Cosmological Perturbations

Quantum fluctuations during inflation provide the initial inhomogeneities for structure formation. Perturbation over the FLRW background are induced by quantum fluctuations of matter (inflaton field) by virtue of Einstein equations

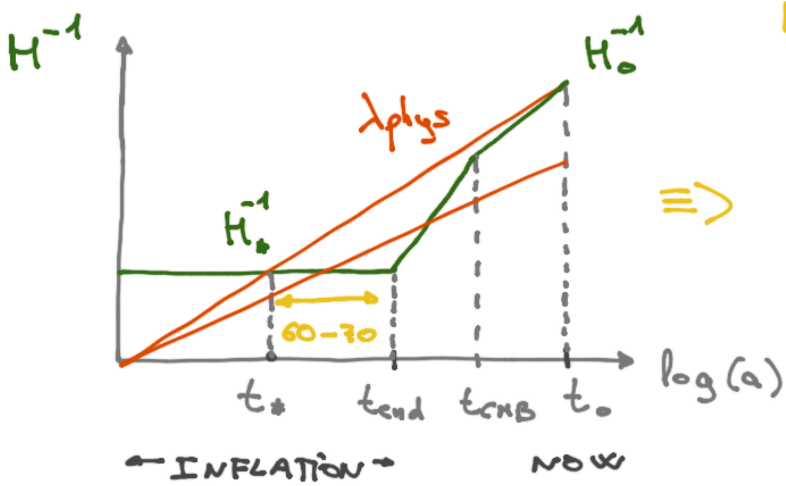
$$\delta G_{\mu\nu} = \kappa^2 \, \delta T_{\mu\nu}$$

These fluctuations on $\delta\phi$ are stretched to "superhorizon" size

$$\lambda_{phys} = \lambda \cdot a(t) > H^{-1}$$

and get "frozen", namely, $|\delta\phi(\vec{x},t)| = cte$.

Afterwards, these fluctuations re-enter de horizon and become dynamical again (see figure)



Large scales were still outside the horizon at recombination when CMB was formed !!

I. Fluctuations of a scalar field

Let us consider a scalar field

"comoving coordinates"

$ds^2 = -dt^2 + a^2(t)\, d\vec{x}^2$

$$\phi(t,\vec{x}) = \phi(t) + \underbrace{\delta\phi(t,\vec{x})}_{\text{linear order}}$$

(quantum origin)

Performing a Fourier transformation to momentum space

$$\delta\phi(t,\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{k}\; \underbrace{\overbrace{\delta\phi_k(t)}^{\text{amplitude}}}_{\substack{\text{mode of momentum } |\vec{k}| \\ \text{(isotropic)}}}\; e^{i\vec{k}\cdot\vec{x}}$$

where $k \equiv |\vec{k}|$.

The Klein-Gordon equation for the fluctuation reads

$$\delta\ddot{\phi}_k + 3H\,\delta\dot{\phi}_k + \underbrace{\frac{k^2}{a^2}\,\delta\phi_k}_{\frac{\nabla^2\delta\phi_k}{a^2} \text{ in momentum space}} + \frac{\partial^2 V}{\partial\varphi^2}\,\delta\phi_k = 0 \qquad (1)$$

It will be convenient to switch to conformal time

$$d\eta \equiv \frac{dt}{a} \implies ds^2 = a^2(\eta)\left[-d\eta^2 + \underbrace{d\vec{x}^2}_{\text{comoving coordinates}}\right]$$

so that, in a de Sitter period with $a = e^{Ht}$ ($H = \text{cte}$), one has

$$\eta = \int \frac{dt}{a} = \int dt\, e^{-Ht} = -\frac{e^{-Ht}}{H} = -\frac{1}{aH}$$

NOTE: Conformal time is therefore negative : $\eta \to 0$ $(t \to \infty)$

It will also be useful to consider a new variable

$$\sigma_K \equiv a\, \delta\phi_K$$

In terms of this variable, the equation (1) reduces to

$$\sigma_K'' + \underbrace{\left(K^2 - \frac{a''}{a}\right)}_{\text{Time-dependent harmonic oscillator !!}} \sigma_K = 0 \qquad (2)$$

with $\sigma_K = \sigma_K(\eta)$, $\sigma_K' = \frac{d\sigma_K}{d\eta}$, etc.

NOTE: Recall that the harmonic oscillator has an equation of motion $\ddot{x} + K^2 x = 0$

Importantly, the equation of motion (2) for the scalar field mode v_k can be obtained from a quadratic action

$$S^{(2)} = \frac{1}{2}\int d\eta\, d^3\vec{x}\, \left[v'^2 - (\nabla v)^2 + \frac{a''}{a} v^2 \right] \qquad (3)$$

upon using the mode expansion

$$\delta\phi(t,\vec{x}) = \frac{1}{(2\pi)^{3/2}}\int d^3\vec{k}\;\; \delta\phi_k(t)\, e^{i\vec{k}\cdot\vec{x}}$$

NOTE: We will come back to the action $S^{(2)}$ later on when studying perturbations of the metric around the FLRW background.

We will search for $v_k(\eta)$ solutions of (2) in two different regimes and then impose a junction condition:

i) Subhorizon limit:

$$\lambda_{phys} = a\,\lambda \ll H^{-1} \iff k_{phys} = \frac{2\pi}{\lambda_{phys}} = \frac{1}{a}\,k \gg H$$

$$\Rightarrow \boxed{k \gg a H = -\frac{1}{\eta}} \iff k\eta \gg -1$$

de Sitter

In this limit

$$k^2 \gg \frac{2}{\eta^2} = \frac{a''}{a}$$

NOTE: $aH = -\frac{1}{\eta} \Rightarrow \dot{a} = a' \cdot \frac{d\eta}{dt} = \frac{a'}{a} = -\frac{1}{\eta}$

$$\Rightarrow \frac{d}{d\eta}\left(\frac{a'}{a}\right) = \frac{a''}{a} - \underbrace{\frac{a'^2}{a^2}}_{\frac{1}{\eta^2}} = \frac{1}{\eta^2}$$

$$\Rightarrow \frac{a''}{a} = \frac{2}{\eta^2}$$

and equation (2) simplifies to

$$v_k'' + k^2 v_k = 0 \quad \Rightarrow \text{Oscilator}$$

$$\rightsquigarrow \lim_{k\eta \gg -1} v_k(\eta) = \frac{1}{\sqrt{2k}}\, e^{-ik\eta}$$

"Bunch-Davies vacuum"

Message: Modes oscillate inside the horizon H^{-1}.

ii) Superhorizon limit

$$\lambda_{phys} = a\,\lambda \gg H^{-1} \iff k_{phys} = \frac{2\pi}{\lambda_{phys}} = \frac{1}{a}\,k \ll H$$

$$\Rightarrow \boxed{k \ll aH \underset{\text{de Sitter}}{=} -\frac{1}{\eta}} \iff k\eta \ll -1$$

In this limit

$$k^2 \ll \frac{2}{\eta^2} = \frac{a''}{a}$$

and equation (2) simplifies to

$$v_k'' - \frac{a''}{a}\,v_k = 0$$

$$\Rightarrow \lim_{k\eta \ll -1} v_k(\eta) = \underbrace{B_k}_{\text{cte}}\,a(\eta)$$

$$\Rightarrow \lim_{k\eta \ll -1} \delta\phi_k = \lim_{k\eta \ll -1} \frac{v_k}{a} = B_k = \underline{\underline{\text{cte}}}$$

Message: Modes are <u>frozen</u> when crossing the horizon H^{-1}.

The junction condition at $k\eta = -1$ or, equivalently,

$$k = aH \quad \approx\!> \text{"Horizon crossing"}$$

fixes $|B_k|$

$$|B_k|\, a = |B_k| \frac{k}{H} = \frac{1}{\sqrt{2k}} \implies |B_k| = \frac{H}{\sqrt{2k^3}}$$

Then, on superhorizon scales, one finds

$$\lim_{k\eta \ll -1} |\delta\phi_k| = |B_k| = \underbrace{\frac{H}{\sqrt{2k^3}}}_{\text{Different modes freeze out with different amplitud}}$$

and define the scalar field power spectrum as

$$P_{\delta\phi}(k) \equiv \frac{k^3}{2\pi^2} \lim_{k\eta \ll -1} \left\langle |\delta\phi_k|^2 \right\rangle = \frac{k^3}{2\pi^2} \frac{H^2}{2k^3} = \left(\frac{H}{2\pi}\right)^2$$

II. Metric perturbations

Idea: Scalar fluctuations give rise to $T_{\mu\nu}$ perturbations and these, by virtue of the Einstein equations, produce metric perturbations.

density perturbation

Idea: Since $|\delta\phi_k|$ translates into $|\delta T^0{}_0| = \delta\rho$ and these are small (as measured at CMB), a linearised analysis of Einstein equations is appropriate.

A general perturbation of the FLRW metric is given by

$$g_{\mu\nu}(t,\vec{x}) = \underbrace{g_{\mu\nu}(t)}_{\text{FLRW}} + \underbrace{\delta g_{\mu\nu}(t,\vec{x})}_{\substack{\text{fluctuation} \\ \text{(linear order)}}}$$

so that

$$ds^2 = a^2(\eta)\left[-(1+2\Phi)\,d\eta^2 + 2\bar{B}_i\,d\eta\,dx^i + (\delta_{ij}+\bar{h}_{ij})\,dx^i dx^j\right]$$

where

$$\bar{B}_i = \frac{\partial B}{\partial x^i} + B_i$$

divergentless $\rightsquigarrow$ $\partial_i B^i = 0$

transverse $\to$ $\partial_i h^{ij} = 0$

traceless $\to$ $h_i{}^i = 0$

and

$$\bar{h}_{ij} = -2\psi\,\delta_{ij} + 2\frac{\partial^2 E}{\partial x^i \partial x^j} + 2\,\partial_{(i}E_{j)} + h_{ij}$$

transverse $\to$ $\partial_i E^i = 0$

NOTE: Counting degrees of freedom (d.o.f)

- General : $\delta g_{\mu\nu} = \delta g_{\nu\mu} \Rightarrow \underbrace{\frac{4\times 5}{2}}_{\text{symmetric}} = 10$ d.o.f

- Parameterisation :

$$\Phi \equiv 1 \text{ d.o.f} \quad (\text{scalar mode})$$

$\bar{B}_i$ $\begin{cases} B \equiv 1 \text{ d.o.f} & (\text{scalar mode}) \\ B_i \equiv 3 - \underset{\text{divergentless}}{1} = 2 \text{ d.o.f} & (\text{vector mode}) \end{cases}$

$\bar{h}_{ij}$ $\begin{cases} \psi \equiv 1 \text{ d.o.f} & (\text{scalar mode}) \\ E \equiv 1 \text{ d.o.f} & (\text{scalar mode}) \\ E_i \equiv 3 - \underset{\text{transverse}}{1} = 2 \text{ d.o.f} & (\text{vector mode}) \\ h_{ij} \equiv \underbrace{\frac{3\times 4}{2}}_{\text{symmetric}} - \underbrace{3}_{\text{transverse}} - \underbrace{1}_{\text{traceless}} = 2 & (\text{tensor modes}) \end{cases}$

$$\rightsquigarrow \text{Total} = \underbrace{(1+1+1+1)}_{\text{scalar modes}} + \underbrace{(2+2)}_{\text{vector modes}} + \underbrace{2}_{\text{tensor modes}}$$

- At linear order scalar, vector and tensor modes decouple in the Einstein equations so one can follow their evolution separately.

Important: Vector perturbations decay $(\sim \frac{1}{a^2})$ in an expanding Universe, so we are no longer considering them.

The most general scalar and tensor perturbations then yield

$$ds^2 = a^2 \left\{ -(1+2\Phi)\, d\eta^2 + \frac{\partial B}{\partial x^i}\, d\eta\, dx^i \right.$$

$$\left. + \left[(1-2\Psi)\, \delta_{ij} + 2 \frac{\partial^2 E}{\partial x^i \partial x^j} + h_{ij} \right] dx^i dx^j \right\}$$

II.1. Scalar modes and curvature power spectrum

We now focus on de 4 scalar d.o.f : E, B, Ψ, Φ

Gauge invariance : Under a general coordinate transformation the various scalar modes transform in a non-trivial manner. Under a specific gauge-fixing [see Postma], the scalar modes (E, B) can be eliminated

$$E = B = 0$$

Moreover, the absence of anisotropic perturbations in the energy momentum tensor

$$\delta T^i{}_j = 0 \quad \text{(always for scalar field inflation)}$$

imposes (via the Einstein equations) the constraint

$$\delta T^i{}_j = 0 \quad \Longrightarrow \quad \Phi = \Psi$$

Message: The remaining metric and scalar field perturbations are related by the other Einstein equations. The scalar field perturbations are responsible for the temperature (density) anisotropies of the CMB, which was produced $\sim 60-70$ e-folds before the end of inflation when $k = aH$ and perturbations got frozen.

An important quanty to be introduced is the so called "comoving curvature perturbation"

$$\mathcal{R} = \Psi + \underbrace{\mathcal{H}}_{\frac{a'}{a}} \frac{\delta\phi}{\phi'} = \Psi + \underbrace{H}_{\frac{\dot a}{a}} \frac{\delta\phi}{\dot\phi}$$

This is the quantity that appears when studying the perturbation of (unperturbed) Einstein-scalar action

$$S^{(0)}[\phi(t), g_{\mu\nu}(t)] = \int d^4x \sqrt{-|g|} \left[\frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu\phi\, \partial_\nu\phi - V(\phi) \right]$$

The perturbed action is given by

$$S[g_{\mu\nu},\phi] = \underbrace{S^{(0)}[\phi(t), g_{\mu\nu}(t)]}_{\substack{\text{unperturbed} \\ \text{(homogeneous terms)}}} + \underbrace{S^{(2)}\left[\delta\phi, \delta g_{\mu\nu}; \phi(t), g_{\mu\nu}(t)\right]}_{\substack{\text{Terms quadratic in the} \\ \text{linear perturbations}}}$$

NOTE: $S^{(1)} = 0 \Longrightarrow$ If $\phi(t)$ and $g_{\mu\nu}(t)$ extremise the action (are solutions)

After some cumbersome manipulations, the quadratic piece $S^{(2)}$ can be expressed as

$$S^{(2)} = \frac{1}{2}\int d\eta\, d^3\vec{x}\left[v'^2 - (\nabla v)^2 + \frac{z''}{z}v^2\right]$$

in terms of a single scalar variable

$$v = a\left(\delta\phi + \frac{\phi'}{\mathcal{H}}\psi\right)$$

$$= a\frac{\phi'}{\mathcal{H}}\mathcal{R} = z\underbrace{\mathcal{R}}_{\text{comoving curvature perturbation}}$$

with

$$z = a\frac{\phi'}{\mathcal{H}} = a\frac{\dot{\phi}}{H}$$

NOTE: $\mathcal{H} = \frac{a'}{a} = \frac{\dot{a}a}{a} = \dot{a} = Ha$; $\phi' = \dot{\phi}a$; $\frac{dt}{d\eta} = a$

Important : We have obtained the exact same action $S^{(2)}$ as when studying the scalar field in (3). There the relevant variable was $v = a\,\delta\phi$ whereas now, after taking into account scalar fluctuations of the metric too, the relevant variable is

$$v = a\left(\delta\phi + \frac{\phi'}{\mathcal{H}}\psi\right) = z\,\mathcal{R}$$

From the variable $v = z\,\mathcal{R}$ we can define the "comoving curvature power spectrum" just by performing a mode expansion and replacing $a \leftrightarrow z$. The result is then given by

$$P_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2}\frac{|v_k|^2}{z^2} = \underbrace{\frac{k^3}{2\pi^2}\frac{|v_k|^2}{a^2}}_{\left(\frac{H}{2\pi}\right)^2}\left(\frac{H}{\dot\phi}\right)^2$$

$$= \frac{1}{8\pi^2}\frac{H^4}{\frac{1}{2}\dot\phi^2}\Bigg|_{k=aH} = \frac{1}{8\pi^2}\frac{H^4}{\mathcal{L}_{kin}}\Bigg|_{k=aH}$$

which is evaluated at horizon exist, namely, when perturbations exit the horizon.

II.2. Tensor modes and tensorial power spectrum

As we saw before, the 2 tensor modes in h_{ij} determine fluctuations of the metric of the form

$$\delta g_{\mu\nu} = a^2 \begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix}$$

with $\partial_i h^{ij} = 0$ (divergenceless) and $h_i{}^i = 0$ (traceless)

It can be shown that these 2 d.o.f are gauge invariant and correspond to the 2 polarizations of the gravitational waves.

When looking at the action $S^{(2)}$ controlling the dynamics of these modes one finds that

$$S^{(2)} = \frac{1}{2} \int d^4x \sqrt{-|g|} \, \frac{1}{2} \, \partial_\sigma h_{ij} \, \partial^\sigma h^{ij}$$

so that tensor modes do not mix with the scalar fields provided the stress-energy tensor is diagonal.

Then, introducing the relevant variable v_{ij} and performing a mode expansion accounting for both polarizations $(\lambda = +, -)$

$$v_{ij} = a\, h_{ij} = \underbrace{\frac{1}{\sqrt{2}}}_{\text{normalisation factor}} \underbrace{\sum_{\lambda=+,-}}_{\text{sum over polarisations}} \int \frac{d^3\vec{k}}{(2\pi)^3} \overbrace{v_{k,\lambda}}^{\text{amplitude}} \underbrace{\epsilon_{ij}(\vec{k};\lambda)}_{\text{polarisation tensor}} e^{i\vec{k}\vec{x}}$$

The tensor modes are then governed by an action $S^{(2)}$ that is analogue to the scalar field case discussed in (3). Taking into account the 2 polarisations and a renormalisation factor that can be traced back to the $\frac{1}{\sqrt{2}}$ factor above, one arrives at a "tensorial power spectrum" of the form

$$P_T(k) = 2 \times 4 \times \left(\frac{H}{2\pi}\right)^2$$

Note: $P_T(k)$ is controlled totally by H^2. During the slow-roll period, $H^2 = \frac{V}{3}$ and only the scale of inflation V_{cte} matters.

III. Inflationary prediction for the CMB

Important: Observable scales now leave the horizon at $N_* \sim 60$ e-folds before the end of inflation $\Rightarrow$ $P_R(k)$ has to be measured at that time:

$$t = t_* \quad \text{when} \quad \phi = \phi_* = \phi(t_*)$$

* Scalar spectral index:

$$P_R(k) = \frac{1}{8\pi^2}\frac{H^4}{\frac{1}{2}\dot{\phi}^2}\Bigg|_{k=aH} \approx \frac{1}{8\pi^2}\frac{H^4\, 3}{\epsilon V}\Bigg|_{k=aH}$$

$$\approx \frac{1}{24\pi^2}\frac{V}{\epsilon}\Bigg|_{k=aH} \equiv A_s$$

where we have used the slow-roll conditions

$$\frac{1}{2}\dot{\phi}^2 = \frac{\epsilon V}{3} \quad \text{and} \quad H^2 = \frac{V}{3}$$

On the other hand, the scale dependence is parameterised by the spectral index n_s.

Taking a power scaling of the form

$$P_{\mathcal{R}}(k) \approx A_s \left(\frac{k}{k_*}\right)^{n_s - 1 + \dots} \quad \overset{\frac{1}{2}\frac{dn_s}{d\ln k}\ln\left(\frac{k}{k_*}\right) + \frac{1}{3!}\frac{d^2 n_s}{d\ln k^2}\left[\ln\left(\frac{k}{k_*}\right)\right]^2}{}$$

$\hookrightarrow k_* =$ reference scale

$$\Rightarrow \ln P_{\mathcal{R}} \propto (n_s - 1) \ln k$$

$\hookrightarrow$ Lowest order

then

$$n_s - 1 \equiv \frac{d\ln P_{\mathcal{R}}}{d\ln k} = \frac{d\ln P_{\mathcal{R}}}{\underbrace{d\ln(aH)}_{d(\ln a + \underbrace{\ln H}_{\sim cte})}} \approx \frac{d\ln P_{\mathcal{R}}}{d\ln a} = \frac{d\ln P_{\mathcal{R}}}{-dN}$$

$\hookrightarrow a \sim e^{-N}$

$-dN = H dt$

$= H \frac{d\phi}{\dot{\phi}}$

$$= \frac{\dot{\phi}}{H} \frac{d\ln P_{\mathcal{R}}}{d\phi} = -\frac{V_\phi}{V} \frac{d\ln P_{\mathcal{R}}}{d\phi} = 2\eta - 6\epsilon$$

$\hookrightarrow H^2 \approx \frac{V}{3}$

(slow-roll) $\dot{\phi} \approx -\frac{V_\phi}{3H}$ $\Bigg\}$ $\frac{\dot{\phi}}{H} = -\frac{V_\phi}{3H^2} = -\frac{V_\phi}{V}$

$$\Rightarrow n_s - 1 \approx 2\eta - 6\epsilon$$

We can further compute :

$$\frac{dn_s}{d\ln k} = -16\epsilon\eta + 24\epsilon^2 + 2\xi^2 \equiv \text{"Running of the spectral index"}$$

where $\xi^2 = \dfrac{V_\phi V_{\phi\phi\phi}}{V^2}$

* Tensor spectral index :

$$P_T(k) = 2 \times 4 \times \left(\frac{H}{2\pi}\right)^2 \Big|_{k=aH} = \frac{2V}{3\pi^2} \Big|_{k=aH} \equiv A_t$$

where we have used the slow-roll condition $H^2 \approx \frac{V}{3}$.

On the other hand, the scale dependence is parameterised by the spectral index n_t. Taking a power scaling of the form

$$\frac{1}{2}\frac{dn_t}{d\ln k}\ln\left(\frac{k}{k_*}\right) + \frac{1}{3!}\frac{d^2 n_t}{d\ln k^2}\left[\ln\left(\frac{k}{k_*}\right)\right]^2 + \dots$$

$$P_t(k) \approx A_t\left(\frac{k}{k_*}\right)^{n_t + \dots} \Rightarrow \ln P_t(k) \propto n_t \ln k$$

k_* = reference scale

lowest order

Then

$$n_t \equiv \frac{d\ln P_t(k)}{d\ln k} \approx -2\epsilon$$

* Tensor to scalar ratio :

Since $P_{\mathcal{R}}(k)$ is measured and $P_t \propto V$ then r sets the scale of inflation V

$$r \equiv \frac{P_t(k)}{P_{\mathcal{R}}(k)} \approx \frac{\frac{2V}{3\pi^2}}{\frac{1}{24\pi^2}\frac{V}{\epsilon}} = 16\,\epsilon \approx -8\,n_t$$

Alternatively :

$$dN = -H\,dt$$

$$r \equiv \frac{P_t(k)}{P_{\mathcal{R}}(k)} = \frac{2\times 4\left(\frac{H}{2\pi}\right)^2}{\frac{1}{8\pi^2}\frac{H^4}{\frac{1}{2}\dot{\phi}^2}} = 8\left(\frac{\dot{\phi}}{H}\right)^2 = 8\left(\frac{d\phi}{dN}\right)^2 \Rightarrow \Delta\phi = \int_0^{N_*} dN \left(\frac{r}{8}\right)^{\frac{1}{2}}$$

→ Lyth bound ($r \approx$ cte during slow-roll) : $\Delta\phi \approx N_*\left(\frac{r}{8}\right)^{\frac{1}{2}}$

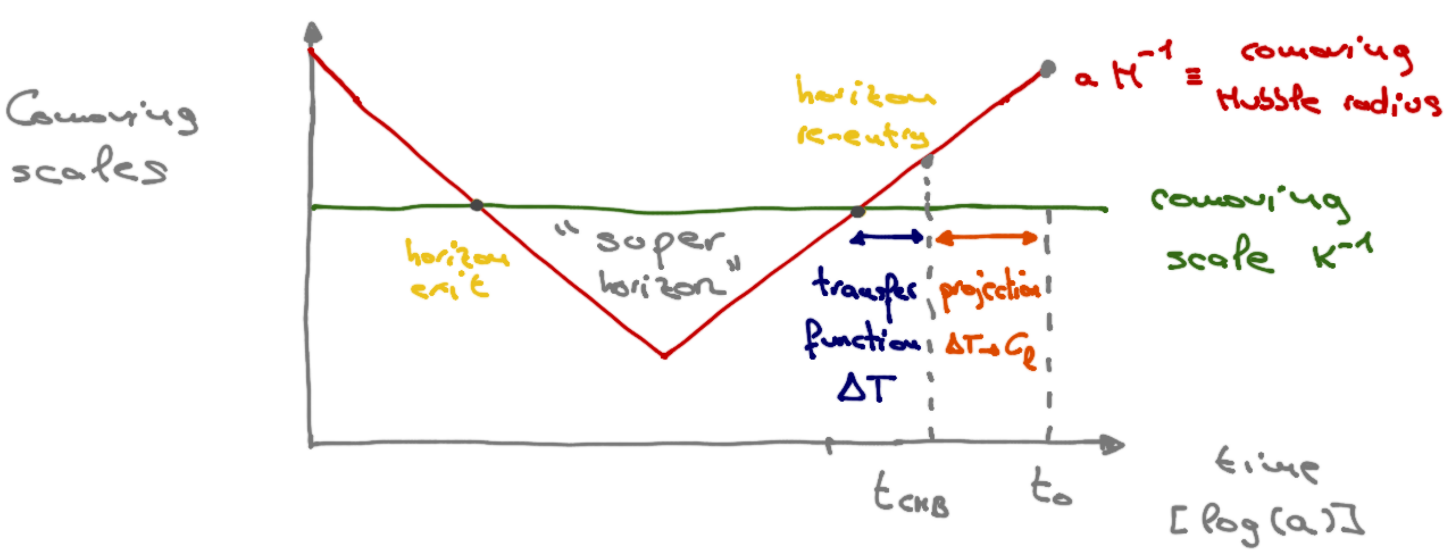
$\approx\!\!>$ Large r to be observable $[r > 10^{-2}, N_* \sim 60]$ $\Rightarrow$ $\underbrace{\Delta\phi > 1}_{\text{transplanckian}}$ [tensor modes ≡ large field inflation]

* The Cosmic Microwave Background (CMB)

The CMB provides a snapshot of the Universe at the time of last scattering $t_{CMB} > t_*$.

Question : How do we transfer the information from the primordial perturbations $P_R(k)$ at $t = t_*$ first to $t = t_{CMB}$ and then to $t = t_0$?

Answer : We must take into account the oscillatory evolution of perturbations that re-entered the horizon before CMB was formed.
This is not necesary for the large scales as modes were still outside the horizon at recombination time t_{CMB}.



$\mapsto \hat{n} \equiv (\theta, \varphi) \equiv S^2$
↳ two-sphere

- Temperature fluctuations $\Delta T(\hat{n})$:

The CMB contains temperature fluctuations ΔT relative to the background temperature $T_0 = 2.7\,K$

As a function on S^2, it can be expanded in a basis of spherical harmonics $Y_{\ell,m}(\hat{n})$

$$\Theta(\hat{n}) \equiv \frac{\Delta T(\hat{n})}{T_0} = \underbrace{\sum_{\ell=0}^{\infty}}_{\substack{\ell=0 \;\Rightarrow\; \text{monopole} \\ \ell=1 \;\Rightarrow\; \text{dipole} \\ \ell=2 \;\Rightarrow\; \text{quadrupole}}} \sum_{m=-\ell}^{\ell} \overbrace{a_{\ell,m}}^{\text{"multipole moments"}} Y_{\ell,m}(\hat{n})$$

The multipole moments $a_{\ell,m}$ can be combined into the rotationally-invariant angular power spectrum C_ℓ^{TT}

$$C_\ell^{TT} = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \langle a^*_{\ell m}, a_{\ell m} \rangle$$

For a small tensor-to-scalar ratio the CMB temperature fluctuations ΔT are dominated by the scalar perturbations $P_{\mathcal{R}}(k)$. Then one has that

$$a_{\ell,m} = 4\pi \, (-i)^\ell \int \frac{d^3\vec{k}}{(2\pi)^3} \underbrace{\Delta_{T\ell}(k)}_{\substack{\text{transfer} \\ \text{function} \\ \text{(horizon re-entry)}}} \underbrace{\mathcal{R}_{\vec{k}}}_{\substack{\text{comoving} \\ \text{curvature} \\ \text{perturbations}}} Y_{\ell,m}(\hat{k})$$

Then, using the identity

$$\sum_{m=-\ell}^{\ell} Y_{\ell,m}(\hat{k}) \, Y_{\ell,m}(\hat{k}') = \frac{2\ell+1}{4\pi} \underbrace{P_\ell(\hat{k}\cdot\hat{k}')}_{\text{Legendre polynomial}}$$

one arrives at

$$\underbrace{C_\ell^{TT}}_{\ell \,\equiv\, \text{angular dependence !!}} = \frac{2}{\pi} \int dk \, k^2 \underbrace{P_{\mathcal{R}}(k)}_{\substack{\text{primordial} \\ \text{inflation}}} \underbrace{\Delta_{T\ell}(k) \, \Delta_{T\ell}(k)}_{\substack{\text{transfer functions} \\ \Downarrow \\ \text{"Anisotropies"}}}$$

It is conventional to plot the redefined angular power spectrum coefficients:

$$\mathcal{C}_\ell \equiv \frac{\ell(\ell+1)}{2\pi} C_\ell^{TT}$$

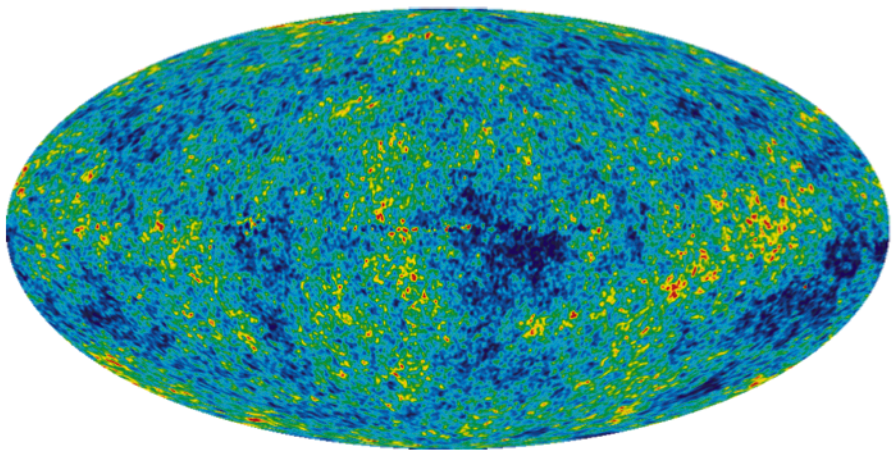
NOTE: The transfer functions $\Delta_{T\ell}(k)$ generically have to be computed numerically. The shape of C_ℓ^{TT} contains information of the initial $P_{\mathcal{R}}(k)$.

- **Large scales** : On large scales, modes were still outside the horizon at $t = t_{CMB}$ => No sub-horizon evolution. Then the large scale CMB (>10 deg) is just the geometric projection of $P_R(k)$ from recombination to us today.

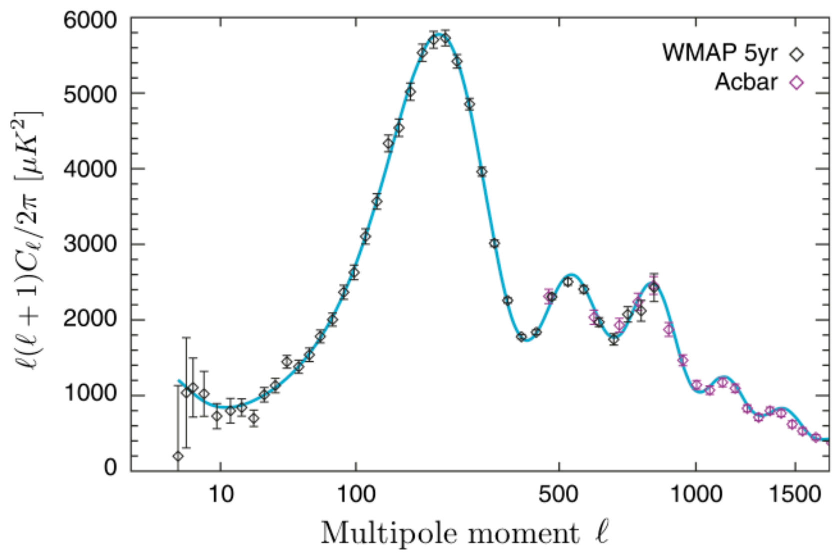
Temperature fluctuations CMB on large scales are dominated by the gravitational Sachs-Wolfe effect of gravitational redshift :

  - Overdense regions => photons invest more energy in escaping and reaching us
    => less energetic
    => cooler

  - Underdense regions => photon invest less energy in escaping and reaching us
    => more energetic
    => hotter



Temperature fluctuations in the CMB. Blue spots represent directions on the sky where the CMB temperature is $\sim 10^{-5}$ below the mean, $T_0 = 2.7$ K. This corresponds to photons losing energy while climbing out of the gravitational potentials of overdense regions in the early universe. Yellow and red indicate hot (underdense) regions. The statistical properties of these fluctuations contain important information about both the background evolution and the initial conditions of the universe.



Angular power spectrum of CMB temperature fluctuations.

* Experimental data [Planck 2018 results X]

Planck satellite : 2018 final data release [1807.06211]

- $\underbrace{P_R}_{k_* = 0.05\ Mpc^{-1}} \approx 21 \times 10^{-10} \quad \Rightarrow \quad \frac{V}{\epsilon} \simeq 24\pi^2 P_R \approx 5 \times 10^{-7}$

- $\underbrace{n_s}_{k_* = 0.05\ Mpc^{-1}} \approx 0.9649 \pm 0.0042$

- $\underbrace{r_{0.002}}_{k_* = 0.002\ Mpc^{-1}} < 0.10$ ≡ No tensor perturbations have been measured

Combining the results above one finds :

$$r = 16\,\epsilon < 0.10 \quad \equiv\!> \quad \epsilon < 6 \times 10^{-3}$$

$$\equiv\!> \quad V = 5 \times 10^{-7}\,\epsilon < 3 \times 10^{-9}$$

$$\equiv\!> \quad H \sim \sqrt{\frac{V}{3}} < 3 \times 10^{-5}$$