hat{F}_{MN}$

- $\hat{F}_{ab} = \hat{e}_a{}^M \hat{e}_b{}^N \hat{F}_{MN}$

$$= \hat{e}_a{}^\mu \hat{e}_b{}^\nu \hat{F}_{\mu\nu} + \hat{e}_a{}^z \hat{e}_b{}^\nu \hat{F}_{z\nu} + \hat{e}_a{}^\mu \hat{e}_b{}^z \hat{F}_{\mu z} + \hat{e}_a{}^z \hat{e}_b{}^z \overbrace{\hat{F}_{zz}}^{0}$$

$$= e^{-2\alpha\phi} F_{ab} + e^{-2\alpha\phi} A_a \partial_b \chi - e^{-2\alpha\phi} A_b \partial_a \chi$$

$$= e^{-2\alpha\phi}\Big[\underbrace{F_{ab} - (\partial_a \chi A_b - \partial_b \chi A_a)}_{\mathcal{F}_{ab} \equiv F_{ab} - 2\partial_{[a}\chi A_{b]}} \Big] = e^{-2\alpha\phi} \mathcal{F}_{ab}$$

- $\hat{F}_{az} = \hat{e}_a{}^M \hat{e}_z{}^N \hat{F}_{MN}$

$$= \hat{e}_a{}^\mu \underbrace{\hat{e}_z{}^\nu}_{0} \hat{F}_{\mu\nu} + \hat{e}_a{}^z \underbrace{\hat{e}_z{}^\nu}_{0} \hat{F}_{z\nu} + \hat{e}_a{}^\mu \hat{e}_z{}^z \hat{F}_{\mu z} + \hat{e}_a{}^z \hat{e}_z{}^z \overbrace{\hat{F}_{zz}}^{0}$$

$$= e^{-(\alpha+\beta)\phi} \partial_a \chi \quad = -\hat{F}_{za}$$

- $\hat{F}_{zz} = 0$

NOTE 2: $\hat{e} = e^{(\alpha D + \beta)\phi} e$

$$(*) = -\frac{1}{4} e^{(\alpha D+\beta)\phi} (2\pi L) \int d^D x \, e \Big[\hat{F}_{ab} \hat{F}^{ab} + \overbrace{\hat{F}_{az} \hat{F}^{az} + \hat{F}_{zb} \hat{F}^{zb}}^{2\hat{F}_{az}\hat{F}^{az}} \Big]$$

$$= -\frac{1}{4} (2\pi L) e^{(\alpha D+\beta)\phi} \int d^D x \, e \Big[e^{-4\alpha\phi} \mathcal{F}_{ab} \mathcal{F}^{ab} + 2 e^{-2(\alpha+\beta)\phi} \partial_a \chi \partial^a \chi \Big]$$

$$\boxed{S_{\hat{B}} = (2\pi L) \int d^D x \, e \Big[-\frac{1}{4} e^{-2\alpha\phi} \mathcal{F}^2 - \frac{1}{2} e^{2(D-2)\alpha\phi} (\partial\chi)^2 \Big]}$$

- The reduction of the scalar field is straightforward. The KK Ansatz reads:

$$\boxed{\hat{\Phi} = \Phi(x)}$$

So that the action reduces as:

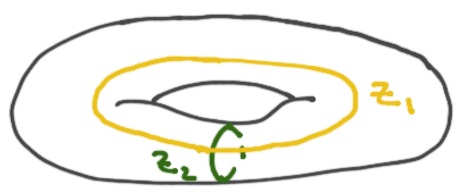
$$\boxed{S_{\hat{\Phi}} = -\frac{1}{2}\int d^{D+1}x \; \hat{e}\; \partial_M \hat{\Phi}\, \partial^M \hat{\Phi} = (2\pi L)\int d^D x\, e\left[-\frac{1}{2}\partial_\mu \Phi\, \partial^\mu \Phi\right]}$$

NOTE 1: $\hat{e} = e^{(D\alpha+\beta)\phi}\; e \overset{\beta=-(D-2)\alpha}{=} e^{2\alpha\phi}\, e$

NOTE 2: $\partial_A \hat{\Phi} = (\hat{e}_a{}^\mu \partial_\mu \Phi, 0) = e^{-\alpha\phi}(\partial_a \Phi, 0)$

IV. Kaluza-Klein reduction on T^2 and SL(2) duality

In this section we are combining the previous results above and will present the KK reduction of gravity in (D+2) dimensions:



$T^2 \equiv$ 2-torus

coordinates (z_1, z_2)

$$\hat{\hat{g}}_{\mathcal{M}\mathcal{N}} \Rightarrow \hat{g}_{MN} + \hat{A}_{M1} + \hat{\phi}_1 \Rightarrow g_{\mu\nu} + A_{\mu 2} + \phi_2 + A_{\mu 1} + \chi + \phi_1$$

step 1 $\qquad\qquad\qquad$ step 2

$\mathcal{M} = M, z_1 \qquad\qquad M = \mu, z_2$

- Reduction along z_1 :

$$S_{D+2} = \frac{1}{2\kappa_{D+2}^2} \int d^D x \, dz_2 \, dz_1 \; \hat{\hat{e}} \; \hat{\hat{R}}$$

$$= \frac{1}{2\kappa_{D+1}^2} \int d^D x \, dz_2 \; \hat{e} \left[\hat{R} - \frac{1}{2} (\partial \hat{\phi}_1)^2 - \frac{1}{4} e^{-2D\alpha_1 \hat{\phi}_1} \hat{F}_1^2 \right] \equiv S_{D+1}$$

with $\kappa_{D+1}^2 = \dfrac{\kappa_{D+2}^2}{2\pi L_1}$ and $\alpha_1^2 = \dfrac{1}{2(D-1)D}$

- Reduction along z_2 :

$$S_D = \frac{1}{2\kappa_D^2} \int d^D x \; e \Big[R - \frac{1}{2} (\partial \phi_2)^2 - \frac{1}{4} e^{-2(D-1)\alpha_2 \phi_2} F_2^2$$

$$- \frac{1}{2} (\partial \phi_1)^2$$

$$+ e^{-2D\alpha_1 \phi_1} \left(- \frac{1}{4} e^{-2\alpha_2 \phi_2} \mathcal{F}_1^2 - \frac{1}{2} e^{2(D-2)\alpha_2 \phi_2} (\partial \chi)^2 \right) \Big]$$

$$= \frac{1}{2\kappa_D^2} \int d^D x \; e \Big[R - \frac{1}{2} (\partial \phi_1)^2 - \frac{1}{2} (\partial \phi_2)^2 - \frac{1}{2} e^{-2D\alpha_1 \phi_1 + 2(D-2)\alpha_2 \phi_2} (\partial \chi)^2$$

$$- \frac{1}{4} e^{-2D\alpha_1 \phi_1 - 2\alpha_2 \phi_2} \mathcal{F}_1^2 - \frac{1}{4} e^{-2(D-1)\alpha_2 \phi_2} F_2^2 \Big]$$

with $\alpha_2^2 = \dfrac{1}{2(D-2)(D-1)}$ and $\mathcal{F}_{\mu\nu 1} = F_{\mu\nu 1} - 2 \partial_{[\mu} \chi A_{\nu] 2}$

The action S_D can be enlightening rewritten as

$$S_D = \frac{1}{2\kappa_D^2}\int d^Dx \; e \left[R - \frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2 - \frac{1}{2}e^{\vec{c}\vec{\phi}}(\partial\chi)^2 - \frac{1}{4}e^{\vec{c}_1\vec{\phi}}\mathcal{F}_1^{\,2} - \frac{1}{4}e^{\vec{c}_2\vec{\phi}}\mathcal{F}_2^{\,2}\right]$$

with

$$\mathcal{F}_{\mu\nu 1} = F_{\mu\nu 1} - 2\,\partial_{[\mu}\chi\, A_{\nu]2}$$
$$\mathcal{F}_{\mu\nu 2} = F_{\mu\nu 2}$$

and

$$\kappa_D^{\,2} = \frac{\kappa_{D+1}^2}{2\pi L_2} = \frac{\kappa_{D+2}^2}{(2\pi L_1)(2\pi L_2)} = \frac{\kappa_{D+2}^2}{\mathrm{Vol}(T^2)}$$

$$\vec{c} = \left[-\sqrt{\frac{2D}{D-1}} \,,\; \sqrt{\frac{2(D-2)}{D-1}} \right]$$

$$\vec{c}_1 = \left[-\sqrt{\frac{2D}{D-1}} \,,\; -\sqrt{\frac{2}{(D-1)(D-2)}} \right]$$

$$\vec{c}_2 = \left[\;0\; ,\; -\sqrt{\frac{2(D-1)}{D-2}} \right]$$

- Let us rotate the two dilatons $\vec{\phi} = (\phi_1, \phi_2)$ to new ones:

$$\phi = -\frac{1}{2}\sqrt{\frac{2D}{D-1}}\,\phi_1 + \frac{1}{2}\sqrt{\frac{2(D-2)}{D-1}}\,\phi_2$$

$$\varphi = -\frac{1}{2}\sqrt{\frac{2(D-2)}{D-1}}\,\phi_1 - \frac{1}{2}\sqrt{\frac{2D}{D-1}}\,\phi_2$$

so the D-dimensional action becomes

$$S_D = \frac{1}{2K_D^2}\int d^Dx\, e\left[R - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2 - \frac{1}{4}e^{q\varphi+\phi}\mathcal{F}_1^{\,2} - \frac{1}{4}e^{q\varphi-\phi}\mathcal{F}_2^{\,2}\right]$$

with $q^2 = \frac{D}{D-2}$ and the $(D+2)$-dimensional metric reads

$$ds^2_{D+2} = e^{-\frac{2}{\sqrt{D(D-2)}}\varphi}\,ds_D^2 + e^{\sqrt{\frac{D-2}{D}}\varphi}\,ds_2^2$$

with

$$ds_2^2 = e^{\phi}(dz_1 + A_{\mu 1}dx^\mu + \chi\, dz_2)^2 + e^{-\phi}(dz_2 + A_{\mu 2}dx^\mu)^2$$

$$\Rightarrow\ ds_2^2\Big|_{\phi=\chi=A_{\mu 1,2}=0} = dz_1^2 + dz_2^2$$

Moduli space: (scalars ≡ "moduli")

- The scalar φ parameterises the volume of volume of T^2 as it appears as a factor in front of ds_2^2.

- The scalar ϕ and χ play different roles. The scalar ϕ parameterises a shape-changing of the torus. It scales the z_1-cycle and the z_2-cycle in opposite manners. The scalar χ varies the angle between the z_1-cycle and the z_2-cycle.

Scalar sector and SL(2) duality

Let's focus on the scalar sector of the above S_D action:

$$\mathcal{L}_{scalar} = -\frac{1}{2}(\partial\varphi)^2 \underbrace{- \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi^2)}_{\mathcal{L}(\phi,\chi)}$$

Global symmetries (or dualities)

i) The scalar φ decouples from the others. It has a global $\mathbb{R}$ shift symmetry

$$\varphi \longrightarrow \varphi + K \qquad \text{with} \quad K \in \mathbb{R}$$

ii) The symmetry analisis for $\mathcal{L}(\phi,\chi)$ is more interesting. To make the symmetry manifest we define a complex modulus field on T^2 as

$$\tau = \chi + i\, e^{-\phi}$$

in terms of which

$$\mathcal{L}(\phi,\chi) = -\frac{1}{2}\left[(\partial\phi)^2 + e^{2\phi}(\partial\chi)^2\right] = -\frac{\partial_\mu\tau\, \partial^\mu\bar{\tau}}{2\,\mathrm{Im}^2(\tau)}$$

Ex: Show that $\mathcal{L}(\phi, \chi)$ is invariant under the global fractional linear transformation:

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}$$

with $ad - bc = 1$. Show that this transformation acts on (ϕ, χ) as:

$$e^{\phi} \longrightarrow e^{\phi'} = (c\chi + d)^2 e^{\phi} + c^2 e^{-\phi}$$
$$\chi e^{\phi} \longrightarrow \chi' e^{\phi'} = (a\chi + b)(c\chi + d) e^{\phi} + ac\, e^{-\phi}$$

iii) As scalars couple to vectors, these must also transform. Let us write a constant 2×2 matrix Λ of the form

$$\Lambda = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{with} \qquad |\Lambda| = ad - bc = 1$$

so that $\Lambda \in SL(2)$. Using this matrix Λ, the transformation

$$\begin{bmatrix} A_2 \\ A_1 + \chi A_2 \end{bmatrix} \longrightarrow (\Lambda^t)^{-1} \begin{bmatrix} A_2 \\ A_1 + \chi A_2 \end{bmatrix}$$

precisely cancells against the transformation of scalars and leaves the action invariant.

As a result, the global symmetry (duality) of the KK reduction of gravity on T^2 turns out to be:

$$G_{global} = \mathbb{R} \times SL(2)$$

Some final remarks:

- If gravity in $(D+n)$ dimensions is reduced on T^n then the duality group becomes $G_{global} = \mathbb{R} \times SL(n)$

- If we start from the unique supergravity theory in 11D and reduce it on T^n then the duality group gets enhanced to the exceptional $G_{global} = E_{n(n)}$

$$S_{11D}^{SUGRA} = \frac{1}{2\kappa_{11D}^2} \int d^{11}x \ \hat{e} \left[\underbrace{\hat{R}}_{\mathbb{R} \times SL(n)} - \frac{1}{2 \times 4!} \underbrace{F^2}_{\text{enhancement to } E_{n(n)}} \right] + \dots$$

where $F^2 = F_{MNPQ} F^{MNPQ}$ with $F_{MNPQ} = \partial_{[M} A_{NPQ]}$.

- Duality transformations allow us to explore diffent regimes of the theory. For example large vs small extra dimensions or weak vs strong coupling.