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Adolfo Guarino Almeida,

Madrid, Mayo 2010.

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Generalised Fluxes, Moduli Fixing and Cosmological Implications

Memoria de Tesis Doctoral realizada por
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Introducción

La Física del siglo XX ha supuesto un gran avance hacia la comprensión de la Naturaleza a diferentes escalas. Esto ha venido dado por la consolidación de la teoría de la Relatividad General (RG) y la Teoría Cuántica de Campos (TCC) como los marcos teóricos en los que describir los fenómenos naturales a grandes y pequeñas distancias respectivamente. Mientras que la RG revolucionó la manera de entender la Gravedad, — el espacio-tiempo adquiría un carácter dinámico en lugar de ser estático — la TCC resultó ser la herramienta adecuada para explorar las leyes que describen la Naturaleza a escalas por debajo del núcleo atómico ($r \leq 10^{-13}$ cm.).

Hoy en día tenemos una TCC para describir la física de las partículas elementales conocida como el Modelo Estándar de la Física de Partículas. Este modelo combina la teoría de las interacciones *electrodébiles* de Glashow-Salam-Weimberg con la *cromodinámica* cuántica propuesta para describir las interacciones entre los constituyentes del núcleo atómico. El Modelo Estándar se compone de tres generaciones de fermiones (quarks y leptones): $(u\,,\,d\,;\,e\,,\,\nu_e)\,,\;(c\,,\,s\,;\,\mu\,,\,\nu_\mu)\,,\;(t\,,\,b\,;\,\tau\,,\,\nu_\tau)$ con una jerarquía de masas entre ellas y de 12 partículas vectoriales que median sus interacciones. Además, el modelo también incluye un escalar masivo que aún no ha sido observado, el bosón de Higgs, cuya masa está relacionada con la ruptura de la simetría *electrodébil*. Las interacciones de estos campos vienen dictadas por el grupo de simetría $G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ del Modelo Estándar.

El Modelo Estándar[1] ha sido testado en experimentos de altas energías llevados a cabo en aceleradores de partículas, resultando ser un marco teórico sólido. Sin embargo no incorpora una descripción cuántica de la Gravedad. Con este objetivo, y a pesar de que fuese formulada originalmente para explicar la gran cantidad de hadrones observados, una nueva teoría en la que los objetos fundamentales tienen extensión (*cuerdas*) entró en escena.

[1] O bien pequeñas variaciones de éste, como la de incluir masas para los neutrinos acordes con las medidas en experimentos de oscilaciones de neutrinos.

De cuerdas a teorías de supergravedad

Hasta la fecha, la Teoría de Cuerdas es una de las más firmes candidatas a ser una descripción unificada de la Naturaleza en tanto en cuanto nos proporciona una teoría cuántica de la Gravedad consistente. Los objetos que componen esta teoría no son partículas puntuales sino cuerdas unidimensionales. Al propagarse en el tiempo, una cuerda dibuja una superficie bidimensional Σ conocida como hoja de mundo. Las cordenadas que especifican la manera en la que esta hoja se embebe en un espacio-tiempo ambiente describen una teoría de campos conforme en dos dimensiones.

La ausencia de *taquiones* y la presencia de fermiones en el espectro reduce a cinco las teorías de cuerdas supersimétricas, las cuales se propagan en un espacio-tiempo diezdimensional (10d). Éstas son las teorías de supercuerdas Tipo IIA, Tipo IIB, Tipo I, Heterótica-SO(32) y Heterótica-$E_8 \times E_8$. Todas ellas son teorías de cuerdas cerradas y contienen un campo de espín 2 g_{MN} el cual se identifica con el gravitón, un tensor antisimétrico B_{MN} y un escalar φ conocido como *dilatón* dentro del espectro 10d de campos sin masa. Estos campos conforman el sector universal de las teorías de supercuerdas.

Las supercuerdas pueden propagarse en presencia de campos de fondo no triviales que aparecen cuando los campos sin masa del sector universal toman un valor esperado no nulo en el vacío (VEV). Sin considerar términos fermiónicos, la acción para la teoría de campos en la hoja de mundo descrita por una supercuerda moviéndose en un fondo para el sector universal viene dada por [1, 2]

$$\begin{aligned} S = & - \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \Big(\sqrt{-h}\, h^{\alpha\beta}\, \partial_\alpha X^M \partial_\beta X^N\, g_{MN}(X) - \epsilon^{\alpha\beta}\, \partial_\alpha X^M \partial_\beta X^N\, B_{MN}(X) \\ & - \alpha' \sqrt{-h}\, \varphi(X)\, R^{(2)} \Big) , \end{aligned} \tag{1}$$

donde σ^α con $\alpha = 0, 1$ denota las cordenadas en la hoja de mundo, y donde $h_{\alpha\beta}$ y $R^{(2)}$ son, respectivamente, la métrica y el escalar de curvatura de la hoja de mundo. El parámetro α' que aparece en la acción está relacionado con la escala de longitud de la cuerda l_s mediante

$$l_s^2 = 2\,\alpha' , \tag{2}$$

por lo que los estados masivos en el espectro de la supercuerda adquieren una (masa)^2

$$M^2 \propto \frac{1}{\alpha'} . \tag{3}$$

Al cuantizar la acción en (1), el número de campos bosónicos X^M en la hoja de mundo viene determinado por la condición de cancelación de la anomalía conforme de la misma manera que la cancelación de anomalías *gauge* condiciona el contenido de campos en las TCC con simetrías *gauge*. En otras palabras, la dimensión del espacio-tiempo viene fijada por esta condición resultando ser $d = 10$ para las teorías de supercuerdas.

Los campos de fondo g_{MN}, B_{MN} y φ correspondientes a excitaciones sin masa de las cuerdas de alrededor se pueden interpretar como acoplos en la teoría de campos conforme. La imposición de invariancia conforme se traduce en la cancelación de las funciones β

$$\beta_g^{MN}(\sqrt{\alpha'}/L) = \beta_B^{MN}(\sqrt{\alpha'}/L) = \beta_\varphi(\sqrt{\alpha'}/L) = 0 \ , \tag{4}$$

donde L es la longitud característica del sistema cuya dinámica se pretende explorar. Considerando la contribución más baja en el parámetro de expansión $\sqrt{\alpha'}/L$, el conjunto de ecuaciones (4) reproduce las ecuaciones de Euler-Lagrange que se derivan de la acción

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} \left(R - \frac{1}{2}\,\partial^M\varphi\partial_M\varphi - \frac{1}{2}e^{-\varphi}|H_3|^2 \right) \ , \tag{5}$$

donde $G = e^{-\frac{\varphi}{2}}g$ es la métrica en el marco de Einstein[2]. Esta acción, válida a nivel árbol en la expansión en lazos de la cuerda, describe la gravedad acoplada a un campo *gauge* B_{MN} (una 2-forma) y a un escalar φ. La constante de acoplo de la supercuerda, g_s, está por tanto relacionada con el valor esperado en el vacío del campo del *dilatón* $\langle\varphi\rangle$

$$g_s = e^{\langle\varphi\rangle} \ , \tag{6}$$

de ahí siendo una cantidad física que ha de fijarse dinámicamente.

No obstante, es importante matizar que la acción en (5) debe entenderse como una acción efectiva que se obtiene al tomar la contribución más baja en el parámetro de expansión $\sqrt{\alpha'}/L$ en las ecuaciones (4). Esto equivale a tomar el límite $\sqrt{\alpha'}/L \to 0$ o $L \gg l_s$ en el cual las cuerdas unidimensionales se ven como partículas puntuales, representando un límite de bajas energías (grandes distancias) con respecto a la escala de la cuerda. En este límite, los modos masivos en el espectro de la supercuerda se desacoplan de la teoría, i.e., $M^2 \to \infty$, dando lugar a una teoría efectiva cuya dinámica se puede aproximar a una teoría clásica de campos con partículas puntuales sin masa. La acción efectiva anterior (5) se generaliza a la de las teorías de supergravedad que describen la dinámica de una teoría clásica de campos supersimétrica con partículas sin masa en diez dimensiones.

Un punto a destacar es que las teorías de supercuerdas predicen necesariamente la existencia de dimensiones extra y de supersimetría, dos propiedades de la Naturaleza que aún no han sido observadas en ningún experimento.

Física en cuatro dimensiones y el problema de los *moduli*

Al *compactificar* supergravedades 10d en un espacio interno seis-dimensional con tamaño característico $R \gg l_s$, uno obtiene extensiones supersimétricas de la Relatividad General en cuatro dimensiones (4d) que se pueden utilizar para describir escalas de energía

[2] El factor $2\kappa^2 = (2\,\pi)^7\,\alpha'^4 = (16\,\pi\,\mathrm{G}_{10})\,e^{-2\varphi}$ se relaciona con la constante de Newton en diez dimensiones G_{10}.

$E \ll 1/R$ a las cuales no hay suficiente resolución para ver las dimensiones compactas. Además, dicha *compactificación* introduce una gran cantidad de campos escalares ϕ^i en la teoría efectiva que parametrizan la forma y el tamaño del espacio interno al igual que el *dilatón* en cuatro dimensiones. Estos campos, conocidos como *moduli*, resultan ser direcciones planas en el potencial efectivo a todo orden en teoría de perturbaciones.

Uno de los principales problemas en Fenomenología de Cuerdas es encontrar mecanismos para estabilizar estos *moduli*, i.e. para que adquieran una masa lo suficientemente grande como para que no hayan sido detectados experimentalmente hasta ahora. De lo contrario, estos *moduli* mediarían interacciones de largo alcance las cuales podrían dar lugar a violaciones del Principio de Equivalencia [3], entrando en conflicto con los tests de precisión de Gravedad. Tras estabilizarse en el mínimo de un potencial efectivo no trivial $V(\phi^i)$, sus valores esperados en el vacío están relacionados con cantidades físicas tales como la constante de acoplo de la cuerda g_s, el volumen del espacio interno V_{int} o la energía del vacío/constante cosmológica Λ. Esto tiene importantes implicaciones fenomenológicas en Física de Partículas y en Cosmología. Por tanto, el estudio de la estabilización de *moduli* es un paso crucial para establecer una conexión entre la fenomenología de bajas energías, la cual está a punto de ser explorada minuciosamente en el LHC (Large Hadron Collider), y construcciones de Teoría de Cuerdas en cuatro dimensiones. En particular, es necesario entender cómo ocurre la ruptura de supersimetría en este contexto para poder continuar con este enfoque "desde arriba hacia abajo" a la hora de conectar las cuerdas con la física de bajas energías.

Durante la última década, uno de los retos principales en Fenomenología de Cuerdas ha sido el de estabilizar los *moduli* en un vacío no supersimétrico de Sitter, aproximadamente Minkowski ($\Lambda \sim 10^{-120}$ en unidades de Planck), que dé lugar a una expansión acelerada del Universo acorde con las observaciones [4, 5]. En parte, esto ha venido motivado por el estudio de las *compactificaciones* de las teorías de cuerdas tipo II en un fondo de flujos generalizados [6]. Además de las intensidades de campo habituales de Neveu-Schwarz-Neveu-Schwarz (NS-NS) y de Ramond-Ramond (R-R) asociadas a los potenciales *gauge* presentes en el espectro de las supercuerdas de tipo II, los flujos generalizados fueron propuestos para restablecer la invariancia de los modelos efectivos 4d bajo las relaciones de dualidad de las supergravedades de tipo II en diez dimensiones. Por ejemplo, ciertos flujos con estructura tensorial conocidos como flujos no geométricos se introdujeron pare restablecer T-dualidad entre *compactificaciones* de las teorías tipo IIA y IIB [7] al igual que S-dualidad en la teoría tipo IIB [8] a nivel efectivo. A este nivel, los flujos desempeñan un papel doble:

i) Por un lado determinan la estructura del álgebra de la supergravedad *gaugeada* que subyace a la *compactificación* [9]. Este álgebra involucra las elecciones de *gauge* del campo B universal así como difeomorfismos de la métrica.

ii) Por otro lado, los flujos inducen un potencial escalar no trivial $V(\phi^i)$ para los *moduli* de la *compactificación* [10, 11], lo cual puede dar lugar a su estabilización completa sin necesidad de requerir efectos menos manejables como son los efectos no perturbativos [12].

Por otra parte se han establecido algunos teoremas de imposibilidad relacionados con la existencia de vacíos de Sitter en las *compactificationes* con flujos [13–15] así como mecanismos para evadirlos [16–22]. En este sentido, las *compactificaciones* con flujos generalizados han resultado esquivar estos teoremas por lo que, en principio, pueden estabilizar los *moduli* en un vacío de Sitter. Uno esperaría que al aumentar el número de flujos, incluyendo por ejemplo flujos no geométricos, pudiera favorecerse la estabilización completa de los *moduli* [23, 24].

Como se mostrará en esta tesis, es posible obtener vacíos Minkowski/de Sitter que rompan supersimetría de forma espontánea y con valores arbitrarios de $(g_s\,,\,\Lambda)$ (salvo restricciones diofánticas sobre los flujos) a partir de *compactificaciones* en *orientifolios* de tipo II sobre *orbifolios* toroidales en presencia de flujos no geométricos [19]. Estos escenarios basados en *compactificaciones* con flujos generalizados resultan ser muy prometedores en lo que se refiere a la construcción de modelos. Sin embargo, la falta de un criterio dinámico para seleccionar un vacío específico entre muchos otros (infinitos), ha dado lugar al concepto del *landscape* de vacíos en Teoría de Cuerdas. Llegados a este punto, uno podría recurrir al Principio Antrópico para excluir aquellas regiones del *landscape* en las que la vida humana no hubiera podido desarrollarse [25].

La dinámica de los *moduli* como origen de *inflación*

Actualmente, una de las principales afirmaciones del Modelo Estándar Cosmológico es que nuestro Universo experimentó un período de expansión exponencial en sus primeros instantes inducido por la dinámica un campo escalar (o varios de ellos). Este proceso, conocido como *inflación*, explicaría de forma elegante la planitud del Universo observada [26] así como la casi invariancia de escala del espectro que se deduce del CMB (Cosmic Microwave Background) a partir de los datos aportados por la colaboración WMAP [27, 28].

Asumiendo un Universo homogéneo e isótropo, la métrica del espacio-tiempo corresponde a una de tipo Friedmann-Robertson-Walker

$$ds^2 = dt^2 - a(t)^2\, d\vec{x}^{\,2} \ , \tag{7}$$

cuya dinámica se codifica totalmente en la función $a(t)$, conocida como factor de escala. La cosmología *inflacionaria* se modeliza mediante una teoría de campos escalares acoplados a la Gravedad, la cual a su vez se describe en términos de un potencial escalar efectivo

$V(\phi^i)$. Tomando los campos constantes en el espacio, las ecuaciones de movimiento para el factor de escala $a(t)$ y para los campos escalares ϕ^i vienen dadas por

$$\begin{aligned} H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\left[\frac{1}{2}\mathcal{G}_{ij}\,\dot{\phi}^i\dot{\phi}^j + V\right] \qquad , \\ \ddot{\phi}^i + 3\,H\,\dot{\phi}^i + \Gamma^i_{jk}\,\dot{\phi}^j\,\dot{\phi}^k + \mathcal{G}^{ij}\,\frac{\partial V}{\partial \phi^j} = 0 \quad , \end{aligned} \tag{8}$$

donde los puntos denotan derivadas temporales y, como es habitual, Γ^i_{jk} representa los símbolos de Christoffel construidos a partir de la métrica $\mathcal{G}_{ij}$ (con $\mathcal{G}^{ij}\mathcal{G}_{jk} = \delta^i_k$) en el espacio de campos. Resolver estas ecuaciones de forma general es una tarea complicada. Ante esto es habitual tomar la aproximación de rodar lento según la cual $H \sim \sqrt{V/3}$ ya que se desprecian los términos de tipo $\ddot{\phi}^i$ y $\dot{\phi}^i\dot{\phi}^j$ en (8).

Los parámetros ϵ y $|\eta|$ que determinan la aproximación de rodar lento se definen, en unidades de Planck, a través de

$$\begin{aligned} \epsilon \quad &\equiv \quad \frac{1}{2\,V^2}\,\mathcal{G}^{ij}\,\frac{\partial V}{\partial \phi^i}\,\frac{\partial V}{\partial \phi^j} \qquad , \\ \eta \quad &\equiv \quad \text{min eigenvalue}\left[\frac{1}{V}\,\mathcal{G}^{ik}\left(\frac{\partial^2 V}{\partial \phi^k\,\partial \phi^j} - \Gamma^l_{kj}\,\frac{\partial V}{\partial \phi^l}\right)\right] \quad . \end{aligned} \tag{9}$$

De acuerdo con los datos experimentales, estos parámetros han de ser $\ll 1$ para que *inflación* tenga lugar en esta aproximación (ver ref. [29]). Esto implica un potencial escalar $V(\phi^i)$ positivo y "artificialmente" plano (casi constante) en alguna región del espacio de campos. De lo contrario, la aproximación deja de ser válida.

Debido a que la Teoría de Cuerdas nos proporciona una gran variedad de campos escalares al *compactificarla* en un espacio interno, i.e. los *moduli*, obtener teorías de campos efectivas que compartan las características anteriores ha sido una de las líneas de investigación más importantes en Fenomenología de Cuerdas. En los últimos años se ha dedicado mucho esfuerzo a tratar de obtener escenarios de *inflación* basados en modelos de estabilización de *moduli* en *compactificaciones* de supercuerdas incluyendo ingredientes adicionales como flujos de fondo, D-*branas* y efectos no perturbativos [30]. En consecuencia, la dinámica de los *moduli* durante el proceso de estabilización resulta ser de máxima importancia desde un punto de vista cosmológico.

Es lógico pensar que *inflación*, al tratarse de un proceso cosmológico, debería poder ser descrito en un marco teórico que incorpore Gravedad de forma natural. En este sentido, la Cosmología de Cuerdas representa un contexto inigualable en el que las construcciones de cuerdas en cuatro dimensiones han de hacer frente a los datos cosmológicos experimentales [31, 32].

Resumen de la tesis

Los contenidos de la presente tesis doctoral se organizan en los siguientes capítulos y apéndices:

- En el capítulo 1 introducimos los conceptos fundamentales sobre los que esta tesis se construye. En particular, los relacionados con las teorías de supergravedad de tipo II, dualidades y *branas*. Lejos de pretender tratar con los aspectos más formales de las teorías de supergravedad, este capítulo se incluye en la tesis con la intención de hacerla lo más autocontenida posible. Los conceptos y resultados que aparecen en este capítulo introductorio se pueden encontrar en los libros [1, 2, 33] y en las revisiones [34, 35].

- En el capítulo 2 fijamos la notación y las convenciones adoptadas en las refs [36, 37] para describir la geometría del *orbifolio* toroidal $\mathbb{T}^6/\mathbb{Z}_2\times\mathbb{Z}_2$ así como del *orientifolio* de tipo IIB con O3/O7-*planos* construido a partir de él. Además obtenemos la acción efectiva $\mathcal{N}=1$ en cuatro dimensiones que describe la dinámica de los *moduli* en este *orientifolio* tras incluir flujos generalizados. Finalmente presentamos las condiciones de consistencia que involucran estos flujos y particularizamos los resultados para el caso de modelos isótropos los cuales serán explorados en profundidad durante la tesis.

- En el capítulo 3 desarrollamos las técnicas necesarias para eliminar grados de libertad superfluos en teorías efectivas $\mathcal{N}=1$ invariantes bajo transformaciones de T-dualidad. Este proceso de reducción de parámetros nos permite simplificar los modelos efectivos de tal manera que podemos obtener analíticamente familias de vacíos supersimétricos AdS_4 con todos los *moduli* estabilizados a valores perturbativos de la constante de acoplo de la cuerda. Todos los resultados que aparecen en este capítulo se pueden encontrar en la ref. [36].

- En el capítulo 4 investigamos escenarios sencillos de estabilización de *moduli* en vacíos de Sitter aproximadamente Minkowski en los que la supersimetría se rompe de manera espontánea. Estudiaremos estos vacíos en el contexto de las teorías *orientifolios* de tipo IIB invariantes bajo T-dualidad las cuales incluyen flujos generalizados, O3/O7-*planos* y D3/D7-*branas*. Este capítulo representa el núcleo central de la tesis y se corresponde con los trabajos en las refs [18, 19].

- En el capítulo 5 estudiamos una teoría de supergravedad construida sobre el *orientifolio* de tipo IIB con O3/O7-*planos* la cual resulta invariante bajo transformaciones de T-dualidad y de S-dualidad. En esta teoría desarrollaremos un método para resolver todas las restricciones sobre los flujos que provienen de la estructura de álgebra que les subyace. Utilizaremos diversas técnicas de Geometría Algebraica que nos permitirán resolver las ligaduras sobre los flujos y obtener de forma sistemática vacíos supersimétricos, centrándonos especialmente en el caso de soluciones Minkowski. Finalizaremos con una digresión sobre el origen $\mathcal{N} = 4$ de los vacíos encontrados. A excepción de ésta, la cual contiene algunos resultados preliminares, el contenido de este capítulo se puede encontrar en la ref. [37].

- En el capítulo 6 presentamos un modelo de *inflación* tomando la aproximación de rodar lento en el contexto de supergravedades de tipo IIB efectivas en presencia de flujos *gauge*. La energía necesaria para elevar el vacío a de Sitter (aproximadamente Minkowski) la proporciona el D-término asociado a un U(1) anómalo formulado de manera consistente e invariante *gauge*. Desarrollaremos un modelo mínimo que incorpora *inflación* topológica eterna ajustándose a las observaciones experimentales. Este modelo evita los problemas habituales que aparecen al intentar obtener *inflación* en modelos de supergravedad, i.e. el problema η y el problema de las condiciones iniciales entre otros. Finalmente apuntamos hacia una aparente persistencia del problema η cuando sólo se considera el efecto de los flujos generalizados como mecanismo para estabilizar todos los *moduli* en un vacío dS. Los resultados en este capítulo se pueden encontrar en la ref. [38] junto con la sección 4.3 en la ref. [19].

- En el apéndice A obtenemos el espectro de estados sin masa (contenido de campos de la supergravedad) para las supercuerdas de tipo II propagándose libremente en un espacio-tiempo Minkowski en diez dimensiones. Un desarrollo similar se puede encontrar en cualquiera de los libros de Teoría de Cuerdas [1,2,33] y en la ref. [35]. En palabras generales, esta tesis es un estudio de la dinámica asociada a estos campos cuando se tienen en cuenta diferentes efectos de cuerda cerrada/abierta.

- El apéndice B recoge las expresiones para los flujos de R-R en términos de los parámetros asociados a translaciones *axiónicas* y a *tadpoles* en la teoría *orientifolio* de tipo IIB invariante bajo transformaciones de T-dualidad estudiada en el capítulo 3.

Introduction

The legacy of the 20th century Physics represents one of the most fascinating attempts to understand how Nature manifests at different scales. It is commonly identified with the consolidation of General Relativity (GR) and Quantum Field Theories (QFT's) as the suitable frameworks in which to respectively investigate large and short scale phenomena. As long as GR revolutionised the way in which Gravity was previously understood, — it abandons the idea of a static spacetime in favour of a dynamical one — QFT's were found to be a powerful tool to explore the laws describing Nature at scales below the size of the atomic nucleus ($r \leq 10^{-13}$ cm.).

Nowadays a well established QFT describing Nature is the Standard Model of Particle Physics, which combines the Glashow-Salam-Weimberg theory of the electroweak interactions and the quantum chromodynamics describing the interactions between the constituents of the atomic nucleus. According to it, there are three generations of fermions (quarks and leptons): $(u\,,\,d\,;\,e\,,\,\nu_e)\,,\;(c\,,\,s\,;\,\mu\,,\,\nu_\mu)\,,\;(t\,,\,b\,;\,\tau\,,\,\nu_\tau)$ at different mass scales, together with 12 vector particles mediating in the interactions between them. In addition, the model also includes a scalar massive boson not yet observed, the so-called Higgs field, whose mass is related to the electroweak symmetry breaking. This field content interacts accordingly to the Standard Model gauge symmetry group $G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ which basically transforms fermions into one another.

The Standard Model[3] has been tested to be a solid framework in which to explain the experimental data extracted from high energy physics experiments at colliders. Nevertheless, it fails when it comes to incorporate a quantum description of Gravity. Although originally introduced as a proposal to explain the vast set of observed hadrons, a theory of one-dimensional extended objects, i.e. *strings*, turns out to serve this purpose.

From strings to supergravity theories

Up to date, String Theory is one of the most promising candidates to be a unified theory of Nature providing us with a consistent quantum theory of Gravity. The fundamental ob-

[3] Or some slight modification of it such as that of including masses for the neutrinos in order to fit the data arising from neutrino oscillation experiments.

jects making up this theory are no longer point-like particles but one-dimensional strings. As a one-dimensional object propagating on time, a string draws a two-dimensional surface Σ known as the *worldsheet* whose embedding coordinates in a target spacetime determine a conformal field theory in two dimensions.

The absence of tachyons and the presence of fermions in the spectrum leaves us with five supersymmetric string theories that propagate in a ten-dimensional (10d) spacetime. These are the Type IIA, Type IIB, Type I, Heterotic-SO(32) and Heterotic-$E_8 \times E_8$ superstring theories. They all are theories of closed strings and contain a spin 2 field g_{MN} which is identified with the graviton, an antisymmetric tensor field B_{MN} and a scalar φ known as *dilaton* within their 10d massless spectrum. These fields are said to constitute the *universal sector* of the superstring theories.

Superstrings can propagate in non-trivial backgrounds arising from non-vanishing vacuum expectation values (VEVs) of the massless fields in the universal sector. Without considering fermi terms, the worldsheet action for a superstring moving in these backgrounds is given by [1, 2]

$$\begin{aligned} S = & - \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \Big(\sqrt{-h}\, h^{\alpha\beta}\, \partial_\alpha X^M \partial_\beta X^N\, g_{MN}(X) - \epsilon^{\alpha\beta}\, \partial_\alpha X^M \partial_\beta X^N\, B_{MN}(X) \\ & - \alpha' \sqrt{-h}\, \varphi(X)\, R^{(2)} \Big) , \end{aligned} \tag{10}$$

where σ^α with $\alpha = 0, 1$ denotes worldsheet coordinates, and $h_{\alpha\beta}$ and $R^{(2)}$ are the metric and the curvature scalar of the worldsheet, respectively. The parameter α' entering the action is related to the string length scale l_s by

$$l_s^2 = 2\,\alpha' , \tag{11}$$

so massive states in the superstring spectrum result with a $(\text{mass})^2$

$$M^2 \propto \frac{1}{\alpha'} . \tag{12}$$

When quantising the action in (10), the number of bosonic fields X^M in the worldsheet is fixed by the cancellation of the conformal anomaly in the same way that the cancellation of the gauge anomalies conditions the field content of quantum field theories with gauge symmetries. In other words, the dimension of the spacetime is fixed by this condition, turning out to be $d = 10$ for superstring theories.

The background fields g_{MN}, B_{MN} and φ corresponding to massless excitations of the strings around can be seen as couplings in the conformal field theory. The requirement of conformal invariance imposes their β-functions to vanish

$$\beta_g^{MN}(\sqrt{\alpha'}/L) = \beta_B^{MN}(\sqrt{\alpha'}/L) = \beta_\varphi(\sqrt{\alpha'}/L) = 0 , \tag{13}$$

where L is the typical length scale of the system whose dynamics is to be explored. At the lowest contribution in the $\sqrt{\alpha'}/L$ parameter expansion, the set of equations (13) turns out to reproduce the Euler-Lagrange equations derived from the spacetime action

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} \left(R - \frac{1}{2}\, \partial^M \varphi \partial_M \varphi - \frac{1}{2} e^{-\varphi} |H_3|^2 \right) \,, \tag{14}$$

where $G = e^{-\frac{\varphi}{2}} g$ is the metric field in the Einstein's frame[4]. This action, valid at tree level in the string loop expansion, describes gravity coupled to a 2-form gauge field B_{MN} and to a scalar φ. The superstring coupling constant, g_s, is then related to the vacuum expectation value of the dilaton field $\langle \varphi \rangle$

$$g_s = e^{\langle \varphi \rangle} \,, \tag{15}$$

hence being a physical quantity that is to be fixed dynamically.

Nonetheless, it is worth noticing that the action in (14) must be understood as an effective action arising when considering the lowest contribution in the $\sqrt{\alpha'}/L$ parameter expansion of the equations (13). Equivalently, when taking the limit of $\sqrt{\alpha'}/L \to 0$ or $L \gg l_s$ in which one-dimensional strings look like point-like objects. This is a limit of low energies (large distances) with respect to the string energy (length) scale. In this limit the massive modes in the superstring spectrum are decoupled from the theory, i.e., $M^2 \to \infty$, and are integrated out, leading to an effective theory whose dynamics can be approximate by a classical field theory of massless (zero size) particles. The above effective action in (14) generalises to that of supergravity theories describing the dynamics of a supersymmetric classical field theory of massless particles in ten dimensions.

A key point to be highlighted is that superstring theories necessarily predict the existence of extra dimensions as well as supersymmetry, two properties of Nature that have not been observed at any experiment yet.

Four-dimensional Physics and the moduli problem

When compactifying 10d supergravities into a six-dimensional internal space with characteristic length $R \gg l_s$, one obtains supersymmetric extensions of General Relativity in four dimensions (4d) that can be used to describe energy scales $E \ll 1/\,R$ at which there is no resolution enough to see the compact dimensions. In addition, the compactification process introduces a plethora of scalar fields ϕ^i in the 4d theory parameterising the shape and size of the internal space as well as the four-dimensional dilaton. These fields, known as moduli, turn out to be flat directions in the effective scalar potential at all orders in perturbation theory.

[4]The factor $2\kappa^2 = (2\,\pi)^7\,\alpha'^4 = (16\,\pi\,\mathrm{G}_{10})\,e^{-2\varphi}$ relates to the G_{10} Newton's constant in ten dimensions.

One of the main problems in String Phenomenology is to find mechanisms for these moduli to be stabilised, i.e. to acquire a mass large enough for them to have so far escaped experimental detection. Otherwise, these moduli would mediate long range interactions which could lead to violations of the Equivalence Principle [3], clashing with precision tests of Gravity. After stabilising at the minimum of a non-trivial effective potential $V(\phi^i)$, their vacuum expectation values relate to physical quantities, such as the string coupling constant g_s, the internal space volume V_{int} or the vacuum energy/cosmological constant Λ. This has important phenomenological implications both in Particle Physics and Cosmology. The study of moduli stabilisation is then a crucial step in establishing a link between low energy phenomenology, which is about to be thoroughly explored at the LHC (Large Hadron Collider), and String Theory constructions in four dimensions. In particular an understanding of how supersymmetry breaking happens in this context is mandatory, in order to proceed with this "top-bottom" approach to linking strings and low energy physics.

During the last decade, the challenge of stabilising the moduli in a non-supersymmetric de Sitter almost Minkowski vacuum ($\Lambda \sim 10^{-120}$ in Planck units) causing the accelerated expansion of the present Universe accordingly to observations [4,5], has become of principal interest in String Phenomenology. This has been partially motivated by the study of type II superstring compactifications in the presence of generalised flux backgrounds [6]. In addition to the ordinary Neveu-Schwarz-Neveu-Schwarz (NS-NS) and the Ramond-Ramond (R-R) field strengths associated to gauge potentials present in the type II superstrings spectrum, generalised fluxes were proposed to restore the invariance of the 4d effective models under duality relations of the original ten-dimensional type II supergravities. For instance, certain tensor fluxes referred to as *non-geometric* fluxes are introduced to restore T-duality between compactifications of types IIA and IIB theories [7] as well as type IIB S-duality [8] at the effective level. At this level, the role played by the fluxes is twofold:

i) On the one hand, they determine the algebra structure of the gauged supergravity underling the compactification [9]. This algebra accounts for gauge choices on the universal Neveu-Schwarz B-field and diffeomorphisms on the metric.

ii) On the other hand, the fluxes induce a non-trivial scalar potential $V(\phi^i)$ for the moduli fields of the compactification [10,11], which can potentially lead to their stabilisation without invoking less manageable non-perturbative effects [12].

In addition, quite restrictive "no-go" theorems concerning the existence of de Sitter vacua in flux compactifications have been stated [13–15] as well as mechanisms to circumvent them [16–22]. To this respect, generalised flux compactifications have been shown to avoid these no-go theorems and hence can potentially stabilise the moduli in a de Sitter vacuum. One expects that enlarging the number of fluxes including the non-geometric ones could help providing complete moduli stabilisation [23,24].

As it will be shown in this thesis, Minkowski/de Sitter moduli flux vacua with arbitrary $(g_s\,,\,\Lambda)$ values (up to Diophantine restrictions upon fluxes) and breaking supersymmetry spontaneously can be obtained from type II orientifold compactifications on toroidal orbifolds in the presence of non-geometric fluxes [19]. In this sense, the above scenarios based on generalised flux compactifications turn out to be very promising as far as model building is concerned. Nevertheless, the lack of a dynamical criterion for choosing an specific flux vacuum among many (infinite) others has led to the idea of a *landscape* of String Theory vacua. An *anthropic* reasoning principle would then act as a censorship mechanism for those regions within the landscape not supporting human life [25].

Moduli dynamics as a seed for inflation

Nowadays, one of the main statements of the Cosmological Standard Model is that our Universe experienced an epoch of exponential cosmic expansion at its initial stages guided by some scalar field(s). This process, known as *inflation*, nicely explains the observed flatness of the Universe [26] as well as the almost scale-invariant spectrum inferred from the CMB (Cosmic Microwave Background) data released by the WMAP collaboration [27,28].

Assuming an homogeneous and isotropic Universe, the spacetime metric corresponds to that of the Friedmann-Robertson-Walker type

$$ds^2 = dt^2 - a(t)^2\, d\vec{x}^{\,2} \ , \tag{16}$$

whose dynamics is totally encoded within the time dependent function $a(t)$ referred to as the *scale factor*. Inflationary cosmology is commonly modelled by a scalar field(s) theory coupled to Gravity, which in turn is characterised by an effective scalar potential $V(\phi^i)$. Provided fields are constant in space, the equations of motion for the scale factor $a(t)$ and the (real) scalar fields ϕ^i are given by

$$\begin{aligned} H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\left[\frac{1}{2}\,\mathcal{G}_{ij}\,\dot{\phi}^i\dot{\phi}^j + V\right] \qquad , \\ \ddot{\phi}^i\, + 3\,H\,\dot{\phi}^i + \Gamma^i_{jk}\,\dot{\phi}^j\,\dot{\phi}^k\, + \mathcal{G}^{ij}\,\frac{\partial V}{\partial \phi^j} = 0 \quad , \end{aligned} \tag{17}$$

where dots denote time derivatives and, as usual, Γ^i_{jk} are the Christoffel symbols derived from the $\mathcal{G}_{ij}$ field-space metric satisfying $\mathcal{G}^{ij}\mathcal{G}_{jk} = \delta^i_k$. Solving these equations in a general way is a difficult task, so, at this point, it is common to use the so-called *slow-roll* approximation according to which $\ddot{\phi}^i$ and $\dot{\phi}^i\dot{\phi}^j$ terms in (17) are neglected and $H \sim \sqrt{V/3}$.

In this approximation, the two slow-roll parameters ϵ and $|\eta|$ defined (in Planck units) through

$$\begin{aligned} \epsilon &\equiv \frac{1}{2\,V^2}\,\mathcal{G}^{ij}\,\frac{\partial V}{\partial \phi^i}\,\frac{\partial V}{\partial \phi^j} \qquad\qquad , \\ \eta &\equiv \text{min eigenvalue}\left[\frac{1}{V}\,\mathcal{G}^{ik}\left(\frac{\partial^2 V}{\partial \phi^k\,\partial \phi^j} - \Gamma^l_{kj}\,\frac{\partial V}{\partial \phi^l}\right)\right] \quad , \end{aligned} \tag{18}$$

are demanded to be $\ll 1$ for inflation to occur. For a good review see ref. [29]. This implies the scalar potential $V(\phi^i)$ to be positive and "artificially" flat (almost constant) somewhere in field space in order to fit the experimental data. Otherwise the approximation is no longer valid.

How to get scalar effective field theories sharing the aforementioned features from string-inspired scenarios has been intensively pursued in String Phenomenology research since String Theory provides us with such scalar fields once it is compactified into an internal manifold, i.e., with moduli fields. In the last years, a lot of effort has been done in trying to derive suitable inflationary scenarios based on moduli stabilisation in superstring compactifications including additional ingredients as background fluxes, D-branes and non-perturbative effects, among others [30]. As a result, the dynamics of the moduli during the stabilisation process becomes of utmost importance from a cosmological viewpoint.

It makes sense to think that inflation, as a cosmological process, should be accommodated within a framework that incorporates Gravity in a natural way. In this respect String Cosmology could serve as the arena in which four-dimensional string constructions and cosmological data confront each other [31,32].

Outline of the thesis

The contents of the present thesis are organised in the following chapters and appendices:

- Chapter 1 is devoted to introduce the fundamental concepts this thesis is built upon. Specifically those concerning type II supergravity theories, dualities and branes. Far from trying to deal with the more formal aspects of supergravity theories, this chapter is included in order to make the thesis project as self-consistent as possible. The ideas and results appearing in this chapter can be found in the textbooks of refs [1,2,33] and the nice reviews of refs [34,35]

- In chapter 2 we set the notation and conventions adopted in refs [36,37] to describe the geometry of the $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ toroidal orbifold as well as of the IIB orientifold theory with O3/O7-planes built from it. In addition, we work out the $\mathcal{N}=1$ four-dimensional effective action describing the moduli dynamics in this orientifold when generalised fluxes are included, derive the set of consistency conditions on such fluxes and particularise the results to the case of isotropic models which will be extensively explored along the thesis.

- In chapter 3 we develop the techniques needed to remove superfluous degrees of freedom when it comes to build T-duality invariant effective theories with $\mathcal{N} = 1$ supersymmetry. This procedure of reducing parameters allows us to rather simplify the resulting effective models such that families of supersymmetric AdS_4 vacua can be analytically obtained with all moduli stabilised at perturbative values of the string coupling constant. All the results appearing in this chapter can be found in ref. [36].

- Chapter 4 is concerned with the task of building minimal scenarios of moduli stabilisation in a de Sitter almost Minkowski vacuum breaking supersymmetry spontaneously. We are dealing with this problem in the previous context of T-duality invariant IIB orientifold theories including generalised fluxes, O3/O7-planes and D3/D7-branes. This chapter represents the central block of the thesis and corresponds with refs [18, 19].

- In chapter 5 we study a simple T and S-duality invariant supergravity theory built upon the IIB orientifold allowing for O3/O7-planes and succeed in developing a systematic method for solving all the flux constraints based on the algebra structure underlying the fluxes. Algebraic geometry techniques are extensively used to solve these constraints and supersymmetric vacua, centering our attention on Minkowski solutions, become systematically computable and are also provided to clarify the methods. We conclude with a digression on the potential $\mathcal{N} = 4$ lifting of the moduli solutions found. With the exception of this digression, which contains some preliminary results, the content of this chapter can be found in ref. [37].

- In chapter 6 we present a model of slow-roll inflation in the context of effective IIB supergravities with gauge fluxes. The uplifting of the potential to generate a de Sitter (almost Minkowski) vacuum is provided by the D-term associated to an anomalous U(1), in a fully consistent and gauge invariant formulation. We develop a minimal working model which incorporates eternal topological inflation and complies with observational constraints, avoiding the usual obstacles to implement successful inflation, i.e. η problem and initial condition problem among others. Finally we point to the apparent persistence of the η problem when considering only the effect of generalised fluxes as the mechanism for stabilising the entire set of moduli in a dS, almost Minkowski, vacuum. The results in this chapter can be found in ref. [38] together with section 4.3 in ref. [19].

- The appendix A contains a derivation of the supergravity massless spectrum (field content) of type II superstrings freely moving in a ten-dimensional Minkowski spacetime. A similar derivation can be found in any of the String Theory textbooks of refs [1, 2, 33] and in ref. [35]. Roughly speaking this thesis is an exploration of the dynamics associated to these fields when different closed/open string ingredients are taken into account, so we have included this derivation for the sake of completeness.

- The appendix B collects the expressions for the R-R fluxes in terms of the axionic shifts and tadpole parameters for the T-duality invariant IIB orientifold theory studied in chapter 3.

Chapter 1

Basics of Type II Superstrings, Dualities and Branes

At large distances $L \gg l_s$, supergravity theories in ten dimensions turn out to be a good approximation to describe the dynamics of systems of characteristic size L in an effective manner.

1.1 Type II supergravity theories in ten dimensions

In this section we introduce the 10d supergravities describing the low energy (large distance) limit of the type II superstring theories at tree level. These effective theories come out with $\mathcal{N} = 2$ supersymmetry in ten dimensions. A derivation of their massless spectrum entering the supergravity action is presented in the appendix A. It contains the graviton field G_{MN}, the antisymmetric tensor field B_{MN} and the dilaton φ furnishing the universal sector of the superstring theories, together with an additional set of p-form gauge fields C_p which depends on considering either the IIA (p odd) or the IIB (p even) theory. Consequently, the massless spectrum also includes the supersymmetric fermionic partners of these bosonic fields. However, we will focus on the bosonic part of the supergravity action which becomes the relevant piece when studying vacuum configurations. Specifically when exploring how supersymmetry breaks down spontaneously in 4d effective theories as well as the relation between the moduli VEVs and the vacuum energy in such theories.

1.1.1 Massive type IIA bosonic supergravity action

Apart from the universal sector common to all superstring theories, the type IIA superstring theory incorporates a 1-form C_1 and a 3-form C_3 gauge potentials within its bosonic massless spectrum. The massive [39] type IIA 10d supergravity action for the bosonic fields in the Einstein's frame consists of three pieces

$$S_{bos} = S_{\text{NS-NS}} + S_{\text{R-R}} + S_{\text{CS}} \ . \tag{1.1}$$

The first term $S_{\text{NS-NS}}$ involves the fields in the universal sector of the spectrum which corresponds to the Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector shown in (A.12). This piece was already introduced in (14),

$$S_{\text{NS-NS}} = \frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{-G}\,\left(R - \frac{1}{2}\,\partial^M\varphi\partial_M\varphi - \frac{1}{2}e^{-\varphi}|H_3|^2\right) \ , \tag{1.2}$$

where the factor $2\kappa^2 = (2\,\pi)^7\,\alpha'^4 = (16\,\pi\,\text{G}_{10})\,e^{-2\varphi}$ relates to the G_{10} Newton's constant and $H_3 = dB_2$ denotes de field strength of the NS-NS 2-form gauge potential B_{MN}. We use the convention that

$$|F_p|^2 \equiv \frac{1}{p!}\,G^{M_1N_1}\cdots G^{M_pN_p}\,F_{M_1\cdots M_p}\,F_{N_1\cdots N_p} \ , \tag{1.3}$$

for a generic p-form F_p.

The second term $S_{\text{R-R}}$ accounts for the fields in the Ramond-Ramond (R-R) sector of the IIA theory as well as for the Romans's mass parameter m. This term is given by

$$S_{\text{R-R}} = -\frac{1}{4\kappa^2}\int d^{10}x\,\sqrt{-G}\,\left(e^{\frac{\varphi}{2}}|\widetilde{F}_4|^2 + e^{\frac{3}{2}\varphi}|\widetilde{F}_2|^2 + e^{\frac{5\varphi}{2}}\,m^2\right) \ , \tag{1.4}$$

where the tilded field strengths appearing in the R-R action are defined as

$$\begin{aligned} \widetilde{F}_2 &= F_2 + mB_2 \ , \\ \widetilde{F}_4 &= F_4 + C_1\wedge H_3 + \frac{m}{2}B_2\wedge B_2 \ , \end{aligned} \tag{1.5}$$

in terms of the standard ones $H_3 = dB_2$ and $F_{p+1} = dC_p$.

The last term S_{CS} does not involve a factor of $\sqrt{-G}$ and corresponds to a topological Chern-Simons term,

$$S_{\text{CS}} = -\frac{1}{4\kappa^2}\int B_2\wedge F_4\wedge F_4 + \frac{m}{3}\,B_2\wedge B_2\wedge B_2\wedge F_4 + \frac{m^2}{20}\,B_2\wedge B_2\wedge B_2\wedge B_2\wedge B_2 \ . \tag{1.6}$$

Finally, the ordinary type IIA supergravity is recovered by setting $m = 0$. In this case the type IIA supergravity can be obtained by dimensional reduction of 11d supergravity on a circle of radius R. The string coupling constant is then related to R by $g_s = e^{\varphi} = R/\sqrt{\alpha'}$. Since M-Theory has the 11d supergravity as its low energy effective action, it can be thought of as a strong-coupling (large R) completion of type IIA superstring theory.

1.1.2 Bosonic action of type IIB supergravity

The bosonic massless spectrum of type IIB superstring theory contains, besides the universal NS-NS sector, a set of even p-forms in the R-R sector. In particular, a fourth-rank antisymmetric self-dual tensor form C_4, a 2-form C_2 and a scalar C_0. Compared to the previous type IIA theory, an additional issue appears when it comes to build a supergravity type IIB action that implements the self-duality condition upon the field strength $F_5 = dC_4$. Nevertheless, the field equations and the supersymmetry transformations of type IIB supergravity can be worked out in a tricky manner [1]. With the equations of motion identified, one looks at the supergravity action that reproduces these equations of motion and complements them with the self-duality condition.

The bosonic part of the 10d type IIB supergravity action in the Einstein's frame consists again of the three terms

$$S_{bos} = S_{\text{NS-NS}} + S_{\text{R-R}} + S_{\text{CS}} \ . \tag{1.7}$$

It contains a term $S_{\text{NS-NS}}$ accounting for the fields in the universal sector. This is again that of eq.(14), namely,

$$S_{\text{NS-NS}} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} \left(R - \frac{1}{2}\, \partial^M \varphi \partial_M \varphi - \frac{1}{2} e^{-\varphi} |H_3|^2 \right) \ . \tag{1.8}$$

The $S_{\text{R-R}}$ term in the action controlling the dynamics of the R-R fields C_0, C_2 and C_4 is this time given by

$$S_{\text{R-R}} = -\frac{1}{4\kappa^2} \int d^{10}x \sqrt{-G} \left(e^{2\varphi} |F_1|^2 + e^{\varphi} |\widetilde{F}_3|^2 + \frac{1}{2} |\widetilde{F}_5|^2 \right) \ , \tag{1.9}$$

where the tilded field strengths appearing in the action are now defined as

$$\begin{aligned} \widetilde{F}_3 &= F_3 - H_3 \wedge C_0 \ , \\ \widetilde{F}_5 &= F_5 + \frac{1}{2} \left(B_2 \wedge F_3 - C_2 \wedge H_3 \right) \ , \end{aligned} \tag{1.10}$$

in terms of the standard ones $H_3 = dB_2$ and $F_{n+1} = dC_n$. Additionally, the self-duality condition

$$\widetilde{F}_5 = \star \widetilde{F}_5 \ , \tag{1.11}$$

with $(\star \widetilde{F})^{MNOPQM} \equiv \dfrac{1}{5! \sqrt{-g}} \varepsilon^{MNOPQM'N'O'P'Q'} \widetilde{F}_{M'N'O'P'Q'}$ has to be supplemented by hand in order to have the correct number of bosonic degrees of freedom.

The type IIB theory also incorporates a topological Chern-Simons term S_{CS} in the action given by

$$S_{\text{CS}} = -\frac{1}{4\kappa^2} \int C_4 \wedge H_3 \wedge F_3 \ . \tag{1.12}$$

Unlike its type IIA counterpart, the type IIB supergravity can not be obtained by dimensional reduction of 11d supergravity on a circle, namely, from the low energy limit of M-Theory. Nonetheless, a strong-coupling completion of certain type IIB supergravity theories (IIB orientifolds to be introduced later) in ten dimensions is known. It is referred to as F-Theory, is formulated in twelve dimensions and makes extensive use of the non-perturbative $\mathrm{SL}(2,\mathbb{Z})$ self-duality of type IIB superstring theory.

1.2 Symmetries of IIB superstring theory

Along this thesis, we will mostly deal with effective four-dimensional supergravities descending from type IIB superstring compactifications. Learning about the symmetries of the ten-dimensional theory they descend from is then essential for a proper understanding of these lower dimensional theories. For this purpose, this section is devoted to briefly introduce the perturbative and non-perturbative symmetries of type IIB superstring theory.

In the Green-Schwarz formalism in the *light-cone* gauge, the worldsheet action of a type IIB superstring freely moving in a ten-dimensional Minkowski spacetime $\mathbb{M}_{1,9}$ is schematically given by [1]

$$S^{\mathrm{IIB}}_{\mathrm{l.c}} = -\frac{1}{4\pi\alpha'}\int_{\Sigma} d^2\sigma \ \left(\partial X^i\right)^2 + \frac{i}{\pi}\int_{\Sigma} d^2\sigma \left(S^a\partial S^a + \tilde{S}^a\partial\tilde{S}^a\right) . \tag{1.13}$$

The action (1.13) involves not only bosonic fields $X^i = X^i_R + X^i_L$ with $i = 1,\ldots,8$ describing right-moving (R) and left-moving (L) excitations of the string along the transverse space, but also fermionic ones S^a (right-moving) and $\tilde{S}^a$ (left-moving) with the same chirality in eight dimensions[1].

After quantising this free field theory imposing closed-string boundary conditions in both bosonic and fermionic worldsheet fields, the massless states in the perturbative spectrum of the type IIB superstring theory turn out to be given by the tensor product of the right-moving S^a_0 and the left-moving fermionic zero modes $\tilde{S}^a_0$. These massless states become the field content entering the type IIB supergravity theory introduced in section 1.1.2. See appendix A for a detailed derivation of these statements.

1.2.1 Perturbative discrete symmetries

Bearing in mind all the aforementioned, the set of discrete transformations on the S^a and $\tilde{S}^a$ spinors leaving the fermionic part of the action in (1.13),

$$S^{\mathrm{IIB}}_{\mathrm{ferm}} = \frac{i}{\pi}\int_{\Sigma} d^2\sigma \left(S^a\partial S^a + \tilde{S}^a\partial\tilde{S}^a\right) , \tag{1.14}$$

[1]The index i refers to transforming as a vector $\mathbf{8_v}$ under the SO(8) transverse group of rotations. Additionally, the index a refers to transforming in the spinorial representation $\mathbf{8_s}$ of SO(8), while that of $\dot{a}$ will refer to transforming in its conjugate representation $\mathbf{8_c}$ of SO(8).

invariant, will give rise to perturbative discrete symmetries of the type IIB superstring theory and, therefore, of the type IIB supergravity theory alike. By a perturbative symmetry we refer to a symmetry that occurs order by order in perturbation theory, but which is believed to be unbroken even non-perturbatively [35].

Worldsheet parity Ω_p

Type IIB superstring theory contains right-moving S^a and left-moving $\tilde{S}^a$ spinors with the same spacetime chirality in the action (1.14). Therefore, it comes out with the symmetry of exchanging its right-moving and left-moving sectors

$$\begin{array}{rccc} \Omega_p: & S^a & \longleftrightarrow & \tilde{S}^a \\ & X^i_R & \longleftrightarrow & X^i_L \end{array} \tag{1.15}$$

This symmetry can equivalently be viewed as an orientation-reversal parity transformation $\sigma \to 2\pi - \sigma$ on the worldsheet coordinate along the string. Under this right $\leftrightarrow$ left sectors swapping, the massless bosons in the IIB spectrum acquire a parity: half of them, namely g, C_2 and φ, are even under Ω_p whereas C_4, B_2 and C_0 result odd.

	EVEN	ODD
Ω_p	$g\,,\,C_2\,,\,\varphi$	$C_0\,,\,B_2\,,\,C_4$

(1.16)

For the fermionic NS-R and R-NS states, Ω_p takes the former into the latter and *vice versa*. Then, one combination of the two gravitinos (and equivalently for the dilatinos χ's) is even and the other is odd.

Modding out a theory by a discrete symmetry that involves the worldsheet orientation-reversal transformation Ω_p, is commonly referred to as an *orientifold*. Truncating the type IIB spectrum to those Ω_p-even states, i.e. the g and φ NS-NS fields together with the R-R 2-form C_2 field, then the closed-string sector of type I supergravity is reproduced. This symmetrisation restriction upon the IIB spectrum gives rise to $\mathcal{N}=1$ supergravity in ten dimension [1]. Notice that the 2-form C_2 in the type I theory comes from the R-R sector unlike the 2-form B_2 appearing in the universal type II bosonic sector. We will be back to this point in section 5.4 when studying the $\mathcal{N}=4$ origin of some gauged supergravities arising from flux compactifications of type IIB superstring theory.

Fermion number $(-1)^{F_R}$ and $(-1)^{F_L}$

The type IIB fermionic action in (1.14) is also invariant under the discrete symmetries

$$\begin{array}{rccc} (-1)^{F_R}: & S^a & \longrightarrow & -S^a\,, \\ (-1)^{F_L}: & \tilde{S}^a & \longrightarrow & -\tilde{S}^a\,, \end{array} \tag{1.17}$$

where F_R and F_L denote the fermion number in the right-moving and left-moving sectors, respectively[2]. This occurs independently for both right- and left-moving sectors, so the massless fields in the IIB spectrum result with the following parities under the $(-1)^{F_R}$ and $(-1)^{F_L}$ discrete transformations,

	EVEN	ODD
$(-1)^{F_R}$	$g\,,\,B_2\,,\,\varphi\,,\,\chi_1\,,\,\psi_1$	$C_0\,,\,C_2\,,\,C_4\,,\,\chi_2\,,\,\psi_2$
$(-1)^{F_L}$	$g\,,\,B_2\,,\,\varphi\,,\,\chi_2\,,\,\psi_2$	$C_0\,,\,C_2\,,\,C_4\,,\,\chi_1\,,\,\psi_1$

(1.18)

Because of the non-commutativity between the worldsheet parity and the fermion number transformations,

$$(-1)^{F_R}\,\Omega_p = \Omega_p\,(-1)^{F_L} \tag{1.19}$$

the complete set of discrete perturbative symmetries of the type IIB theory consists of the eight-element non-abelian dihedral group D_4, i.e. the group of symmetries of a square.

1.2.2 Non-perturbative SL$(2,\mathbb{Z})$ self-duality

Together with the perturbative symmetries introduced above, namely, the worldsheet orientation-reversal and the fermion number symmetries, the full type IIB superstring theory has been conjectured to have an additional SL$(2,\mathbb{Z})$ self-duality [40,41]. This conjecture stems from the invariance of the effective field theory describing the IIB massless spectrum, i.e. type IIB ten-dimensional supergravity, under a non-compact global symmetry SL$(2,\mathbb{R})$ [1].

This symmetry transforms the *axion-dilaton* field defined by $S \equiv C_0 + i\,e^{-\varphi}$ in a non-linear manner

$$S \;\rightarrow\; \frac{a\,S+b}{c\,S+d} \qquad \text{with} \qquad a\,d-b\,c=1\;, \tag{1.20}$$

so that a weakly-coupled region of the moduli space having $g_s = (\text{Im}S)^{-1} < 1$, see eq.(15), can be mapped to a strongly-coupled one with $g_s > 1$ by means of an SL$(2,\mathbb{R})$ transformation. Therefore, the SL$(2,\mathbb{Z})$ self-duality of type IIB superstring theory acts on the moduli space of the theory[3]. This fact makes this self-duality impossible to be checked order by order in perturbation expansion since perturbative states (oscillation states) at one point in the moduli space of the theory may map to perturbative and non-perturbative ones, as solitons or bound states, not corresponding to oscillation states of the string in the dual theory.

[2]The bosonic states with an even number of creation operators in the NS sector of (A.8) are then even, while those fermionic states in the R sector of (A.9) result odd.

[3]This will be no longer the case for other dualities in String Theory. As we will see, type IIA superstring theory compactified on a circle of radius R is dual to IIB superstring theory compactified on a circle of radius R^{-1} at the same value of the string coupling constant g_s.

The action of the SL$(2,\mathbb{Z})$ self-duality on the rest of the massless bosonic fields in the IIB spectrum is given by

$$\begin{pmatrix} B_2 \\ C_2 \end{pmatrix} \to \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} \quad , \quad G \to G \quad , \quad C_4 \to C_4 \, , \tag{1.21}$$

where $G_{MN} = e^{-\frac{\varphi}{2}} g_{MN}$ is the metric field in the Einstein's frame. This group is generated by the transformations

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow \; S \to S+1 \qquad \text{and} \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow \; S \to \frac{-1}{S} \; , \tag{1.22}$$

together with a reflection that leaves the axiodilaton invariant

$$R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \; S \to S \; . \tag{1.23}$$

The SL$(2,\mathbb{Z})$ reflection in (1.23) acts upon the type IIB 2-forms as $(B_2, C_2) \to -(B_2, C_2)$ so, according to (1.16) and (1.18), such an action can be identified with that of an orientifold $R = (-1)^{F_L}\,\Omega_p$.

The above self-duality of the type IIB theory is commonly referred to in the literature as *S-duality*.

SL$(2,\mathbb{R})$ covariant formulation of type IIB supergravity

The IIB bosonic action in (1.7) defined in terms of equations (1.8), (1.9) and (1.12) is not manifestly covariant under the SL$(2,\mathbb{R})$ symmetry previously discussed. For it to be, let us define a doublet of 2-form gauge potentials $\mathbb{B}_2$ with field strength $\mathbb{H}_3 = d\mathbb{B}_2$, together with a symmetric SL$(2,\mathbb{R})$ matrix $\mathbb{M}$ made up of scalars

$$\mathbb{B}_2 = \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} \qquad , \qquad \mathbb{M} = e^{\varphi} \begin{pmatrix} |S|^2 & -C_0 \\ -C_0 & 1 \end{pmatrix} \; . \tag{1.24}$$

Under a constant SL$(2,\mathbb{R})$ transformation given by

$$\Lambda = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \; , \tag{1.25}$$

the fields appearing in (1.24) will be transformed according to

$$\mathbb{B}_2 \longrightarrow \Lambda\, \mathbb{B}_2 \qquad \text{and} \qquad \mathbb{M} \longrightarrow \left(\Lambda^{-1}\right)^t \, \mathbb{M}\, \Lambda^{-1} \; . \tag{1.26}$$

Using these quantities, the IIB bosonic action in (1.1) can be rewritten in a manifestly SL$(2,\mathbb{R})$ covariant form

$$\begin{aligned} S_{bos} \;&=\; \frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{-G}\,\left(R - \frac{1}{12}\,\mathbb{H}^t_{MNP}\,\mathbb{M}\,\mathbb{H}^{MNP} + \frac{1}{4}\,\mathrm{Tr}\Big(\partial^M\mathbb{M}\,\partial_M\mathbb{M}^{-1}\Big)\right) \\ &-\; \frac{1}{8\kappa^2}\left(\int d^{10}x\,\sqrt{-G}\,|\widetilde{F}_5|^2 + \int \varepsilon_{ij}\, C_4\wedge \mathbb{H}_3^{(i)}\wedge\mathbb{H}_3^{(j)}\right) \; , \end{aligned} \tag{1.27}$$

since the Einstein-frame metric G and the 4-form C_4 are $\mathrm{SL}(2,\mathbb{R})$ invariant, and then, also the $\widetilde{F}_5$ field strength of eq.(1.10).

1.3 Toroidal compactifications and target space dualities

In this section we introduce the basics of a closed string that propagates in a ten-dimensional Minkowski spacetime $\mathbb{M}_{1,9}$ in which one of the spacial directions, say the j-th one, is taken to be a circle of radius R. In other words, it has been *compactified*.

Following the conventions in appendix A, the embedding of a closed string propagating in $\mathbb{M}_{1,9}$ is encoded into the bosonic fields

$$X^M(\tau,\sigma) = X_R^M(\tau-\sigma) + X_L^M(\tau+\sigma) \qquad \text{with} \qquad M=0,\ldots,9 \; . \tag{1.28}$$

The coordinate τ is the worldsheet time coordinate whereas σ denotes the coordinate along the string $0 \leq \sigma < \pi$.

The bosonic field $X^j(\tau,\sigma)$ describing the propagation of the string along the compact j-th direction can have a periodic boundary condition with a non vanishing winding number w,

$$X^j(\tau,\sigma+\pi) = X^j(\tau,\sigma) + 2\pi R\, w \qquad \text{with} \qquad w \in \mathbb{Z} \; , \tag{1.29}$$

counting the number of times that the string winds around the circle. The momentum along this direction results also quantised in terms of an integer k named the Kaluza-Klein excitation number, i.e. $p = k/R$.

When imposing the boundary condition (1.29) upon the j-th spacial dimension, the corresponding bosonic field has the mode expansion of

$$X_R^j = \frac{x^j}{2} + \alpha'\, p_R\,(\tau-\sigma) + \alpha^j\text{-oscillators} \quad , \quad X_L^j = \frac{x^j}{2} + \alpha'\, p_L\,(\tau+\sigma) + \tilde{\alpha}^j\text{-oscillators} \tag{1.30}$$

with right- and left-moving momenta given by

$$p_R = \frac{k}{R} - \frac{wR}{\alpha'} \qquad , \qquad p_L = \frac{k}{R} + \frac{wR}{\alpha'} \; . \tag{1.31}$$

The mode expansion for the rest of bosonic fields X^i (with $i \neq j$-th) describing the propagation along the non-compact coordinates is shown in (A.5).

At distance scales $L \gg R$, it is not possible to resolve the compact dimension so we are left with a 9-dimensional effective theory involving a tower of states whose mass depends on the momenta (p_R, p_L) in the compactified j-th dimension. This mass is given by

$$\alpha'\, M^2 = \frac{\alpha'}{2}(p_R^2 + p_L^2) + 2\,(N+\tilde{N}) = \left(\frac{k\sqrt{\alpha'}}{R}\right)^2 + \left(\frac{wR}{\sqrt{\alpha'}}\right)^2 + 2\,(N+\tilde{N}) \; , \tag{1.32}$$

where N and $\tilde{N}$ account for the bosonic and fermionic oscillators in the mode expansion of the worldsheet fields.

For a generic value of the ratio $R/\sqrt{\alpha'}$, the massless states correspond to $k = w = 0$. However, at the limit case of $R/\sqrt{\alpha'} \to \infty$ $\left(R/\sqrt{\alpha'} \to 0\right)$ we are left with a tower of light states with zero winding (momentum) and arbitrary momentum (winding) which would enter the lower dimensional physics. We will be back to this question when studying non-geometric flux compactifications of type IIB superstring theory on T-fold spaces[4]. Specifically, when discussing the large internal volume commonly assumed to neglect α' corrections, i.e., the radius of the internal space R is larger than the string length scale $l_s = \sqrt{2\,\alpha'}$ rendering field theories a good approximation to study the dynamics of String Theory. For a large internal volume, $R/\sqrt{\alpha'} \gg 1$, the *stringy* winding modes acquire a large mass and are decoupled from the light spectrum.

Transforming simultaneously

$$R \leftrightarrow \frac{\alpha'}{R} \qquad \text{and} \qquad k \leftrightarrow w \;, \tag{1.33}$$

the α^j-oscillators change their sign while the $\tilde{\alpha}^j$-oscillators are left invariant. Then, the j-th right-moving and left-moving fields do it as $X_R^j \to -X_R^j$ and $X_L^j \to X_L^j$, representing a target-space duality of the theory known as *T-duality* [34].

By spacetime supersymmetry, the chirality of the right-moving worldsheet fermion is changed when applying a T-duality transformation in the j-th direction[5]. This translates into the fact that: T-duality takes the type IIB (chiral) theory on a circle of radius R_B to a type IIA (non-chiral) theory on the dual circle of radius $R_{\rm A} = \alpha'/R_{\rm B}$ and *vice versa*.

It is worth noticing that the mass formula in (1.32), and hence the spectrum, is invariant under the T-duality transformation of (1.33). This illustrates how target-space dualities can emerge at the effective level when compactifying String Theory to obtain lower dimensional physics.

1.3.1 Moduli space and T-duality in toroidal backgrounds

Let us now describe the propagation of a type II superstring in a more general spacetime geometry M. In particular we will consider the case of a ten-dimensional spacetime of the form

$$M = \mathbb{M}_{(1,9-d)} \times \mathbb{T}^d \;. \tag{1.35}$$

[4] The geometry of these T-fold spaces incorporates circles of size R as well as their duals of size $1/R$.

[5] The right-moving S^a and left-moving $\tilde{S}^{a(\dot{a})}$ worldsheet fermions in type II theories transform as

$$\begin{aligned} \text{IIB}: &\quad S^a \to \Gamma^j_{a\dot{b}}\, S^{\dot{b}} \quad , \quad \tilde{S}^a \to \tilde{S}^a \\ \text{IIA}: &\quad S^a \to \Gamma^j_{a\dot{b}}\, S^{\dot{b}} \quad , \quad \tilde{S}^{\dot{a}} \to \tilde{S}^{\dot{a}} \end{aligned} \tag{1.34}$$

where the Γ's are the eight-dimensional Dirac matrices satisfying $(\Gamma^j)^2 = 1$.

$\mathbb{M}_{(1,9-d)}$ denotes $(10-d)$-dimensional Minkowski spacetime with coordinates x^μ where $\mu = 0, \dots, 9-d$, whereas $\mathbb{T}^d$ is a d-dimensional torus with coordinates $x^i \simeq x^i + 2\pi R$ where $i = 1, \dots, d$.

Turning on a background for the metric G_{ij} as well as for the antisymmetric tensor B_{ij} on $\mathbb{T}^d$, gives rise to more generic string backgrounds known as *toroidal backgrounds*. As it was stated in the introduction, these background fields can be introduced into the worldsheet action in a consistent manner, provided it has conformal symmetry.

Including them, the bosonic fields describing the propagation of the string along the d compact dimensions in $\mathbb{T}^d$ now have periodic boundary conditions

$$X^i(\tau, \sigma + \pi) = X^i(\tau, \sigma) + 2\pi R\, w^i \qquad \text{with} \qquad w^i \in \mathbb{Z} \qquad \text{and} \qquad i = 1, \dots d\,, \tag{1.36}$$

leading to the mode expansion of the bosonic fields along the toroidal directions of

$$X^i_R = \frac{x^i}{2} + \alpha'\, p^i_R\,(\tau - \sigma) + \alpha^i\text{-oscillators} \quad , \quad X^i_L = \frac{x^i}{2} + \alpha'\, p^i_L\,(\tau + \sigma) + \tilde{\alpha}^i\text{-oscillators} \tag{1.37}$$

where this time the right- and left-moving momenta are given by

$$p_{R,\,i} = \frac{k_i}{R} - \frac{R}{\alpha'}\,(G_{ij} + B_{ij})\, w^j \qquad , \qquad p_{L,\,i} = \frac{k_i}{R} + \frac{R}{\alpha'}\,(G_{ij} - B_{ij})\, w^j\ . \tag{1.38}$$

Once more, at distance scales $L \gg R$, the theory looks like a $(10-d)$-dimensional theory where the mass of a state with momenta $\left(p^i_R, p^i_L\right)$ in the compactified dimensions is given by

$$\alpha'\, M^2 = Z^t\, \mathcal{H}\, Z + 2\,(N + \tilde{N}) \qquad , \quad Z = \left(\frac{w^i R}{\sqrt{\alpha'}}\,,\ \frac{k_i \sqrt{\alpha'}}{R} \right)\ , \tag{1.39}$$

with N and $\tilde{N}$ accounting again for bosonic and fermionic oscillators.

The matrix $\mathcal{H}(G,B) \in \text{O}(d,d;\mathbb{R})$ that determines the mass of the states according to (1.39),

$$\mathcal{H}(G,B) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}\ , \tag{1.40}$$

is a $2d \times 2d$ *generalised* metric built from the NS-NS background fields. This matrix can be interpreted as the metric of a doubled internal space [42]. Even a generalised worldsheet action has been proposed [43, 44] to describe strings propagating in a doubled torus $\mathbb{T}^{2d}$.

Notice this time that if inverting the size of the circles in $\mathbb{T}^d$, i.e. $R \leftrightarrow \dfrac{\alpha'}{R}$, the mass formula in (1.39) is invariant under the transformation

$$\mathcal{H} \;\leftrightarrow\; \mathcal{H}^{-1} \qquad \text{with} \qquad k_i \;\leftrightarrow\; w^i\ , \tag{1.41}$$

which is the natural generalisation of (1.33). Furthermore the right- and left-moving momenta in (1.38) are invariant under

$$B_{ij} \quad \leftrightarrow \quad B_{ij} + \frac{\alpha'}{R^2} N_{ij} \qquad \text{with} \qquad k_i \quad \leftrightarrow \quad k_i + N_{ij}\, w^j \quad , \tag{1.42}$$

where N_{ij} is an arbitrary antisymmetric matrix. This B-shift symmetry is new in the sense that it only appears when $d > 1$ because of the antisymmetry of N_{ij}.

The two transformations in (1.41) and (1.42) are symmetries of the spectrum and jointly generate the $\mathrm{O}(d, d; \mathbb{Z})$ group[6], although the T-duality group of the effective theory in $(10 - d)$ dimensions is reduced to the subgroup $\mathcal{G} = \mathrm{SO}(d, d; \mathbb{Z})$ which preserves the chirality of the spinors. At this stage, there is no longer difference between IIA and IIB theories once they have been compactified on $\mathbb{T}^d$. This underlies the idea of introducing the so-called *generalised fluxes* to restore T-duality at the level of the effective action in the lower dimensional theory.

The G_{ij} and B_{ij} background fields specifying the generalised metric $\mathcal{H} \in \mathrm{O}(d, d; \mathbb{R})$ are understood, after compactifying to $(10 - d)$ dimensions, as d^2 degrees of freedom appearing as scalar fields in the lower dimensional theory. These scalars play the roll of parameters connecting between different theories and are referred to as the *moduli space* of the compactification. The moduli space contains $d(d + 1)/2$ parameters coming from the internal components of the metric together with $d(d - 1)/2$ ones coming from the internal components of the antisymmetric B-field. The only restriction to be imposed is the positiveness of the symmetric part of the matrix $G_{ij} + B_{ij}$. This determines the moduli space to be the coset space

$$\mathcal{M}_0 = \frac{\mathrm{O}(d, d; \mathbb{R})}{\mathrm{O}(d; \mathbb{R}) \times \mathrm{O}(d; \mathbb{R})} \ . \tag{1.43}$$

This is the moduli space of toroidal compactifications to $(10 - d)$ dimensions. The isotropy group $\mathrm{O}(d; \mathbb{R}) \times \mathrm{O}(d; \mathbb{R})$ in the above coset space reflects the invariance of the theory under separate rotations of the p^i_R and p^i_L momenta. Two different points in the moduli space associated to $\mathrm{O}(d, d; \mathbb{R})$ elements which are related by an element of the isotropy group give rise to the same term $p^2_R + p^2_L$ entering the mass spectrum[7], so they must be identified as physically equivalent. Therefore, the set of inequivalent toroidal compactifications of type II superstrings will be given by the quotient $\mathcal{M} = \mathcal{M}_0/\mathrm{SO}(d, d; \mathbb{Z})$.

Finally, the simplest toroidal background is that consisting on a square torus $G = \mathbb{I}_{d\times d}$ together with a vanishing B-field background, this is $\mathcal{H} = \mathbb{I}_{2d\times 2d}$. In this case, the

[6]The inversion transformation in (1.41) changes the relative chirality between the right- and left-moving fermions S and $\tilde{S}$ in type II theories.

[7]Two generic points in the moduli space connected by a $\mathrm{O}(d, d; \mathbb{R})$ rotation give rise to the same $p^2_R - p^2_L$ quantity since the momenta (p^i_R, p^i_L) in toroidal compactifications live on an (Lorentzian) even and self-dual lattice $\Gamma_{d,d}$ with Lorentzian signature, i.e., a Narain lattice.

massless d-dimensional spectrum in the lower dimensional theory can be obtained by decomposing the SO(8) representations of the 10d massless fields with respect to the surviving Lorentz group. In going from ten dimensions to 4d effective theories by means of compactifying on $\mathbb{T}^6$ one obtains the non-chiral $\mathcal{N}=8$, $d=4$ supergravity field theory at energies $E \ll 1/R \ll 1/\sqrt{\alpha'}$. As we will see in the next sections, and although chirality is mandatory in order to relate four-dimensional strings to the real world, six-dimensional toroidal backgrounds provide us with simple 4d supergravity models in which to explore the dynamics of the moduli fields.

1.3.2 Toroidal orbifolds

Compactifications of type II superstrings on $\mathbb{T}^6$ preserve all the 32 supercharges of the original supergravity in ten dimensions and produce effective theories with $\mathcal{N}=8$ supersymmetry in four dimensions. Due to the large amount of supersymmetry, the structure of these 4d effective theories is very rigid. For instance, the scalar potential generated by $\mathrm{SO}(p,q,r)$ gaugings with $p+q+r=8$ possesses de Sitter extrema which turn out to be unstable (see [45] and references therein). As we reduce the number of supersymmetries during the compactification process, the structure of the resulting 4d effective theories becomes richer and their associated phenomenology more attractive [46].

A simple possibility to obtain 4d effective theories with a less number of unbroken supersymmetries is to compactify String Theory on 6d toroidal orbifolds. Given a toroidal internal space $\mathbb{T}^6$, an orbifold $\mathbb{T}^6/G$ can be constructed quoting it by a finite symmetry group G acting on the target space. After the action of G on $\mathbb{T}^6$, different points in the original (ambient) torus result identified and some singularities may appear associated to the fixed points of the orbifold group action. Toroidal orbifolds can be interpreted as singular limits of Calabi-Yau spaces. Even though there are singularities, strings propagate consistently in these spaces provided that a *twisted sector* is taken into account together with the *untwisted* (invariant under the action of G) one. Far from the singularities, toroidal orbifolds represent smooth target spaces whose local structure is directly inherited from the ambient space $\mathbb{T}^6$. A nice property of toroidal orbifolds is that they are as simple as tori but after compactifying String Theory on them one obtains effective theories with reduced supersymmetry[8].

Let us say a few comments about the twisted sector in toroidal orbifold compactifications. This sector arises from oscillations of strings that are open strings in the ambient space $\mathbb{T}^6$ but become closed strings in the orbifold $\mathbb{T}^6/G$. In other words, open strings starting and ending at points in $\mathbb{T}^6$ that are identified under the action of an element of the orbifold group. These "closed" strings are localised necessarily around the fixed points

[8]The amount of unbroken supersymmetry corresponds to the components of the original supercharge that are invariant under the action of the orbifold group G.

of the orbifold group action and the twisted spectrum they produce simply contains the moduli associated to *blowing up* these singularities.

1.3.3 Toroidal orientifolds

As we stated in the previous section, toroidal orbifolds $\mathbb{T}^6/G$ can be viewed as singular limits of Calabi-Yau 3-folds (CY_3). Compactification of type II superstrings on these CY_3 spaces gives rise to four-dimensional supergravity with $\mathcal{N}=2$ supersymmetry. This amount of supersymmetry is still large from a phenomenological viewpoint. However, it can be further reduce to $\mathcal{N}=1$ by modding out the theory with the orientation-reversal parity transformation Ω_p, the fermion number projector for left-moving fermions $(-1)^{F_L}$ and an internal space involution σ which has to be an isometry of CY_3. The resulting theory is referred to as a Calabi-Yau *orientifold* and comes out with $\mathcal{N}=1$ supersymmetry in four dimensions.

The space involution σ leaves the non-compact 4d spacetime invariant. Apart from it, the action of σ may also leave invariant certain submanifolds of the internal space. The product of one of these submanifolds with the ordinary 4d spacetime is referred to as an Op-plane, where the label p denotes the number of spacial dimensions filled by the O-plane. The main features of an O-plane are that it has a negative tension and couples to a R-R gauge potential C_{p+1}, although it does not have an associated dynamics.

There exist three sorts of (type II) CY_3 orientifold theories depending on how does the pullback σ^* of the orientifold involution σ act upon the Kähler form J and the holomorphic 3-form Ω encoding the geometry of the CY_3 space [47,48]:

$$
\begin{array}{llcll}
i) & \text{Type IIB with O3/O7-planes} & \longrightarrow & \sigma^* J = J \quad , & \sigma^*\Omega = -\Omega \quad , \\
ii) & \text{Type IIA with O6-planes} & \longrightarrow & \sigma^* J = -J \quad , & \sigma^*\Omega = \bar{\Omega} \quad , \\
iii) & \text{Type IIB with O9/O5-planes} & \longrightarrow & \sigma^* J = J \quad , & \sigma^*\Omega = \Omega \quad .
\end{array}
$$

Specially in the case of toroidal orbifolds with (generalised) fluxes [7,8], the above type II orientifold theories have been shown to be related one to another by a chain of T-duality transformations (mirror symmetry) [49,50], so we will often refer to them, with some abuse of language, as (T-) *duality frames*.

1.4 D-branes in type II superstring theory

Even though String Theory was conceived as a theory of fundamental one-dimensional objects, it was eventually found to also contain higher dimensional objects called D-branes. These extended objects can be understood from two approaches: the microscopic approach according to which D-branes correspond to the place where the open strings can terminate and the macroscopic one where D-branes emerge as 1/2-BPS solitonic solutions of the supergravity equations of motion.

In the microscopic description, a Dp-brane fills a $(p+1)$ region within the 10d spacetime (p spacial directions plus time) where the ends of an open string can move. In other words, the open string carries momentum along these directions (Neumann boundary conditions). On the contrary, an open string has a specified position (Dirichlet boundary conditions) along the $(9-p)$ coordinates in the space transverse to the Dp-brane[9]. After quantisation, an open string ending on a Dp-brane gives rise to a set of massless states that consists of a U(1) gauge boson, $(9-p)$ real scalars and a set of fermion superpartners confined to the *worldvolume* of the Dp-brane. These massless states furnish a U(1) vector supermultiplet with respect to $32/2 = 16$ supercharges in $(p+1)$ dimensions. Additionally, Dp-branes can emit/absorb closed strings that propagate in the whole 10d spacetime, also known as the "bulk".

From a macroscopic approach, Dp-branes in type II superstring theories represent certain non-trivial vacuum configurations for the fields entering the type II supergravities. A Dp-brane admits the interpretation of an extended 1/2-BPS object located in the background which breaks half of the supersymmetry and couples to the bosonic closed-string modes propagating in the "bulk". In other words, Dp-branes interact via the exchange of bosonic fields belonging to the type II massless spectrum. These are fields in the universal NS-NS sector, i.e., g, B_2 and φ, together with p-forms C_p in the R-R sector.

It will result useful to use the 10d democratic formulation according to which one defines additional R-R gauge potentials C_{8-p} in such a way that the equations of motion of the original C_p gauge fields become the Bianchi identities of the new C_{8-p} ones and *vice versa*. This is

$$\begin{array}{lllll} d\star F_{p+1} & = & dF_{9-p} & = & \star j_p \quad , \\ dF_{p+1} & = & d\star F_{9-p} & = & \star j_{8-p} \quad , \end{array} \tag{1.44}$$

where j_p and j_{8-p} are the electric (magnetic) and the magnetic (electric) currents that couple to the gauge potential C_p (C_{8-p}). These currents are then sourced by D$(p-1)$-branes and D$(7-p)$-branes, respectively.

In the static gauge, the bosonic Dp-brane action consists of two parts

$$S_{\mathrm{D}p} = S_{\mathrm{DBI}} + S_{\mathrm{CS}} \ . \tag{1.45}$$

- The Dirac-Born-Infeld action: It describes how the Dp-brane couples to the universal NS-NS closed-string sector. Denoting ξ the $(p+1)$ worldvolume coordinates, the

[9] An open string has no a non-trivial dynamics along these $(9-p)$ coordinates, but only an oscillatory motion.

DBI action is written as[10]

$$S_{\rm DBI} = -T_{{\rm D}p} \int d^{p+1}\xi \, \sqrt{-{\rm det}\left(g + B_2 + (2\pi\alpha')\, F + (2\pi\alpha')^2\,(\,\partial\vec{\phi}\,)^2\right)}\;, \qquad (1.46)$$

where $F = dA$ is the field strength for the U(1) gauge boson living on the worldvolume of the Dp-brane and $\vec{\phi}$ accounts for the $(9-p)$ scalars (moduli fields) parameterising the position of the Dp-brane in the transverse space. In the string frame, the tension $T_{{\rm D}p}$ for a Dp-brane relates to the string coupling constant as $T_{{\rm D}p} \propto g_s^{-1}$. Therefore, D$p$-branes are non perturbative states that become very heavy at weak coupling and behave as rigid objects.

- The Chern-Simons action: A Dp-brane is a R-R charged extended object that couples to the set of p-form fields in the R-R closed-string sector. The CS piece in the bosonic action accounts for these couplings and, in a flat spacetime, is given by

$$S_{\rm CS} = \mu_p \int \left(C\, e^{B_2+(2\pi\alpha')F}\right)_{p+1} = \mu_p \int C_{p+1} + \dots \;, \qquad (1.47)$$

where μ_p is the Dp-brane electric charge under the R-R gauge potential C_{p+1}. The field C denotes the formal sum over all the R-R p-form potentials in the theory. Only the $(p+1)$-form piece of the integrand is kept at each step in the expansion of $e^{B_2+(2\pi\alpha')F}$. Therefore, in the presence of a non vanishing $B_2+(2\pi\alpha')F$ background, a Dp-brane also has induced charges corresponding to less dimensional D$(p-2n)$-branes with n being a positive integer.

In the 10d democratic formulation of type IIB (type IIA) superstring theory there will be D-branes coupling to the set of C_{2p} (C_{2p+1}) R-R fields with $p = 0,\dots,4$ ($p = 0,\dots,3$). More specifically,

- Type IIB theory: The C_{2p} with $p = 0,\dots,4$ R-R gauge potential has D$(2p-1)$-branes as electric sources and D$(7-2p)$-branes as magnetic sources,

R-R POTENTIAL	ELECTRIC SOURCE	MAGNETIC SOURCE
C_0	D(-1)-brane	D7-brane
C_2	D1-brane	D5-brane
C_4	D3-brane	D3-brane
C_6	D5-brane	D1-brane
C_8	D7-brane	D(-1)-brane

(1.48)

The D(-1)-brane is an object that is localised in time as well as in space, so it has the interpretation of a D-instanton.

[10] All the tensors are understood to be pulled back to the worldvolume of the Dp-brane.

- Type IIA theory: The C_{2p+1} with $p = 0, \ldots, 3$ R-R gauge potential has D($2p$)-branes as electric sources and D($6-2p$)-branes as magnetic sources.

R-R POTENTIAL	ELECTRIC SOURCE	MAGNETIC SOURCE
C_1	D0-brane	D6-brane
C_3	D2-brane	D4-brane
C_5	D4-brane	D2-brane
C_7	D6-brane	D0-brane

(1.49)

Additionally, one could consider D8-branes which would couple electrically to a C_9 R-R gauge potential with field strength $F_{10} = dC_9$. In ten dimensions, this field strength is non dynamical and relates to the Romans's mass parameter $F_{10} = \star m$ of the type IIA massive supergravity presented in section 1.1.1.

With the discover of the D-branes as non perturbative states in type II superstring theories, new possibilities to obtain non-abelian gauge theories in which to embed the Standard Model gauge group popped up. An important result is that, as well as the DBI piece (1.46) of the action can be regarded as a "gravitational" interaction for the Dp-brane, the CS piece (1.47) represents a "gauge" interaction. In this sense Dp-branes were found to be BPS-*saturated* states of the theory. This implies that a Dp-brane preserves one half of the original supersymmetry and that $T_{\mathrm{D}p} = \mu_p \, g_s^{-1}$, namely, its mass *equals* its charge. Roughly speaking, as an outcome of this matching, the gravitational force and the gauge force between two Dp-branes do exactly cancel each other. This is commonly referred to as the *no-force* condition and allows us to put N Dp-branes together without falling into problems of stability. The field theory in the worldvolume of a stack of N coincident Dp-branes enjoys an enhanced U(N) non-abelian gauge symmetry. In order to account for the non-abelian structure, the bosonic effective action results slightly modified with respect to that of the single Dp-brane we have introduced in this section.

The fact of being able to obtain non-abelian gauge theories from D-branes gave rise to a model building revolution trying to obtain the Standard Model of Particle Physics and Cosmology from type II superstring compactifications including D-branes. In order to derive four-dimensional semi-realistic effective models from type II superstring constructions, they should encompass, among many others, the following aspects of Supertring Phenomenology:

i) To obtain the Standard Model gauge group $G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ with three chiral quark-lepton generations. Most of the effort in this direction has been focused on four scenarios: 1) D3/D7-brane systems at singularities [51–54]. 2) Intersecting D6-brane models [55–69]. 3) D-branes with worldvolume magnetic fields (magnetised) and background fluxes [70–73]. 4) F-Theory Grand Unification Theories (GUT's) [74–78].

ii) To stabilise the moduli fields of the compactification [6–8, 10, 11, 49, 73, 79–100]. Specially the case of having moduli stabilisation in a de Sitter (dS) almost Minkowski (Mkw) vacuum breaking supersymmetry spontaneously. Type II toroidal orientifolds including D-branes, O-planes and (generalised) flux backgrounds are simple scenarios in which this goal seems to be more affordable [12,13,16–23,37,98,101–110].

iii) To reproduce an inflationary regimen at the first stages of the Universe [26]. Some attempts to get inflation from string-based scenarios [30,31] have based on: 1) D-brane inflation [111–127], where the inflaton field corresponds to a modulus parameterising the distance between branes. 2) Modular inflation [13–15, 29, 128–137], where the role of the inflaton is played by some of the geometry/structure moduli fields of the internal space.

This thesis goes along the lines of *ii*) and *iii*). Although most of it will be about moduli stabilisation in generalised flux compactifications, other related topics as gauged supergravities and modular inflation will also be directly or indirectly explored.

1.4.1 T-duality and D-branes

The fact of having a D-brane wrapping some compact dimension of the internal space is intimately linked to having applied a T-duality trasformation upon an open string which originally did not propagate along such a dimension but wound itself around it.

When applied to open strings, T-duality maps a Neumann boundary condition carrying momentum $p = k/R$ along a circle of radius R into a Dirichlet boundary condition carrying a winding number $w = k$ along a circle of radius $\tilde{R} = \alpha'/R$ in the T-dual geometry (and *vice versa*). Starting with an open string that propagates in ten dimensions, which can be understood as having a D9-brane wrapping the whole (compact) internal space, and by applying successive T-dualities along the six internal space coordinates, we will end up with a D3-brane localised at a point of the internal space in the T-dual geometry.

Finally, Dp-branes can also be helpful when it comes to break part of the supersymmetry of the type II superstring vacua. In the absence of Dp-branes, T-duality exchanges between the type IIA and type IIB theories without breaking their $\mathcal{N} = 2$ supersymmetry in ten dimensions. This will be no longer the case once Dp-branes are added. Due to their 1/2-BPS state nature, at least half of the supersymmetry will be broken in type II superstring vacua that include Dp-branes.

1.4.2 S-duality and (p, q)-branes

The $\mathrm{SL}(2, \mathbb{Z})$ self-duality of the type IIB supergravity acts upon the BPS branes of the type IIB theory in a much more complicated manner. Recall that it represents a strong-weak coupling self-duality of the IIB theory.

According to the S-duality transformation of the C_4 and (B_2, C_2) gauge potentials in (1.21), one has that:

- D3-branes are singlets under S-duality transformations since they are the electric and magnetic sources of the R-R field C_4, which is an $\mathrm{SL}(2,\mathbb{Z})$ singlet.
- D1-branes are transformed into (p,q) 1-branes, commonly referred to as (p,q)-strings, under S-duality transformations. A (p,q)-string has charges (p,q) with respect to the gauge fields (B_2, C_2), which form an $\mathrm{SL}(2,\mathbb{Z})$ doublet. The $(0,1)$-string is then identified with the D1-string whereas the $(1,0)$-string is known as the fundamental F1-string.
- S-duality transforms D5-branes into (p,q) 5-branes which are the magnetic duals of the (p,q)-strings. The $(0,1)$ and $(1,0)$ 5-branes are identified with the D5-branes and NS5-branes respectively.

7-branes

A classification of the 7-brane supergravity solutions can be found in [138,139]. According to (1.48), the D7-branes are magnetic sources of the C_0 gauge potential entering the type IIB axiodilaton field $S \equiv C_0 + i\, e^{-\varphi}$ whose transformation under S-duality is displayed in (1.20). Additionally, the D7-branes are electric sources of the R-R gauge potential C_8 which is one of the components of an $\mathrm{SL}(2,\mathbb{Z})$ triplet $(C_8, C_8', \tilde{C}_8)$ of 8-forms [139–141].

Generically, a 7-brane couples (electrically) to the triplet of 8-forms $(C_8, C_8', \tilde{C}_8)$ with charges p^2, r and q^2, respectively. These charges determine the $\mathrm{SL}(2,\mathbb{Z})$ monodromy matrix

$$e^{Q} = \cosh\left(\sqrt{\det Q}\right)\, \mathbb{I}_{2\times 2} + \frac{\sinh\left(\sqrt{\det Q}\right)}{\sqrt{\det Q}}\, Q \qquad \text{with} \qquad Q \equiv \begin{pmatrix} r/2 & p^2 \\ -q^2 & -r/2 \end{pmatrix} , \tag{1.50}$$

experienced by the axiodilaton, i.e., $S \to e^{Q}\, S$, when circling once around the location of the 7-brane in its two-dimensional transverse space.

The set of $\mathrm{SL}(2,\mathbb{Z})$ orbits is then characterised by the value of $\det Q$. The orbit including the $(1,0,0)$ and $(0,0,1)$ 7-branes, which are identified with the D7-brane and the NS7-brane respectively, is that of

$$\det Q = p^2\, q^2 - \frac{r^2}{4} = 0\ . \tag{1.51}$$

This implies that a 7-brane lying on the orbit (1.51) has $r = \pm 2\, p\, q$. This 7-brane is referred to as a (p,q) 7-brane and induces the $\mathrm{SL}(2,\mathbb{Z})$ monodromy matrix

$$M_{(p,q)} = \begin{pmatrix} 1 \pm p\, q & p^2 \\ -q^2 & 1 \mp p\, q \end{pmatrix} \tag{1.52}$$

upon the axiodilaton S. The point to be highlighted is that if applying an $\mathrm{SL}(2,\mathbb{Z})$ self-duality transformation upon a D7-brane we will end up with a (p,q) 7-brane. This fact plays a central role in F-Theory [142].

F-Theory

The original observation questioned the perturbative limit $g_s \ll 1$ of type IIB supergravity when D7-branes are present. In the D7-brane supergravity solution [143], the axiodilaton field $S(z)$ depends on the single complex coordinate z parameterising the position of the D7-brane in its two-dimensional transverse space. The field equation of the axiodilaton admits the solution

$$S(z) \sim \frac{1}{2\pi i} \log(z - z_0) \ , \tag{1.53}$$

near the location $z = z_0$ of the D7-brane. When circling around it, the axiodilaton (1.53) experiences the monodromy $S \to S + 1$ obtained from (1.52) by fixing $p = 1$ and $r = q = 0$. Even though $g_s \ll 1$ close to the D7-brane, the theory will necessarily become strongly coupled in some region of the transverse space, rendering the weak coupling limit no longer valid.

Based on the ideas introduced in [144, 145], an auxiliary twelve-dimensional theory known as F-Theory has been developed as a geometrisation of the type IIB $\mathrm{SL}(2,\mathbb{Z})$ self-duality. This strong-weak coupling duality of the type IIB theory is identified with the $\mathrm{SL}(2,\mathbb{Z})$ modular group of an auxiliar $\mathbb{T}^2$ in F-Theory. It is in this sense that F-Theory appears as a non-perturbative completion of the type IIB theory.

Compactifications of F-Theory on a complex Calabi-Yau $(n+1)$-fold that admits elliptic fibration, i.e. it has the structure of a fiber bundle with a complex compact n-dimensional manifold B_n as the base and a real $\mathbb{T}^2$ as the fiber, correspond by definition to type IIB compactifications on B_n with general (p,q) 7-branes wrapping real codimension two cycles of B_n. The axiodilaton field is identified with the modular parameter in the fiber of the F-Theory construction, so it varies when moving along the basis B_n. In this picture, the (p,q) 7-branes correspond to real codimension two loci of B_n where the elliptic fibration degenerates into an A-D-E singularity. The local geometry around such a singularity is encoded into the $\mathrm{SL}(2,\mathbb{Z})$ monodromy suffered by the axiodilaton when cycling around it.

In the case of F-Theory compactified on K3 elliptic fibrations, it was shown by Sen [146] that there exists a weak coupling limit in which the axiodilaton turns out to be constant. In this limit, known as the Sen's limit, each of the singularities of the elliptic fibration comes out with a multiplicity and corresponds to a stack of (p,q) 7-branes. The $\mathrm{SL}(2,\mathbb{Z})$ monodromy induced upon the axiodilaton when circling around one of these singularities is given by

$$M = (M_A)^4\,(M_B M_C) = -\mathbb{I}_{2\times 2}\ , \tag{1.54}$$

so it results in the reflection $R = (-1)^{F_L}\,\Omega_p$ of (1.23) and gives rise to an orientifold theory [146,147]. The M_A, M_B and M_C matrices can be read from (1.52) taking $A = (1,0)$, $B = (1,-1)$ and $C = (1,1)$. The above stack A^4BC of 7-branes then manifests as four D7-branes (A^4) together with a single O7-plane (a bound state BC of 7-branes). Therefore, the Sen's construction of F-Theory compactified on K3 represents a special point in the moduli space that admits a description in terms of a perturbative type IIB orientifold theory. The above derivation was further generalised to F-Theory compactifications on complex Calabi-Yau $(n+1)$-folds in [148].

One of the most appealing features of F-Theory from a Particle Physics point of view is its suitability when it comes to obtain exceptional Lie groups in which to achieve the spinorial (matter) representation for an SO(10) GUT as well as the top-quark Yukawa coupling $\mathbf{10}\,\mathbf{10}\,\mathbf{5_H}$ for an SU(5) GUT, two properties that can not be obtained in perturbative type IIB orientifold models [74, 76, 142, 149, 150]. This is precisely because of the existence of a wider set of (p,q) 7-branes configurations in F-Theory which lead to the realization of these exceptional Lie groups as non-abelian gauge symmetries of the theory.

Chapter 2

$\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ IIB Orientifold with O3/O7-planes

In the chapter 1 we have briefly introduced the background material needed to go along this thesis, such as the basics of the type II supergravity theories, the duality relations and the non-perturbative O-planes and D-branes objects present in these theories.

This thesis explores the closed-string moduli dynamics in $\mathcal{N} = 1$ type IIB orientifold models including O3-planes and O7-planes, arising from compactifications on the $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ toroidal orbifold in the presence of generalised flux backgrounds. These generalised fluxes are needed to achieve invariance of the four-dimensional effective models under T-duality and S-duality transformations.

For this reason we devote the present chapter to introduce the geometrical aspects of the $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ toroidal orbifold as well as of the type IIB orientifold with O3/O7-planes built upon it. Next we derive the four-dimensional supergravity effective action describing the moduli dynamics in this orientifold when generalised flux backgrounds are included and also work out the set of constraints that the fluxes must satisfy in order to obtain consistent effective models. Finally we will restrict ourselves to the simplified case of isotropic flux configurations in order to obtain more tractable supergravity models which will be deeply analysed in the forthcoming chapters of the thesis.

2.1 The $\mathbb{Z}_2 \times \mathbb{Z}_2$ toroidal orbifold

The orbifold geometric action $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ automatically leads to the factorised six-torus of figure 2.1 whose basis of 1-forms is denoted η^a with $a = 1, \ldots, 6$. In the following we will use Greek indices α, β, γ for horizontal "$-$" x-like directions (η^1, η^3, η^5) and Latin indices i, j, k for vertical "$|$" y-like directions (η^2, η^4, η^6) in the 2-tori.

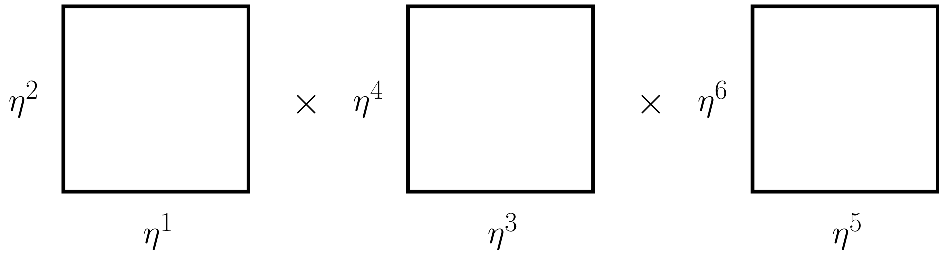

Figure 2.1: $\mathbb{T}^6 = \mathbb{T}^2_1 \times \mathbb{T}^2_2 \times \mathbb{T}^2_3$ torus factorisation and the basis of 1-forms.

The orbifold quotient group generators act on the tangent 1-forms η^a as

$$\begin{aligned} \theta_1 &: (\eta^1\,, \eta^2\,, \eta^3\,, \eta^4\,, \eta^5\,, \eta^6) \;\to\; (\eta^1\,, \eta^2\,, -\eta^3\,, -\eta^4\,, -\eta^5\,, -\eta^6)\,, \\ \theta_2 &: (\eta^1\,, \eta^2\,, \eta^3\,, \eta^4\,, \eta^5\,, \eta^6) \;\to\; (-\eta^1\,, -\eta^2\,, \eta^3\,, \eta^4\,, -\eta^5\,, -\eta^6)\,. \end{aligned} \tag{2.1}$$

and, clearly, there is another order-two element[1] $\theta_3 = \theta_1\theta_2$.

Under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold action, the invariant 3-forms are just those with one leg in each 2-torus[2]

$$\begin{array}{llllllll} \alpha_0 = \eta^{135} & , & \alpha_1 = \eta^{235} & , & \alpha_2 = \eta^{451} & , & \alpha_3 = \eta^{613} & , \\ \beta^0 = \eta^{246} & , & \beta^1 = \eta^{146} & , & \beta^2 = \eta^{362} & , & \beta^3 = \eta^{524} & , \end{array} \tag{2.2}$$

where, e.g. $\eta^{135} = \eta^1 \wedge \eta^3 \wedge \eta^5$. On the other hand, the invariant 2-forms and their dual 4-forms are

$$\begin{array}{llllll} \omega_1 = \eta^{12} & , & \omega_2 = \eta^{34} & , & \omega_3 = \eta^{56} & , \\ \tilde{\omega}^1 = \eta^{3456} & , & \tilde{\omega}^2 = \eta^{1256} & , & \tilde{\omega}^3 = \eta^{1234} & . \end{array} \tag{2.3}$$

whereas there are neither 1-forms nor 5-forms invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold group.

We choose the orientation and normalisation

$$\int_{\mathrm{M}_6} \eta^{123456} = \mathcal{V}_6\,, \tag{2.4}$$

where the positive constant $\mathcal{V}_6$ gives the volume of the internal space that we generically denote M_6. Notice that the cohomology basis satisfies

$$\int_{\mathrm{M}_6} \alpha_0 \wedge \beta^0 = -\mathcal{V}_6 \quad , \quad \int_{\mathrm{M}_6} \alpha_I \wedge \beta^J = \int_{\mathrm{M}_6} \omega_I \wedge \tilde{\omega}^J = \mathcal{V}_6\, \delta_I^J \quad , \quad I, J = 1, 2, 3\,, \tag{2.5}$$

so the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold symmetry restricts the period matrix τ^{ij} to be diagonal.

Up to normalisation, the Kähler form J and the holomorphic 3-form Ω that encode the geometry of the internal space can be written in a basis of invariant (untwisted) forms.

[1] The orbifold group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{\mathbb{1}, \theta_1, \theta_2, \theta_1\theta_2\}$.

[2] This also occurs in the compactification with an extra $\mathbb{Z}_3$ cyclic permutation of the three 2-tori that was studied in [7, 95]. In that case there are only two geometric moduli, namely the overall Kähler and complex structure parameters.

- The holomorphic 3-form is given by

$$\Omega = (\eta^1+\tau_1\,\eta^2)\wedge(\eta^3+\tau_2\,\eta^4)\wedge(\eta^5+\tau_3\,\eta^6) = \alpha_0+\tau_K\,\alpha_K+\beta^K\,\frac{\tau_1\tau_2\tau_3}{\tau_K}+\beta^0\,\tau_1\tau_2\tau_3\ , \quad (2.6)$$

with the $H^3(\mathrm{M}_6,\mathbb{Z})$ cohomology basis displayed in (2.2).

- The Kähler 2-form can be expanded in the basis of 2-forms of (2.3) as

$$J = A_1\,\omega_1 + A_2\,\omega_2 + A_3\,\omega_3\ , \quad (2.7)$$

where A_I denotes the area of the I-th 2-torus.

The geometric moduli $\{\tau_I, A_I\}_{I=1,2,3}$ descend from the internal components of the 10d metric and constitute the moduli space of possible metrics (far from the singularities)

$$ds^2 = \sum_{I=1}^{3}\frac{A_I}{\mathrm{Im}\,\tau_I}\left[|\tau_I|^2\left(\eta^{2I-1}\right)^2+\left(\eta^{2I}\right)^2-2\,(\mathrm{Re}\,\tau_I)\,\eta^{2I-1}\,\eta^{2I}\right]\ , \quad (2.8)$$

in the $\mathbb{T}^6/\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold. This moduli space consists of one Kähler ($A_I\in\mathbb{R}$) and one complex structure ($\tau_I\in\mathbb{C}$) parameter for each 2-torus T^2_I.

Compactification of type IIB superstring theory on the $\mathbb{T}^6/\mathbb{Z}_2\times\mathbb{Z}_2$ toroidal orbifold produces four-dimensional supergravities that preserve $1/4$ of the original supersymmetry in ten dimensions. The effective theories built in this way correspond to a $\mathcal{N}=2\ ,\ d=4$ extended supergravity which is again non-chiral.

Finally, in the case of the orbifold group $G=\mathbb{Z}_2\times\mathbb{Z}_2$, each non-trivial generator $\theta_{I=1,2,3}$ of G has $4\times4=16$ fixed points, so the total number of singularities is $3\times16=48$. The twisted sector of the spectrum then contains 48 additional moduli fields localised around the singularities of the $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold.

2.2 The $\mathbb{T}^6/\mathbb{Z}_2\times\mathbb{Z}_2$ IIB orientifold with O3/O7-planes

Let us now move to consider the type II toroidal orientifold $\mathbb{T}^6/\,(\mathbb{Z}_2\times\mathbb{Z}_2)\,\cup\,(-1)^{F_L}\,\Omega_p\,\sigma$, where the action of the orientifold involution σ upon the internal space coordinates translates into an additional $\mathbb{Z}_2$ reflection[3]

$$\sigma\ :\ (\eta^1\,,\eta^2\,,\eta^3\,,\eta^4\,,\eta^5\,,\eta^6)\ \rightarrow\ (-\eta^1\,,-\eta^2\,,-\eta^3\,,-\eta^4\,,-\eta^5\,,-\eta^6)\ , \quad (2.10)$$

[3]The action of σ upon the internal space coordinates for each of the three type II orientifolds presented in section 1.3.3 is given by

$$\begin{array}{rcllcl}
\sigma_{i)} & : & (\eta^1,\eta^2,\eta^3,\eta^4,\eta^5,\eta^6) & \rightarrow & (-\eta^1,-\eta^2,-\eta^3,-\eta^4,-\eta^5,-\eta^6) & ,\\
\sigma_{ii)} & : & (\eta^1,\eta^2,\eta^3,\eta^4,\eta^5,\eta^6) & \rightarrow & (\ \eta^1,-\eta^2,\ \eta^3,-\eta^4,\ \eta^5,-\eta^6) & ,\\
\sigma_{iii)} & : & (\eta^1,\eta^2,\eta^3,\eta^4,\eta^5,\eta^6) & \rightarrow & (\ \eta^1,\ \eta^2,\ \eta^3,\ \eta^4,\ \eta^5,\ \eta^6) & .
\end{array} \quad (2.9)$$

hence corresponding to a type IIB theory including O3-planes and O7-planes, as it was stated in section 1.3.3. Clearly, the invariant 3-forms in (2.2) are all odd under the orientifold involution σ, whereas the invariant 2-forms and their dual 4-forms in (2.3) are even.

The orientifold group action consists of the combined action of the orientifold reflection σ in (2.10) and the $\{\mathbb{1}, \theta_1, \theta_2, \theta_3\}$ generators of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold group. This is a $\mathbb{Z}_2^3$ symmetry. It creates then O-planes of the following types:

- $\sigma\,\mathbb{1}$: The action of this element upon the internal space coordinates is that of the orientifold involution σ shown in (2.10). It reflects the coordinates of all the 2-tori $\mathbb{T}_I$, creating 64 O3-planes located at its $4 \times 4 \times 4 = 64$ fixed points. One of these O3-planes is shown in figure 2.2.

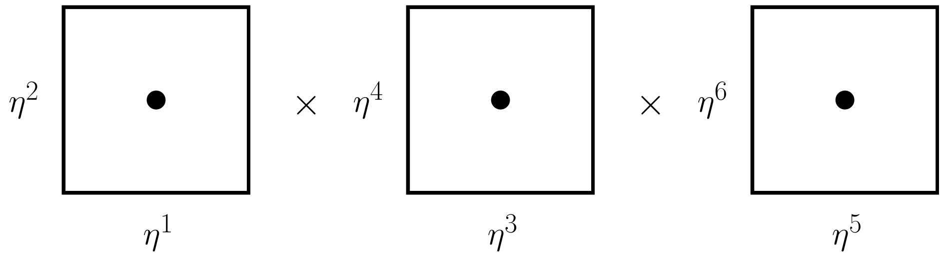

Figure 2.2: O3-plane located at the point $\left(\frac{1}{2}\,,\,\frac{1}{2}\right) \times \left(\frac{1}{2}\,,\,\frac{1}{2}\right) \times \left(\frac{1}{2}\,,\,\frac{1}{2}\right)$ in the internal space that is fixed under the action of the orientifold group element $\sigma\,\mathbb{1}$.

- $\sigma\,\theta_I$: Acting with this element on the internal space reflects the coordinates of the 2-tori $\mathbb{T}_I$. Therefore, it creates four O7-planes located at its 4 fixed 4-tori. One of these O7-planes is shown in figure 2.3.

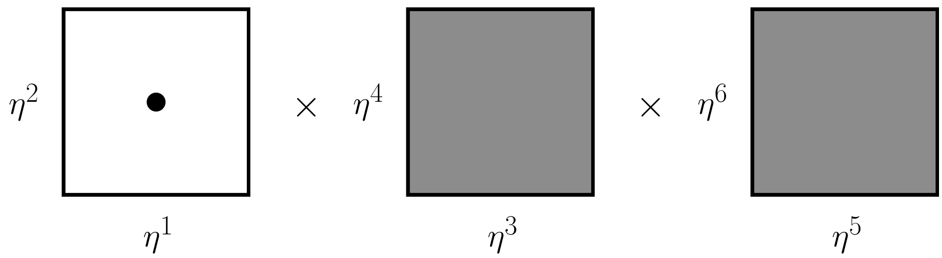

Figure 2.3: O7-plane wrapping the 4-torus $\left(\frac{1}{2}\,,\,\frac{1}{2}\right) \times \mathbb{T}_2 \times \mathbb{T}_3$ in the internal space that is fixed under the action of the orientifold group element $\sigma\,\theta_1$.

Modding out the theory with the worldsheet symmetry $(-1)^{F_L}\,\Omega_p$ makes the fields in the bosonic sector of the type IIB supergravity to acquire a parity under its action. From (1.16) and (1.18), the 2-forms B_2 and C_2 are found to be odd while the rest of the bosonic fields result to be even.

	EVEN	ODD
$(-1)^{F_L}\,\Omega_p$	$g\,,\,\varphi\,,\,C_0\,,\,C_4$	$B_2\,,\,C_2$

(2.11)

Notice that the action of this worldsheet symmetry is that of the type IIB SL$(2,\mathbb{R})$ self-duality reflection $R = (-1)^{F_L}\,\Omega_p$ in (1.23) setting the Sen's weak coupling limit of F-theory (see section 1.4.2) and allowing for a perturbative type IIB orientifold description of the theory.

The moduli space of type IIB orientifolds with O3/O7-planes [50] consists of three Kähler moduli T_I, three complex structure moduli U_I and the axiodilaton S modulus defined by

$$\begin{aligned} U_I &= \tau_I \,, \\ S &= C_0 + i\,e^{-\varphi} \,, \\ \mathcal{J} &= C_4 + \frac{i}{2}\,e^{-\varphi}\,J \wedge J + (C_2 - i\,S\,B_2) \wedge B_2 \;=\; \sum_{I=1}^{3} T_I\,\tilde{\omega}^I \,, \end{aligned} \tag{2.12}$$

where τ_I are the complex structure parameters entering the expansion (2.6) of the holomorphic 3-form Ω, and J is the fundamental Kähler 2-form of (2.7). Due to the absence of odd elements (under the action of σ) in the basis of 2-foms (2.3), we have that

$$B_2 = C_2 = 0 \quad \Longrightarrow \quad \mathcal{J} = C_4 + \frac{i}{2}\,e^{-\varphi}\,J \wedge J \,, \tag{2.13}$$

where $\mathcal{J}$ is known as the complexified Kähler 4-form. The Kähler moduli can then be explicitly written as

$$T_I = \frac{1}{\mathcal{V}_6}\int_{\mathrm{M}_6} C_4 \wedge \omega_I + i\,e^{-\varphi}\,A_J\,A_K \qquad \text{with} \qquad I \neq J \neq K \,. \tag{2.14}$$

This compactification preserves $\mathcal{N}=1$ supersymmetry in four dimensions. In this case the moduli space dynamics is encoded within a Kähler potential given by

$$K = -\,\log\left(-i\,(S-\bar{S})\right) - \sum_{I=1}^{3}\log\left(-i\,(U_I-\bar{U}_I)\right) - \sum_{I=1}^{3}\log\left(-i\,(T_I-\bar{T}_I)\right) \,, \tag{2.15}$$

which is valid to first order in the string and sigma model perturbative expansions. The Kähler potential in (2.15) codifies the metric in the moduli field space, namely, the Kähler metric

$$K_{ij} = \frac{\partial^2 K}{\partial\Phi_i\,\partial\bar{\Phi}_j} \,, \tag{2.16}$$

with $\Phi \equiv (\,U_1\,,\,U_2\,,\,U_3\,,\,S\,,\,T_1\,,\,T_2\,,\,T_3\,)$. This metric K_{ij} determines the non-canonically normalised kinetic term for the seven moduli fields of the compactification, whose dynamics is totally described in terms of the Lagrangian density

$$\mathcal{L}_{moduli} = K_{ij}\,\partial_\mu\Phi_i\,\partial^\mu\bar{\Phi}_j - V(\Phi) \,, \tag{2.17}$$

with $V(\Phi)=0$. The Lagrangian in (2.17) would perilously entail to have seven free complex scalars in Nature which couple to Gravity[4].

[4]For the sake of clarity we have omitted from the Lagrangian in (2.17) the global factor $\sqrt{-g_{(4)}}$ depending on the four-dimensional $g_{\mu\nu}$ metric.

As we mentioned in the introduction, one of the main problems in String Phenomenology is that of finding mechanisms to generate a non-vanishing potential

$$V(\Phi) \neq 0 \,, \tag{2.18}$$

involving the moduli fields. For instance, compactifications of type II superstrings in the presence of (generalised) background fluxes, which is the topic covered in this thesis, have been found to induce a non trivial $V(\Phi) \neq 0$ interaction between moduli fields. This interaction opens the possibility for the moduli to be stabilised at a minimum of the flux-induced scalar potential.

2.3 Generalised fluxes and effective action

Our starting point will be the four-dimensional effective action controlling the dynamics of the moduli fields when including a constant background for the internal components of the field strengths entering the type IIB supergravity of the section 1.1.2.

Under the orientifold involution action in (2.10) there are neither σ-even 1-forms nor σ-even 5-forms in the $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ toroidal orbifold, so a background for the σ-even R-R field strengths F_1 and $\widetilde{F}_5$ is forbidden in the type IIB orientifold with O3/O7-planes. The next step is then to switch on background fluxes $\bar{H}_3$ and $\bar{F}_3$ for the NS-NS and R-R 3-forms,

$$\begin{aligned} H_3 &= dB_2 + \bar{H}_3 \,, \\ \widetilde{F}_3 &= F_3 - H_3 \wedge C_0 + \bar{F}_3 \,. \end{aligned} \tag{2.19}$$

Since both H_3 and $\widetilde{F}_3$ are σ-odd under the orientifold involution action, the allowed constant background fluxes can be expanded as

$$\begin{aligned} \bar{H}_3 &= b_3\,\alpha_0 + b_2^{(I)}\,\alpha_I + b_1^{(I)}\,\beta^I + b_0\,\beta^0 \quad , \\ \bar{F}_3 &= a_3\,\alpha_0 + a_2^{(I)}\,\alpha_I + a_1^{(I)}\,\beta^I + a_0\,\beta^0 \quad , \end{aligned} \tag{2.20}$$

in terms of the basis of invariant 3-forms shown in (2.2). All flux coefficients are integers because the integrals of $\bar{H}_3$ and $\bar{F}_3$ over 3-cycles are quantised. To avoid subtleties with exotic orientifold planes we take all fluxes to be even [81, 151].

Working with only the two 3-form fluxes, i.e. the NS-NS $\bar{H}_3$ and the R-R $\bar{F}_3$, we have the standard form of the flux-induced $\mathcal{N}=1$ superpotential derived by Gukov, Vafa and Witten in ref. [79],

$$W_{\text{GVW}} = \int_{\text{M}_6} (\bar{F}_3 - S\,\bar{H}_3) \wedge \Omega \,. \tag{2.21}$$

Plugging the expansion (2.20) for the background fluxes and that of (2.6) with $U_I = \tau_I$ for the holomorphic 3-form into (2.21), the resulting superpotential after integrating over the internal space M_6 takes the form of

$$W = P_1(U_I) + P_2(U_I)\, S \ , \tag{2.22}$$

involving only the S and the U_I moduli fields of the compactification. The polynomials $P_1(U_I)$ and $P_2(U_I)$ appearing in (2.22) are cubic polynomials in the complex structure moduli given by

$$\begin{aligned} P_1(U_I) &= a_0 - \sum_{K=1}^{3} a_1^{(K)}\, U_K + \sum_{K=1}^{3} a_2^{(K)} \frac{U_1 U_2 U_3}{U_K} - a_3\, U_1 U_2 U_3 \quad , \\ P_2(U_I) &= -b_0 + \sum_{K=1}^{3} b_1^{(K)}\, U_K - \sum_{K=1}^{3} b_2^{(K)} \frac{U_1 U_2 U_3}{U_K} + b_3\, U_1 U_2 U_3 \quad . \end{aligned} \tag{2.23}$$

The $\mathcal{N} = 1$ supergravity theory defined by the Kähler potential in (2.15) and the superpotential in (2.22) has a no-scale structure [152] due to the lack of Kähler moduli T_I in the latter. The inclusion of non-perturbative effects depending on these moduli, such as gaugino condensation [153], was proposed as a mechanism to potentially stabilise them [12, 101, 109, 154].

However, considerable discussion has been done on the effect of applying T-duality transformations on type II orientifolds with background fluxes as well as its repercussion on the moduli stabilisation problem [6–8, 95, 155].

2.3.1 Fluxes and T-duality

As argued originally in [82, 156], applying one T-duality transformation T_a to the NS-NS fluxes $\bar{H}_{abc}$ can give rise to geometric fluxes ω^a_{bc} that correspond to structure constants of the isometry algebra of the internal space.

In the presence of $\bar{H}_{abc}$ and ω^a_{bc} fluxes, the Lie algebra $\mathfrak{g}$ of the supergravity group $\mathcal{G}$ is spanned by isometry Z_a and gauge X^a generators [157]

$$\begin{aligned} [Z_a\,,\, Z_b] &= \omega^c_{ab}\, Z_c + \bar{H}_{abc}\, X^c \ , \\ \left[Z_a\,,\, X^b\right] &= -\,\omega^b_{ac}\, X^c \ , \\ \left[X^a\,,\, X^b\right] &= 0 \ , \end{aligned} \tag{2.24}$$

with $a = 1, \ldots, 6$. These generators enter the expansion of the isometry $A^{(G)}_\mu = G_\mu{}^p\, Z_p$ and the gauge $A^{(B)}_\mu = B_{\mu p}\, X^p$ vector bosons (1-forms) in four dimensions descending from the reduction of the metric and the B-field with fluxes [90, 157–159]. Notice that the fluxes play the role of structure constants of the algebra in (2.24).

Performing further T-dualities T_b and T_c leads to *generalised* fluxes denoted Q^{ab}_c and R^{abc} in ref. [7],

$$\bar{H}_{abc} \xrightarrow{T_a} \omega^a_{bc} \xrightarrow{T_b} Q^{ab}_c \xrightarrow{T_c} R^{abc} \ . \tag{2.25}$$

The Q_c^{ab} are called non-geometric fluxes because the resulting metric after two T-dualities yields a background that is locally but not globally geometric [6, 95]. Compactifications with R^{abc} fluxes are not even locally geometric but these fluxes are necessary to maintain T-duality between type IIA and type IIB [7]. As a result, starting with a NS-NS background flux $\bar{H}_3$ turned on, successive T-dualities on the six circular dimensions take the space from one having a well defined metric everywhere to one with successively more and more "pathological" descriptions, ultimately losing any notion of a locally definable metric.

The above set $\{ \bar{H}_{abc} \, , \, \omega_{bc}^a \, , \, Q_c^{ab} \, , \, R^{abc} \}$ of generalised NS-NS fluxes would in principle determine an extension of the supergravity algebra in (2.24) to a new one being invariant under T-duality transformations. This T-duality invariant algebra was proposed in ref. [7] to have the commutation relations

$$\begin{array}{rcrcr} [Z_a, Z_b] & = & \bar{H}_{abc}\, X^c & + & \omega_{ab}^c\, Z_c \, , \\ \left[Z_a\, , X^b\right] & = & -\,\omega_{ac}^b\, X^c & + & Q_a^{bc}\, Z_c \, , \\ \left[X^a, X^b\right] & = & Q_c^{ab}\, X^c & + & R^{abc}\, Z_c \, , \end{array} \tag{2.26}$$

involving this time the entire set of generalised fluxes as structure constants. We will be back to this point at the end of the chapter 5 when exploring the $\mathcal{N} = 4$ origin of the supergravity theories including non-geometric fluxes [155, 160, 161].

In our $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ type IIB orientifold with O3/O7-planes, the geometric ω_{bc}^a and the R^{abc} fluxes must be σ-even under the orientifold involution in (2.10) and are thus totally absent. On the other hand, the non-geometric Q-flux must be σ-odd and is fully permitted. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry only allows 24 components of the flux tensor Q_c^{ab}, namely those with one leg on each 2-torus shown in figure 2.1. This set of non-geometric fluxes is displayed in table 2.1. All the components of the tensor Q-flux are integers that we take to be even.

Type	Components	Fluxes
$Q_{-}^{--} \equiv Q_{\alpha}^{\beta\gamma}$	$Q_1^{35}\, , \, Q_3^{51}\, , \, Q_5^{13}$	$\tilde{c}_1^{(1)}\, , \, \tilde{c}_1^{(2)}\, , \, \tilde{c}_1^{(3)}$
$Q_{\vert}^{\vert -} \equiv Q_{k}^{i\beta}$	$Q_4^{61}\, , \, Q_6^{23}\, , \, Q_2^{45}$	$\hat{c}_1^{(1)}\, , \, \hat{c}_1^{(2)}\, , \, \hat{c}_1^{(3)}$
$Q_{\vert}^{-\vert} \equiv Q_{k}^{\alpha j}$	$Q_6^{14}\, , \, Q_2^{36}\, , \, Q_4^{52}$	$\check{c}_1^{(1)}\, , \, \check{c}_1^{(2)}\, , \, \check{c}_1^{(3)}$
$Q_{\vert}^{--} \equiv Q_{k}^{\alpha\beta}$	$Q_2^{35}\, , \, Q_4^{51}\, , \, Q_6^{13}$	$c_0^{(1)}\, , \, c_0^{(2)}\, , \, c_0^{(3)}$
$Q_{-}^{\vert\vert} \equiv Q_{\gamma}^{ij}$	$Q_1^{46}\, , \, Q_3^{62}\, , \, Q_5^{24}$	$c_3^{(1)}\, , \, c_3^{(2)}\, , \, c_3^{(3)}$
$Q_{-}^{\vert -} \equiv Q_{\gamma}^{i\beta}$	$Q_5^{23}\, , \, Q_1^{45}\, , \, Q_3^{61}$	$\check{c}_2^{(1)}\, , \, \check{c}_2^{(2)}\, , \, \check{c}_2^{(3)}$
$Q_{-}^{-\vert} \equiv Q_{\beta}^{\gamma i}$	$Q_3^{52}\, , \, Q_5^{14}\, , \, Q_1^{36}$	$\hat{c}_2^{(1)}\, , \, \hat{c}_2^{(2)}\, , \, \hat{c}_2^{(3)}$
$Q_{\vert}^{\vert\vert} \equiv Q_{k}^{ij}$	$Q_2^{46}\, , \, Q_4^{62}\, , \, Q_6^{24}$	$\tilde{c}_2^{(1)}\, , \, \tilde{c}_2^{(2)}\, , \, \tilde{c}_2^{(3)}$

Table 2.1: Non-geometric Q-flux components in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold.

The effect of this new non-geometric Q-flux can be incorporated into the superpotential by contracting it with the complexified Kähler 4-form $\mathcal{J}$ in (2.13). Specifically,

$$(Q\cdot\mathcal{J})_{abc}=\frac{1}{2}Q^{de}_{[a}(\mathcal{J})_{bc]de}\quad\Rightarrow\quad\int_{\mathrm{M}_6}(Q\cdot\mathcal{J})\wedge\Omega\subset W\ ,\tag{2.27}$$

so the T-duality invariant four-dimensional effective theory involving the R-R flux $\bar{F}_3$ and the NS-NS fluxes $\bar{H}_3$ and Q is described by the Kähler potential in (2.15) together with the superpotential

$$W_{\text{T-dual}}=\int_{\mathrm{M}_6}(\bar{F}_3-S\,\bar{H}_3+Q\cdot\mathcal{J})\wedge\Omega\ .\tag{2.28}$$

The contraction $Q\cdot\mathcal{J}$ can be expanded in the basis of 3-forms displayed in (2.2). This expansion takes the form of

$$Q\cdot\mathcal{J}=T_K\left(c_3^{(K)}\,\alpha_0-\mathcal{C}_2^{(IK)}\,\alpha_I-\mathcal{C}_1^{(IK)}\,\beta^I+c_0^{(K)}\,\beta^0\right)\ ,\tag{2.29}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are the non-geometric Q-flux matrices

$$\mathcal{C}_1=\begin{pmatrix}-\tilde{c}_1^{(1)} & \check{c}_1^{(3)} & \hat{c}_1^{(2)}\\ \hat{c}_1^{(3)} & -\tilde{c}_1^{(2)} & \check{c}_1^{(1)}\\ \check{c}_1^{(2)} & \hat{c}_1^{(1)} & -\tilde{c}_1^{(3)}\end{pmatrix}\quad,\quad\mathcal{C}_2=\begin{pmatrix}-\tilde{c}_2^{(1)} & \check{c}_2^{(3)} & \hat{c}_2^{(2)}\\ \hat{c}_2^{(3)} & -\tilde{c}_2^{(2)} & \check{c}_2^{(1)}\\ \check{c}_2^{(2)} & \hat{c}_2^{(1)} & -\tilde{c}_2^{(3)}\end{pmatrix}\ .\tag{2.30}$$

Using the expansion for the NS-NS $\bar{H}_3$ and the R-R $\bar{F}_3$ fluxes in (2.20) and substituting those of (2.29) and (2.6) into (2.28), the T-duality invariant superpotential reads

$$W_{\text{T-dual}}=W_{\text{GVW}}+\sum_{K=1}^{3}P_3^{(K)}(U_I)\,T_K\ ,\tag{2.31}$$

where the piece W_{GVW} was previously shown in (2.22). Compared to it, the invariance under T-duality transformations introduces a new cubic polynomial $P_3(U_I)$ depending again on the complex structure moduli

$$P_3^{(K)}(U)\ =\ c_0^{(K)}+\sum_{L=1}^{3}\mathcal{C}_1^{(LK)}\,U_L-\sum_{L=1}^{3}\mathcal{C}_2^{(LK)}\frac{U_1U_2U_3}{U_L}-c_3^{(K)}U_1U_2U_3\ .\tag{2.32}$$

The main feature of the flux superpotential in (2.31) is that it now depends on all the untwisted closed string moduli. As we will see in chapters 3 and 4, this T-duality invariant supergravity theory deserves special attention since it induces a scalar potential $V(\Phi)\neq 0$ for the moduli fields that possesses supersymmetric AdS_4 as well as non-supersymmetric Minkowski and de Sitter vacua without flat (massless) directions in field space.

2.3.2 Fluxes and S-duality

Taking the T-duality invariant type IIB supergravity theory of the previous section as the starting point, we now wish to impose an additional symmetry, that of S-duality. This SL$(2,\mathbb{Z})$ self-duality of the type IIB theory was shown in section 1.2.2 to have a non linear action on the axiodilaton field S given by

$$S \;\rightarrow\; \frac{a\,S+b}{c\,S+d} \qquad \text{with} \qquad a\,d-b\,c=1\ . \tag{2.33}$$

For the effective theory to be invariant under this transformation, the superpotential must transform in a particular way,

$$W(S) \rightarrow W\left(\frac{a\,S+b}{c\,S+d}\right) = \frac{1}{c\,S+d}W(S)\,. \tag{2.34}$$

This implies that the fluxes must themselves transform in such a way as to satisfy this and they must transform in multiplets. Therefore, having a non-trivial $\bar{H}_3$ or $\bar{F}_3$ flux means allowing for both 3-form fluxes being non-zero following such a transformation in S,

$$\bar{F}_3 - S\,\bar{H}_3 \rightarrow \bar{F}_3 - \left(\frac{a\,S+b}{c\,S+d}\right)\bar{H}_3 = \frac{1}{c\,S+d}\Big((d\,\bar{F}_3 - b\,\bar{H}_3) - S(a\,\bar{H}_3 - c\,\bar{F}_3)\Big)\,. \tag{2.35}$$

In terms of $\bar{H}_3$ and $\bar{F}_3$, we have that the S-duality action on the 3-form fluxes is given by

$$\begin{pmatrix}\bar{H}_3\\ \bar{F}_3\end{pmatrix} \rightarrow \begin{pmatrix} d & c\\ b & a\end{pmatrix}\begin{pmatrix}\bar{H}_3\\ \bar{F}_3\end{pmatrix}\ , \tag{2.36}$$

which, of course, coincides with that of (1.21) for the gauge potentials. Similarly, the non-geometric Q-flux needs to be partnered with another flux of the same tensor type and we are forced to turn on another non-geometric flux, P, which is multiplied by the axiodilaton in order to give the same doublet mixing [8],

$$\begin{pmatrix}P\\ Q\end{pmatrix} \rightarrow \begin{pmatrix} d & c\\ b & a\end{pmatrix}\begin{pmatrix}P\\ Q\end{pmatrix}\ . \tag{2.37}$$

With the inclusion of this additional non-geometric P-flux, we are lead to a both T-duality and S-duality invariant four-dimensional effective theory. It involves the $\bar{H}_3$, $\bar{F}_3$, Q and P fluxes and is described by the Kähler potential in (2.15) and the superpotential

$$W_{\text{T/S-dual}} = \int_{\text{M}_6}\Big(\bar{F}_3 - S\,\bar{H}_3 + (Q - S\,P)\cdot\mathcal{J}\Big)\wedge\Omega\,. \tag{2.38}$$

As it happened with the $Q\cdot\mathcal{J}$ contraction in (2.29), the $P\cdot\mathcal{J}$ one can also be expanded in the basis (2.2) of 3-forms as,

$$P\cdot\mathcal{J} = T_K\left(d_3^{(K)}\,\alpha_0 - \mathcal{D}_2^{(IK)}\,\alpha_I - \mathcal{D}_1^{(IK)}\,\beta^I + d_0^{(K)}\,\beta^0\right)\ , \tag{2.39}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are the new non-geometric P-flux matrices,

$$\mathcal{D}_1 = \begin{pmatrix} -\tilde{d}_1^{(1)} & \check{d}_1^{(3)} & \hat{d}_1^{(2)} \\ \hat{d}_1^{(3)} & -\tilde{d}_1^{(2)} & \check{d}_1^{(1)} \\ \check{d}_1^{(2)} & \hat{d}_1^{(1)} & -\tilde{d}_1^{(3)} \end{pmatrix} \quad , \quad \mathcal{D}_2 = \begin{pmatrix} -\tilde{d}_2^{(1)} & \check{d}_2^{(3)} & \hat{d}_2^{(2)} \\ \hat{d}_2^{(3)} & -\tilde{d}_2^{(2)} & \check{d}_2^{(1)} \\ \check{d}_2^{(2)} & \hat{d}_2^{(1)} & -\tilde{d}_2^{(3)} \end{pmatrix} . \tag{2.40}$$

Analogously to its Q-flux counterpart, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold symmetry only allows for 24 independent components of the non-geometric flux tensor P_c^{ab}. These components can also be read off from table 2.1 by the exchanges $Q \leftrightarrow P$ and $c \leftrightarrow d$. After substituting the expansions (2.20), (2.29), (2.39) and (2.6) and performing the integral over the internal space, the T- and S-duality invariant superpotential of (2.38) takes the form

$$W_{\text{T/S-dual}} = W_{\text{T-dual}} + S \sum_{K=1}^{3} P_4^{(K)}(U)\, T_K , \tag{2.41}$$

with the $W_{\text{T-dual}}$ piece being that of (2.31). To restore invariance under S-duality transformations, the superpotential in (2.41) incorporates an additional cubic polynomial $P_4(U_I)$ depending again on the complex structure moduli

$$P_4^{(K)}(U_I) \;\; = \;\; -d_0^{(K)} - \sum_{L=1}^{3} \mathcal{D}_1^{(LK)} U_L + \sum_{L=1}^{3} \mathcal{D}_2^{(LK)} \frac{U_1 U_2 U_3}{U_L} + d_3^{(K)} U_1 U_2 U_3 \, . \tag{2.42}$$

Therefore, the T- and S-duality invariant $\mathcal{N}=1$ effective supergravity described in this section involves $(8+24)+(8+24)=64$ flux parameters coming from the $\left(\bar{F}_3\,,\, Q\right)$ and the $\left(\bar{H}_3\,,\, P\right)$ fluxes that are generically allowed.

2.4 Flux algebra and Jacobi identities

In the absence of fluxes, compactifications of the type II ten-dimensional supergravities on $\mathbb{T}^6$ orientifolds yield a $\mathcal{N}=4\,,\ d=4$ supergravity. Without considering additional vector multiplets coming from D-branes, its deformations produce $\mathcal{N}=4$ gauged supergravities [160] specified by two constant embedding tensors, $\xi_{\alpha A}$ and $f_{\alpha ABC}$, under the global symmetry

$$\text{SL}(2,\mathbb{Z}) \times \text{SO}(6,6,\mathbb{Z}) \; , \tag{2.43}$$

where $\alpha = \pm$ and $A, B, C = 1, \ldots, 12$. These embedding tensors are interpreted as flux parameters, so the fluxes become the gaugings of the $\mathcal{N}=4$ gauged supergravity [9].

In the $\mathbb{Z}_2 \times \mathbb{Z}_2$ toroidal orbifold the global symmetry (2.43) is broken to the $\text{SL}(2,\mathbb{Z})^7$ group and the tensor $\xi_{\alpha A}$ is projected out. Compactifying the type IIB supergravity on this orbifold produces a $\mathcal{N}=2$ supergravity further broken to $\mathcal{N}=1$ in the type IIB orientifold theory with O3/O7-planes.

2.4.1 Spinorial embedding of the generalised fluxes

As it was shown in ref. [8], the $\bar{F}_3$, $\bar{H}_3$, Q and P sixty four flux components can be embedded into a spinorial **128** representation of SO(7,7,$\mathbb{Z}$) which in turn decomposes as two Weyl spinors transforming respectively in the **64** (left) and the **64′** (right) representations. Accordingly to the SU(7) tensorial structure of these Weyl representations,

$$\begin{aligned} \mathbf{64} &= \mathbf{1} + \mathbf{7} + \mathbf{21} + \mathbf{35} \quad , \\ \mathbf{64'} &= \mathbf{1'} + \mathbf{7'} + \mathbf{21'} + \mathbf{35'} \quad , \end{aligned} \tag{2.44}$$

the embedding of the fluxes is displayed in table 2.2.

As introduced in ref. [155], and subsequently deeper understood in ref. [161], the $(\bar{F}_3\,,\,Q)$ pair of fluxes belongs to the SL(2,$\mathbb{Z}$)-electric part of the flux algebra whereas the $(\bar{H}_3\,,\,P)$ pair belongs to the SL(2,$\mathbb{Z}$)-magnetic part. The same occurs with the fluxes* (red print) in table 2.2 which correspond to heterotic fluxes [8,162,163], complete the **128** spinorial representation of SO(7,7,$\mathbb{Z}$) and restore an SL(2,$\mathbb{Z}$)^7 modular invariance at the level of the effective theory[5]. This fact severely modifies the structure of the generalised flux algebra in (2.26) proposed in ref. [7]. Nevertheless, we will postpone further discussion of this matter to the end of chapter 5.

Besides to the ones imposed by the symmetries of the compactification, e.g. the $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold symmetry, the set of generalised fluxes is subject to additional complex algebraic relations. These constraints appear either as the Jacobi identities of the flux algebra that the generalised fluxes give rise to or as tadpole cancellation conditions required by consistency of the effective theory when sources such as O-planes and D-branes are included.

2.4.2 T-duality invariant supergravity and $(\bar{H}_3, Q)$ flux algebra

Let us explore the constraints on the fluxes arising as Jacobi identities of the flux algebra in the case of the IIB supergravity theory described in the previous section and simplified by setting $P = 0$. Not including the effect of this non-geometric P-flux, the effective supergravity results invariant only under T-duality transformations.

The geometrical properties of the T-duality invariant type II compactifications to four dimensions have been explored in [9, 23, 43, 44, 90, 164–170]. These compactifications are naturally described as Scherk-Schwarz reductions [158] on a doubled torus, $\mathbb{T}^{12}$, twisted

[5] An SL(2,$\mathbb{Z}$)^7 global symmetry would translate into an effective theory invariant under SL(2,$\mathbb{Z}$) modular transformations on each of its seven untwisted moduli fields. Then, the SL(2,$\mathbb{Z}$)^7-invariant superpotential has to contain all the multi lineal couplings between the seven moduli up to a degree seven term given by $b_0^*\, S\, U_1 U_2 U_3 T_1 T_2 T_3$ [8]. Notice that the effective theory specified by the superpotential in (2.41) is not invariant under modular transformations on the three Kähler moduli T_I, so the fluxes* in table 2.2 are not present in our type IIB orientifold theory.

REP				FLUXES			
1				a_0			
7$'$			$a_1^{(K)}$	b_0	$c_0^{(K)}$		
21		$a_2^{(K)}$	$b_1^{(K)}$	$\mathcal{C}_1^{(KL)}$	$d_0^{(K)}$	$c_3^{*(K)}$	
35$'$	a_3	$b_2^{(K)}$	$\mathcal{C}_2^{(KL)}$	$\mathcal{D}_1^{(KL)}$	$\mathcal{C}_2^{*(KL)}$	$d_3^{*(K)}$	a_3^*
35	b_3	$c_3^{(K)}$	$\mathcal{D}_2^{(KL)}$	$\mathcal{C}_1^{*(KL)}$	$\mathcal{D}_2^{*(KL)}$	$a_2^{*(K)}$	b_3^*
21$'$		$d_3^{(K)}$	$c_0^{*(K)}$	$\mathcal{D}_1^{*(KL)}$	$a_1^{*(K)}$	$b_2^{*(K)}$	
7			$d_0^{*(K)}$	a_0^*	$b_1^{*(K)}$		
1$'$				b_0^*			

Table 2.2: Embedding of the flux parameters into the spinorial **128** representation of $\text{SO}(7,7,\mathbb{Z})$. The (black print) ordinary fluxes are those entering the superpotential in (2.41) while the (red print) fluxes* are not present in the type IIB orientifold theory under consideration. It is worth noticing that the $\text{SL}(2,\mathbb{Z})$-electric $\bar{F}_3 \equiv (a_0\,,\, a_1^{(K)}\,,\, a_2^{(K)}\,,\, a_3)$ and $Q \equiv (c_0^{(K)}\,,\, \mathcal{C}_1^{(KL)}\,,\, \mathcal{C}_2^{(KL)}\,,\, c_3^{(K)})$ fluxes as well as the $\text{SL}(2,\mathbb{Z})$-magnetic $\bar{H}_3 \equiv (b_0\,,\, b_1^{(K)}\,,\, b_2^{(K)}\,,\, b_3)$ and $P \equiv (d_0^{(K)}\,,\, \mathcal{D}_1^{(KL)}\,,\, \mathcal{D}_2^{(KL)}\,,\, d_3^{(K)})$ fluxes fit into components of the two **64** and **64**$'$ Weyl spinors.

under the supergravity group $\mathcal{G}$. A stringy feature of these reductions is that the coordinates in $\mathbb{T}^{12}$ account for the ordinary coordinates and their duals, so both momentum and winding modes of the string are treated on equal footing, see section 1.3.1. Furthermore, the fluctuations of the internal components of the metric and the B field are jointly described [157] in terms of the $\text{O}(6,6,\mathbb{R})$ doubled space metric in (1.40). In this framework, a T-duality transformation can be interpreted as an $\text{SO}(6,6,\mathbb{Z})$ rotation on the background [164].

Recalling the introductory section 1.3.3, the three type II orientifolds are found to be related by a chain of (three) T-duality transformations,

$$\text{type IIB with O3/O7} \;\leftrightarrow\; \text{type IIA with O6} \;\leftrightarrow\; \text{type IIB with O9/O5}\,. \tag{2.45}$$

Each of these (T-) duality frames projects out half of the flux entries [8]. The type IIB orientifold allowing for O3/O7-planes projects the geometric ω and the non-geometric R fluxes out of the effective theory. This duality frame is particularly suitable when classifying the supergravity algebras, since it does not forbid certain components in all the fluxes, as it happens with the type IIA orientifold allowing for O6-planes, but certain fluxes as a whole[6]. By performing three T-duality transformations, the type IIB flux models with

[6] This is also the case for the type IIB orientifold allowing for O9/O5-planes, which forbids the $\bar{H}_3$ and Q fluxes. The generalised fluxes mapping between the O3/O7 and O9/O5 type IIB orientifold theories reads $Q_c^{ab} \leftrightarrow \omega_{ab}^c$ together with $\bar{H}_{abc} \leftrightarrow R^{abc}$.

O3/O7-planes are mapped to type IIA compactifications with O6-planes in the presence of the entire set of generalised $\bar{H}_3$, ω , Q and R fluxes. We will make use of this mapping later on in chapter 4.

In the IIB with O3/O7-planes duality frame, the conjectured supergravity algebra in (2.26) simplifies to

$$\begin{array}{rcl} [Z_a, Z_b] & = & \bar{H}_{abc}\, X^c \quad , \\ \left[Z_a\, , X^b\right] & = & Q_a^{bc}\, Z_c \quad , \\ \left[X^a, X^b\right] & = & Q_c^{ab}\, X^c \quad , \end{array} \tag{2.46}$$

and the effective supergravity theory admits a description in terms of a reduction on a T-fold space [43,44,164,165]. From now on, we will refer to the IIB orientifold allowing for O3/O7-planes as the *T-fold description* of the effective supergravity theory. One observes that the algebra in (2.46) comes up with a gauge-isometry $\mathbb{Z}_2$-graded structure involving the subspaces spanned by the gauge X^a and the isometry Z_a generators as the grading subspaces.

In the T-fold description, the supergravity group $\mathcal{G}$ has a six-dimensional subgroup $\mathcal{G}_{gauge}$ whose algebra $\mathfrak{g}_{gauge}$ involves the vector fields X^a coming from the reduction of the B-field. The $\mathfrak{g}_{gauge}$ is completely determined by the non-geometric Q-flux forced to satisfy the quadratic XXX-type Jacobi identity $Q^2=0$ from (2.46),

$$Q_x^{[ab}\, Q_d^{c]x} = 0 \; . \tag{2.47}$$

From the general structure of (2.46), the remaining Z_a vector fields coming from the reduction of the metric are the generators of the reductive and symmetric coset space $\mathcal{G}/\mathcal{G}_{gauge}$ [171]. Provided a Q-flux, the mixed gauge-isometry brackets in (2.46) are given by the co-adjoint action Q^* of Q and the $\mathcal{G}/\mathcal{G}_{gauge}$ coset space is determined by the $\bar{H}_3$ flux restricted by the $\bar{H}_3 Q = 0$ constraint

$$\bar{H}_{x[bc}\, Q_{d]}^{ax} = 0 \; , \tag{2.48}$$

coming from the quadratic XZZ-type Jacobi identity from (2.46). Any point in the coset space remains fixed under the action of the isotropy subgroup $\mathcal{G}_{gauge}$ of $\mathcal{G}$ [172], so an effective supergravity theory is defined by specifying both the supergravity algebra $\mathfrak{g}$ as well as the subalgebra $\mathfrak{g}_{gauge}$ associated to the isotropy subgroup of the coset space $\mathcal{G}/\mathcal{G}_{gauge}$.

The constraints in (2.47) and (2.48) on the Q and $\bar{H}_3$ fluxes can also be interpreted in terms of a nilpotency condition $\mathcal{D}^2=0$ on the operator $\mathcal{D}=\bar{H}_3\wedge + Q\,\cdot$ introduced in ref. [95].

2.4.3 S-duality on top of T-duality

Under the ansatz of systematically applying S-duality transformations upon the flux constraints of the T-duality invariant supergravity theory, we shall focus on the S-dualisation of the Jacobi identities in (2.47) and (2.48).

- Applying the self-duality $\mathrm{SL}(2,\mathbb{Z})$ transformation of (2.37) on the non-geometric Q-flux, the $Q^2 = 0$ Jacobi identity in (2.47) gives rise to an $\mathrm{SL}(2,\mathbb{Z})$-triplet of constraints involving the Q and P fluxes,

$$Q_x^{[ab} Q_d^{c]x} = 0 \quad , \quad P_x^{[ab} P_d^{c]x} = 0 \quad , \quad Q_x^{[ab} P_d^{c]x} + P_x^{[ab} Q_d^{c]x} = 0 \ , \tag{2.49}$$

which, as before, we will denote as $Q^2 = 0$, $P^2 = 0$ and $QP + PQ = 0$. The first condition results in that of (2.47) and the second one reduces to (2.47) under $Q \to P$. The third element of the triplet gives a mixing between the Q and P fluxes.

- We now turn to the second Jacobi constraint $\bar{H}_3 Q = 0$ in (2.48) and consider what effect S-duality has on it. The result is that it is extended to mix the four types of fluxes in the IIB orientifold theory with O3/O7-planes,

$$\bar{H}_{x[bc}\, Q_{d]}^{ax} - \bar{F}_{x[bc}\, P_{d]}^{ax} = 0 \ . \tag{2.50}$$

This constraint is an $\mathrm{SL}(2,\mathbb{Z})$-singlet and we will refer to it as $\bar{H}_3 Q - \bar{F}_3 P = 0$.

At this point we want to emphasise that the Jacobi constraints in (2.49) have been obtained by applying an S-duality transformation to the Jacobi constraint $Q^2 = 0$ of the T-duality invariant effective supergravity theory. Starting however with the $\mathrm{SL}(2,\mathbb{Z})^7$-duality invariant flux algebra involving the $\mathrm{SL}(2,\mathbb{Z})$-electric $(\bar{F}_3\, ,\, Q)$ and the $\mathrm{SL}(2,\mathbb{Z})$-magnetic $(\bar{H}_3\, ,\, P)$ background fluxes as structure constants, these conditions result slightly modified to $Q^2 = P^2 = 0$ together with $QP = PQ = 0$ [155,161]. Nevertheless, these modified constraints can be understood as a particular case of (2.49). We will be back to this point at the end of the chapter 5 and will explain where does this mismatch stem from [161].

2.5 Tadpole cancellation conditions

In the absence of background fluxes, the R-R field strengths are constrained by the Bianchi identities in (1.44). In the case of the type IIB supergravity of section 1.1.2, these are

$$dF_1 = 0 \quad , \quad d\widetilde{F}_3 = \star j_6 \quad , \quad d\widetilde{F}_5 = \star j_4 \ , \tag{2.51}$$

where the magnetic currents j_6 and j_4 account for D5-branes and D3-branes coupling to the R-R gauge potentials C_2 and C_4, respectively.

When including the background fluxes needed to restore T-duality invariance in the type IIB orientifold theory with O3/O7-planes, these Bianchi identities are promoted to a new one of the type $\mathcal{D}\bar{F} = \mathcal{S}$. Now $\mathcal{S}$ is a generalised form due to sources that are assumed smeared instead of localised and $\mathcal{D}$ is again the covariant derivative operator

$\mathcal{D} = \bar{H}_3 \wedge + Q\cdot$ of the previous section. Since there is only a non-trivial R-R flux background $\bar{F}_3$, one expects combinations of fluxes of the types $\bar{H}_3 \wedge \bar{F}_3$ (6-form) and $Q \cdot \bar{F}_3$ (2-form). These can be understood as tadpole cancellation conditions on the R-R 4-form C_4 and 8-form C_8 that respectively couple to O3/D3 and O7/D7 sources.

In the type IIB orientifold that we are considering there is a flux-induced C_4 tadpole due to the Chern-Simons term in (1.12)

$$\int_{\mathrm{M}_4\times\mathrm{M}_6} C_4 \wedge \bar{H}_3 \wedge \bar{F}_3 \ . \tag{2.52}$$

There are further C_4 tadpoles due to O3-planes and to D3-branes that can also be added. The total orientifold charge is -32, equally distributed among 64 O3-planes located at the fixed points of the orientifold involution in (2.10). Each D3-brane has charge $+1$ and if they are located in the bulk, as opposed to fixed points of $\mathbb{Z}_2^3$, images must be included. Adding the sources to the flux tadpole of (2.52) leads to the cancellation condition

$$a_0\, b_3 - a_1^{(K)}\, b_2^{(K)} + a_2^{(K)}\, b_1^{(K)} - a_3\, b_0 = N_3 \ , \tag{2.53}$$

where $N_3 = 32 - N_{\mathrm{D}3}$ and $N_{\mathrm{D}3}$ is the total number of D3-branes.

The non-geometric Q and the R-R $\bar{F}_3$ fluxes combine to produce a flux-induced tadpole for the R-R gauge potential C_8. This tadpole depends on the 2-form contraction $Q \cdot \bar{F}_3$ and is given by the coupling term

$$\int_{\mathrm{M}_4\times\mathrm{M}_6} C_8 \wedge (Q \cdot \bar{F}_3) \ . \tag{2.54}$$

Expanding $(Q \cdot \bar{F}_3)$ in the basis of 2-forms of (2.3) yields coefficients

$$(Q \cdot \bar{F}_3)_I = a_0\, c_3^{(I)} + a_1^{(K)}\, \mathcal{C}_2^{(KI)} - a_2^{(K)}\, \mathcal{C}_1^{(KI)} - a_3\, c_0^{(I)} \quad , \quad I = 1,2,3 \ . \tag{2.55}$$

This means that there are induced tadpoles for C_8 components of type $C_8 \sim d\mathrm{vol}_4 \wedge \widetilde{\omega}^I$, where $d\mathrm{vol}_4$ is the spacetime volume 4-form and $\widetilde{\omega}^I$ is the 4-form dual to ω_I. On the other hand, there are also C_8 tadpoles due to O7_I-planes that have a total charge $+32$ for each I. As discussed before, due to the orbifold group $\mathbb{Z}_2 \times \mathbb{Z}_2$, there are O$7_I$-planes located at the 4 fixed tori of $\sigma\theta_I$, where θ_I are the three order-two elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. At the end we find the three tadpole cancellation conditions

$$a_0\, c_3^{(I)} + a_1^{(K)}\, \mathcal{C}_2^{(KI)} - a_2^{(K)}\, \mathcal{C}_1^{(KI)} - a_3\, c_0^{(I)} = N_{7_I} \quad , \quad I = 1,2,3 \ , \tag{2.56}$$

where $N_{7_I} = -32 + N_{\mathrm{D}7_I}$ and $N_{\mathrm{D}7_I}$ is the number of D7_I-branes that are generically allowed.

2.5.1 S-duality and tadpole cancellation conditions

The tadpole constraints once S-duality invariance is included were derived in [8]. There are two kinds of flux-induced tadpoles accordingly to their behaviour under self-duality $\mathrm{SL}(2,\mathbb{Z})$ transformations on the fluxes.

- There is a flux-induced tadpole for each of the components of the $\mathrm{SL}(2,\mathbb{Z})$-triplet $(C_8, C_8', \tilde{C}_8)$ of 8-forms introduced in section 1.4.2. These three tadpoles,

$$\int_{\mathrm{M}_4\times\mathrm{M}_6} C_8\wedge(Q\cdot\bar{F}_3) \quad , \quad \int_{\mathrm{M}_4\times\mathrm{M}_6} C_8'\wedge(Q\cdot\bar{H}_3+P\cdot\bar{F}_3) \quad , \quad \int_{\mathrm{M}_4\times\mathrm{M}_6} \tilde{C}_8\wedge(P\cdot\bar{H}_3)\ , \tag{2.57}$$

  transform as an $\mathrm{SL}(2,\mathbb{Z})$-triplet and arise from the original $(Q\cdot\bar{F}_3)$ one in (2.54). Using the 2-form expansions

$$\begin{aligned}
(Q\cdot\bar{H}_3)_I &= b_0\, c_3^{(I)} + b_1^{(K)}\,\mathcal{C}_2^{(KI)} - b_2^{(K)}\,\mathcal{C}_1^{(KI)} - b_3\, c_0^{(I)} \ ,\\
(P\cdot\bar{F}_3)_I &= a_0\, d_3^{(I)} + a_1^{(K)}\,\mathcal{D}_2^{(KI)} - a_2^{(K)}\,\mathcal{D}_1^{(KI)} - a_3\, d_0^{(I)} \ ,\\
(P\cdot\bar{H}_3)_I &= b_0\, d_3^{(I)} + b_1^{(K)}\,\mathcal{D}_2^{(KI)} - b_2^{(K)}\,\mathcal{D}_1^{(KI)} - b_3\, d_0^{(I)} \ ,
\end{aligned} \tag{2.58}$$

  the new second and third tadpole cancellation conditions in (2.57) for the R-R gauge potentials C_8' and $\tilde{C}_8$ read

$$\begin{aligned}
& b_0\, c_3^{(I)} + b_1^{(K)}\,\mathcal{C}_2^{(KI)} - b_2^{(K)}\,\mathcal{C}_1^{(KI)} - b_3\, c_0^{(I)} + \\
& a_0\, d_3^{(I)} + a_1^{(K)}\,\mathcal{D}_2^{(KI)} - a_2^{(K)}\,\mathcal{D}_1^{(KI)} - a_3\, d_0^{(I)} = N_{7_I}'
\end{aligned} \tag{2.59}$$

  and

$$b_0\, d_3^{(I)} + b_1^{(K)}\,\mathcal{D}_2^{(KI)} - b_2^{(K)}\,\mathcal{D}_1^{(KI)} - b_3\, d_0^{(I)} = \tilde{N}_{7_I}\ . \tag{2.60}$$

  The quantities N_{7_I}' and $\tilde{N}_{7_I}$ are respectively related to the number of I7-branes (bound states of D7-branes and NS7-branes [139]) and NS7-branes which can be added to the system wrapping the I^{th} 4-cycle dual to the 2-torus $\mathbb{T}_I^2$.

- There is also the $\mathrm{SL}(2,\mathbb{Z})$-singlet tadpole of (2.52) for the R-R gauge potential C_4 which does not transform under the type IIB $\mathrm{SL}(2,\mathbb{Z})$ self-duality.

In addition, further relations involving Jacobi-like contractions and tadpole-like contractions of background fluxes take place in this type IIB orientifold theory, as originally notice in ref. [8]. These relations are given by

$$\bar{H}_3 Q = 0 \ \Rightarrow\ Q\cdot\bar{H}_3 = 0 \qquad \text{and} \qquad \bar{F}_3 P = 0 \ \Rightarrow\ P\cdot\bar{F}_3 = 0\ . \tag{2.61}$$

As a consequence, satisfying the $\bar{H}_3 Q - \bar{F}_3 P = 0$ Jacobi identity of (2.50) in a piecewise manner, i.e. $\bar{H}_3 Q = \bar{F}_3 P = 0$, will imply the vanishing $N_{7_I}' = 0$ of the flux-induced tadpoles for the R-R gauge potential C_8'.

2.6 The "*isotropic*" supergravity flux models

Thus far we have worked out a four-dimensional $\mathcal{N}=1$ supergravity theory which describes the dynamics of seven moduli fields $\Phi \equiv (U_1, U_2, U_3, S, T_1, T_2, T_3)$ and that results invariant under both T- and S-duality transformations. Given the Kähler potential in (2.15) and the superpotential in (2.41), the moduli evolve according to the standard $\mathcal{N}=1$ scalar potential

$$V(\Phi) = e^K \left(\sum_{\Phi} K^{\Phi\bar{\Phi}} |D_\Phi W|^2 - 3|W|^2 \right) , \tag{2.62}$$

where $K^{\Phi\bar{\Phi}}$ is the inverse of the Kähler metric in (2.16), and $D_\Phi W = \frac{\partial W}{\partial \Phi} + \frac{\partial K}{\partial \Phi} W$ is the Kähler derivative. Moduli fields are stabilised at the minimum of the potential energy, taking a vacuum expectation value Φ_0 determined by the extremisation conditions

$$\left. \frac{\partial V(\Phi)}{\partial \Phi} \right|_{\Phi=\Phi_0} = 0. \tag{2.63}$$

This effective theory depends on 64 flux parameters although, as we have already seen, they are further restricted by Jacobi identities and tadpole cancellation conditions. In any case, finding moduli flux vacua in this generic setup is rather cumbersome.

To make it more affordable, we will impose an additional $\mathbb{Z}_3$ symmetry on the fluxes under the exchange $1 \to 2 \to 3$ in the factorisation

$$\mathbb{T}^6 = \mathbb{T}^2_1 \times \mathbb{T}^2_2 \times \mathbb{T}^2_3 , \tag{2.64}$$

of the three 2-tori making up the internal space. We will refer to this restriction as *isotropic flux backgrounds* or, with some abuse of language, as the *isotropic model*. Notice that this isotropy assumption is realised on the flux backgrounds instead of on the internal space, as it was done in [7,95]. In that case there are only O3-planes and two geometric moduli, namely the overall Kähler and complex structure parameters.

Concretely, the aforementioned *isotropic model* ansatz is realized on the components of the flux backgrounds as

$$\begin{array}{cccc}
c_0^{(I)} \equiv c_0 \;, & \tilde{c}_1^{(I)} \equiv \tilde{c}_1 \;, & \hat{c}_1^{(I)} \equiv \hat{c}_1 \;, & \check{c}_1^{(I)} \equiv \check{c}_1 \;, \\
c_3^{(I)} \equiv c_3 \;, & \tilde{c}_2^{(I)} \equiv \tilde{c}_2 \;, & \hat{c}_2^{(I)} \equiv \hat{c}_2 \;, & \check{c}_2^{(I)} \equiv \check{c}_2 \;, \\
d_0^{(I)} \equiv d_0 \;, & \tilde{d}_1^{(I)} \equiv \tilde{d}_1 \;, & \hat{d}_1^{(I)} \equiv \hat{d}_1 \;, & \check{d}_1^{(I)} \equiv \check{d}_1 \;, \\
d_3^{(I)} \equiv d_3 \;, & \tilde{d}_2^{(I)} \equiv \tilde{d}_2 \;, & \hat{d}_2^{(I)} \equiv \hat{d}_2 \;, & \check{d}_2^{(I)} \equiv \check{d}_2 \;, \\
b_1^{(I)} \equiv b_1 \;, & b_2^{(I)} \equiv b_2 \;, & a_1^{(I)} \equiv a_1 \;, & a_2^{(I)} \equiv a_2 \;,
\end{array} \tag{2.65}$$

so it simplifies the supergravity effective models by reducing the number of flux parameters from 64 to 24. The isotropic fluxes are summarised in tables 2.3, 2.4 and 2.5.

$\bar{F}_{---}$	$\bar{F}_{\vert--}$	$\bar{F}_{-\vert\vert}$	$\bar{F}_{\vert\vert\vert}$	$\bar{H}_{---}$	$\bar{H}_{\vert--}$	$\bar{H}_{-\vert\vert}$	$\bar{H}_{\vert\vert\vert}$
a_3	a_2	a_1	a_0	b_3	b_2	b_1	b_0

Table 2.3: Isotropic $\bar{F}_3$ and $\bar{H}_3$ flux components in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold.

Q_{-}^{--}	$Q_{\vert}^{\vert-}$	$Q_{\vert}^{-\vert}$	$Q_{\vert}^{--}$	$Q_{-}^{\vert\vert}$	$Q_{-}^{\vert-}$	$Q_{-}^{-\vert}$	$Q_{\vert}^{\vert\vert}$
$\tilde{c}_1$	$\hat{c}_1$	$\check{c}_1$	c_0	c_3	$\check{c}_2$	$\hat{c}_2$	$\tilde{c}_2$

Table 2.4: Isotropic non-geometric Q-flux components in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold.

P_{-}^{--}	$P_{\vert}^{\vert-}$	$P_{\vert}^{-\vert}$	$P_{\vert}^{--}$	$P_{-}^{\vert\vert}$	$P_{-}^{\vert-}$	$P_{-}^{-\vert}$	$Q_{\vert}^{\vert\vert}$
$\tilde{d}_1$	$\hat{d}_1$	$\check{d}_1$	d_0	d_3	$\check{d}_2$	$\hat{d}_2$	$\tilde{d}_2$

Table 2.5: Isotropic non-geometric P-flux components in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold.

The ansatz of isotropic fluxes is compatible with vacua in which the geometric moduli are also isotropic, namely,

$$U_1 = U_2 = U_3 \equiv U \qquad \text{and} \qquad T_1 = T_2 = T_3 \equiv T \ . \tag{2.66}$$

This means that there is only one overall complex structure modulus U and one Kähler modulus T together with the axiodilaton S. In this case, the Kähler potential in (2.15) and the superpotential in (2.41) reduce to

$$\begin{aligned} K &= -3 \log\left(-i\,(U-\bar{U})\right) - \log\left(-i\,(S-\bar{S})\right) - 3\log\left(-i\,(T-\bar{T})\right) \ , \\ W &= P_1(U) + S\,P_2(U) + T\left(P_3(U) + S\,P_4(U)\right) \ , \end{aligned} \tag{2.67}$$

where the isotropic $P_{1,2,3,4}(U)$ cubic polynomials now depend on the single complex structure modulus U and have the simple expressions

$$P_1(U) = a_0 - 3\,a_1\,U + 3\,a_2\,U^2 - a_3\,U^3 \ , \tag{2.68}$$

$$P_2(U) = -b_0 + 3\,b_1\,U - 3\,b_2\,U^2 + b_3\,U^3 \ , \tag{2.69}$$

$$P_3(U) = 3\,\left(c_0 + (\hat{c}_1 + \check{c}_1 - \tilde{c}_1)\,U - (\hat{c}_2 + \check{c}_2 - \tilde{c}_2)\,U^2 - c_3\,U^3\right) \ , \tag{2.70}$$

$$P_4(U) = -3\,\left(d_0 + (\hat{d}_1 + \check{d}_1 - \tilde{d}_1)\,U - (\hat{d}_2 + \check{d}_2 - \tilde{d}_2)\,U^2 - d_3\,U^3\right) \ . \tag{2.71}$$

The Kähler potential and the superpotential in (2.67) define the four-dimensional $\mathcal{N}=1$ effective supergravity which we will deal with during the next three chapters of the thesis.

2.6.1 Isotropic Jacobi identities and tadpole cancellation conditions

Making the isotropic flux ansatz of (2.65) leaves us with a much more simplified set of Jacobi identities as well as tadpole cancellation conditions.

We will start with the SL$(2,\mathbb{Z})$-triplet of Jacobi identities in (2.49). In terms of flux components, the $Q^2=0$ constraints are written as

$$\begin{array}{lcl} \hat{c}_2\,\tilde{c}_1-\tilde{c}_1\,\check{c}_2+\check{c}_1\,\hat{c}_2-c_0\,c_3=0 & , & c_3\,\tilde{c}_1-\check{c}_2^2+\tilde{c}_2\,\hat{c}_2-\hat{c}_1\,c_3=0 \quad , \\ c_3\,c_0-\check{c}_2\,\hat{c}_1+\tilde{c}_2\,\check{c}_1-\hat{c}_1\,\tilde{c}_2=0 & , & c_0\,\tilde{c}_2-\check{c}_1^2+\tilde{c}_1\,\hat{c}_1-\hat{c}_2\,c_0=0 \quad , \end{array} \tag{2.72}$$

plus one additional copy of each condition with $\check{c}_i\leftrightarrow\hat{c}_i$. An important result is that saturating[7] this ideal with respect to the conditions $\check{c}_i\neq\hat{c}_i$ automatically implies that $\tilde{c}_i$ is complex. Therefore, it must be that

$$\check{c}_1=\hat{c}_1\equiv c_1 \qquad \text{and} \qquad \check{c}_2=\hat{c}_2\equiv c_2\ . \tag{2.73}$$

Using the result in (2.73), the Jacobi constraints satisfied by the non-geometric Q-flux background become

$$\begin{array}{rcl} c_0\,(c_2-\tilde{c}_2)+c_1\,(c_1-\tilde{c}_1) & = & 0 \quad , \\ c_2\,(c_2-\tilde{c}_2)+c_3\,(c_1-\tilde{c}_1) & = & 0 \quad , \\ c_0c_3-c_1c_2 & = & 0 \quad . \end{array} \tag{2.74}$$

A similar reasoning can be done concerning the $P^2=0$ constraints in (2.49). These conditions are exactly those in (2.74) when replacing $c\to d$. In addition, the third element of the triplet in (2.49) gives a mixing between the Q and P fluxes which, in terms of their entries, reduces to

$$\begin{array}{rcl} c_3d_0-c_2d_1-c_1d_2+c_0d_3 & = & 0 \quad , \\ c_1(d_1-\tilde{d}_1)+c_0(d_2-\tilde{d}_2)+d_0(c_2-\tilde{c}_2)+d_1(c_1-\tilde{c}_1) & = & 0 \quad , \\ c_3(d_1-\tilde{d}_1)+c_2(d_2-\tilde{d}_2)+d_2(c_2-\tilde{c}_2)+d_3(c_1-\tilde{c}_1) & = & 0 \quad . \end{array} \tag{2.75}$$

Therefore, the conditions in (2.74) (equivalently for the P-flux components) and (2.75) determine the form of the flux-induced polynomials $P_3(U)$ and $P_4(U)$ in (2.70) and (2.71).

Let us now consider the $\bar{H}_3Q-\bar{F}_3P=0$ Jacobi identity of (2.50). It is an SL$(2,\mathbb{Z})$-singlet and involves the entire set of fluxes. Inserting the isotropic flux backgrounds, we find that the resulting conditions are written in terms of the flux entries as

$$\begin{array}{rcrcl} c_0\,b_2-c_2\,b_0+(c_1-\tilde{c}_1)\,b_1 & - & d_0\,a_2+d_2\,a_0-(d_1-\tilde{d}_1)\,a_1 & = & 0 \quad , \\ c_0\,b_3-c_2\,b_1+(c_1-\tilde{c}_1)\,b_2 & - & d_0\,a_3+d_2\,a_1-(d_1-\tilde{d}_1)\,a_2 & = & 0 \quad , \\ c_1\,b_2-c_3\,b_0-(c_2-\tilde{c}_2)\,b_1 & - & d_1\,a_2+d_3\,a_0+(d_2-\tilde{d}_2)\,a_1 & = & 0 \quad , \\ c_1\,b_3-c_3\,b_1-(c_2-\tilde{c}_2)\,b_2 & - & d_1\,a_3+d_3\,a_1+(d_2-\tilde{d}_2)\,a_2 & = & 0 \quad . \end{array} \tag{2.76}$$

[7] This can be done using a computational algebra program as *Singular* [173] and solving over the real field. In ref. [7], an analogous result is obtained manipulating this set of polynomial constraints by hand.

The above constraints restrict the R-R $\bar{F}_3$ and the NS-NS $\bar{H}_3$ fluxes that respectively determine the polynomials $P_1(U)$ and $P_2(U)$ in (2.68) and (2.69).

Finally, the tadpole cancellation relations also become simpler in the isotropic case. Substituting the isotropic flux backgrounds one obtains

$$a_0\, b_3 - 3\, a_1\, b_2 + 3\, a_2\, b_1 - a_3\, b_0 \quad = \quad N_3 \ , \tag{2.77}$$

as the tadpole cancellation condition for the R-R gauge potential C_4. In addition, the constraints in (2.56), (2.59) and (2.60) depending on $I = 1, 2, 3$ reduce to the three constraints

$$a_0\, c_3 + a_1\, (2\, c_2 - \tilde{c}_2) - a_2\, (2\, c_1 - \tilde{c}_1) - a_3\, c_0 \quad = \quad N_7 \ , \tag{2.78}$$

$$\begin{aligned} b_0\, c_3 + b_1\, (2\, c_2 - \tilde{c}_2) - b_2\, (2\, c_1 - \tilde{c}_1) - b_3\, c_0 \quad &+ \\ a_0\, d_3 + a_1\, (2\, d_2 - \tilde{d}_2) - a_2\, (2\, d_1 - \tilde{d}_1) - a_3\, d_0 \quad &= \quad N_7' \end{aligned} \tag{2.79}$$

and

$$b_0\, d_3 + b_1\, (2\, d_2 - \tilde{d}_2) - b_2\, (2\, d_1 - \tilde{d}_1) - b_3\, d_0 \quad = \quad \tilde{N}_7 \ , \tag{2.80}$$

which are the tadpole cancellation conditions associated to the SL$(2,\mathbb{Z})$-triplet of R-R gauge potentials $(C_8, C_8', \tilde{C}_8)$. All these tadpole conditions further restrict the R-R fluxes. Nevertheless, our approach will consider the net charges N_3, N_7, N_7' and $\tilde{N}_7$ to be free parameters.

2.6.2 Roots structure of non-geometric flux-induced polynomials

The restriction in (2.73) upon the non-geometric Q-flux background simplifies the corresponding flux-induced polynomial in (2.70) to

$$P_3(U) = 3\, \left(c_0 + (2\, c_1 - \tilde{c}_1)\, U - (2\, c_2 - \tilde{c}_2)\, U^2 - c_3\, U^3 \right) \ , \tag{2.81}$$

with the flux coefficients being integer parameters.

Additionally, the $Q^2 = 0$ system of equations in (2.74) is easy to solve explicitly. The solution variety has three disconnected pieces of different dimensions. The first piece has dimension four and it is characterised by fluxes

$$\begin{array}{lllll} c_3 = \lambda_p\, k_2 & , & c_2 = \lambda_p\, k_1 & , & \tilde{c}_1 = \lambda_q\, k_2 + \lambda k_1 \quad , \\ c_1 = \lambda_q\, k_2 & , & c_0 = \lambda_q\, k_1 & , & \tilde{c}_2 = \lambda_p\, k_1 - \lambda k_2 \quad . \end{array} \tag{2.82}$$

Here $\lambda = 1$, (k_1, k_2) are two integers not zero simultaneously, and (λ_p, λ_q) are two rays given by

$$\lambda_p = 1 + \frac{p}{\text{GCD}(k_1, k_2)} \qquad \text{and} \qquad \lambda_q = 1 + \frac{q}{\text{GCD}(k_1, k_2)} \ , \tag{2.83}$$

where $p,\, q \in \mathbb{Z}$. By convention $\text{GCD}(n, 0) = |n|$. With coefficients given by the fluxes in (2.82), the polynomial $P_3(U)$ turns out to factorise as

$$P_3(U) = 3\,(k_1 + k_2\,U)\,(\lambda_q - \lambda\,U - \lambda_p\,U^2)\ . \tag{2.84}$$

Notice that we have taken into account that the non-geometric fluxes are integers. The second piece of solutions is three-dimensional, the set of fluxes can still be characterised by (2.82) and $P_3(U)$ by (2.84), but with $\lambda \equiv 0$ and $\lambda_p \equiv 1$. Finally, the third piece has only two dimensions with fluxes and $P_3(U)$ specified by setting $\lambda \equiv 0$, $\lambda_p \equiv 0$ and $\lambda_q \equiv 1$.

As a byproduct of the previous analysis we have isolated the real root of $P_3(U)$ that always exists. In the next chapter we will investigate the T-duality invariant supergravity theory and will explain how the nature of the remaining two roots is correlated with the type of $\mathfrak{g}_{gauge}$ subalgebra in (2.46) spanned by the X^a generators. For example, we will see that in the third piece of solutions with $k_2 = 0$, the $\mathfrak{g}_{gauge}$ turns out to be nilpotent.

Finally the above derivation can be repeated for the case of the non-geometric P-flux whose flux-induced polynomial in (2.71) also simplifies to

$$P_4(U) = -3\,\left(d_0 + (2\,d_1 - \tilde{d}_1)\,U - (2\,d_2 - \tilde{d}_2)\,U^2 - d_3\,U^3\right)\ . \tag{2.85}$$

The roots structure of $P_4(U)$ when imposing the $P^2 = 0$ Jacobi identity is exactly the same as for its counterpart in (2.84). In this sense, the $P^2 = 0$ constraint can be interpreted as the Jacobi identity associated to a P-flux algebra which relates to the type of roots of $P_4(U)$. However an interpretation of the remaining $QP + PQ = 0$ condition in (2.49) turns out to be more involved and needs further discussion. We will go into these questions in chapter 5.

Chapter 3

Supersymmetric Vacua in T-duality Invariant Flux Models

In chapter 2 we derived the $\mathcal{N} = 1$ effective supergravity theory in four dimensions arising from the $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ type IIB orientifold including O3/O7-planes. In addition we considered the effect of including the doublet of gauge fluxes $(\bar{F}_3, \bar{H}_3)$ as well as of non-geometric fluxes (Q, P) transforming under the SL$(2,\mathbb{Z})$ self-duality of type IIB supergravity (weak-strong coupling duality). For the sake of simplicity, we concentrated on a family of supergravity flux models that appears after imposing an isotropy $\mathbb{Z}_3$ symmetry upon the fluxes and worked out the Jacobi identities and the tadpole cancellation conditions that the fluxes must satisfy for consistency of the models. These models are found to be invariant under the action of T-duality and S-duality transformations which reflect at the effective level as modular transformations on the complex structure and on the dilaton moduli fields respectively.

Here we want to consider only the effect of the gauge fluxes $(\bar{F}_3, \bar{H}_3)$ together with a non-geometric Q-flux. In other words, we are fixing the non-geometric P-flux background to zero, namely, the vanishing $d_A = 0$ of the fluxes in table 2.5. The symbolic index A runs over all the entries of the non-geometric P-flux tensor. As a result, the supergravity flux models arising from this $P = 0$ simplification are no longer invariant under SL$(2,\mathbb{Z})$ modular transformations upon the axiodilaton modulus S.

Setting $P = 0$ in the Jacobi identities and the tadpole cancellation conditions restricting the isotropic flux configurations, these result further simplified. While the flux constraints in (2.74) coming from the Jacobi identity $Q^2 = 0$ are left intact, those in

(2.76) arising from $\bar{H}_3 Q - \bar{F}_3 P = 0$ get simplified to

$$\begin{array}{rcl} c_0\, b_2 - c_2\, b_0 + (c_1 - \tilde{c}_1)\, b_1 & = & 0 \quad , \\ c_0\, b_3 - c_2\, b_1 + (c_1 - \tilde{c}_1)\, b_2 & = & 0 \quad , \\ c_1\, b_2 - c_3\, b_0 - (c_2 - \tilde{c}_2)\, b_1 & = & 0 \quad , \\ c_1\, b_3 - c_3\, b_1 - (c_2 - \tilde{c}_2)\, b_2 & = & 0 \quad . \end{array} \tag{3.1}$$

Because of the first relation in (2.61), the fact of taking $P = 0$ automatically imposes the flux-induced tadpole cancellations $N_7' = \tilde{N}_7 = 0$ in (2.79) and (2.80). In the weak coupling limit, 7-branes will then look like ordinary O7-planes and D7-branes in these type IIB orientifold models, as it was explained in section 1.4.2.

Our objective in this chapter is the study and classification of supersymmetric vacua in the aforementioned type IIB orientifold flux models.

3.1 Flux algebra and NS-NS flux-induced polynomials

We will start by discussing the solution to the Jacobi identities satisfied by the $\bar{H}_3$ and the non-geometric Q fluxes. The key idea is twofold:

- First, the gauge generators X^a in (2.46) coming from the reduction of the B-field span a six-dimensional subalgebra $\mathfrak{g}_{gauge}$ whose structure constants are precisely the Q_c^{ab} non-geometric fluxes.
- Second, when these Q-fluxes are invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold symmetry described in section 2.1, the subalgebra $\mathfrak{g}_{gauge}$ is rather constrained. We expect only a few subalgebras to be allowed in the isotropic case and our strategy is to identify them.

In this way we will manage to provide explicit parameterisations for the non-geometric Q-flux backgrounds that satisfy the Jacobi identity $Q^2 = 0$ in (2.47). Once this is achieved, we will also be able to find the corresponding NS-NS $\bar{H}_3$ fluxes that fulfil the $\bar{H}_3 Q = 0$ Jacobi identities in (3.1).

We want to consider in detail the set of isotropic non-geometric Q-fluxes given in table 2.4 plus the conditions $\check{c}_1 = \hat{c}_1 \equiv c_1$ and $\check{c}_2 = \hat{c}_2 \equiv c_2$ of (2.73). In this case the subalgebra $\mathfrak{g}_{gauge}$ simplifies to

$$\begin{array}{rcl} \left[X^{2I-1}, X^{2J-1}\right] & = & \epsilon_{IJK} \left(\tilde{c}_1\, X^{2K-1} + c_0\, X^{2K}\right) \ , \\ \left[X^{2I-1}, X^{2J}\right] & = & \epsilon_{IJK} \left(c_2\, X^{2K-1} + c_1\, X^{2K}\right) \ , \\ \left[X^{2I}, X^{2J}\right] & = & \epsilon_{IJK} \left(c_3\, X^{2K-1} + \tilde{c}_2\, X^{2K}\right) \ , \end{array} \tag{3.2}$$

where $I, J, K = 1, 2, 3$. The Jacobi identities of this Q-algebra are given in (2.74). To reveal further properties, it is instructive to compute the Cartan-Killing metric, denoted

$\mathcal{M}_Q$, with components

$$\mathcal{M}_Q^{ab} = Q_c^{ad}\, Q_d^{bc} \ . \tag{3.3}$$

For the above $\mathfrak{g}_{gauge}$ of isotropic fluxes we find that the six-dimensional matrix $\mathcal{M}_Q$ is block-diagonal, namely

$$\mathcal{M}_Q = \text{diag}\,(\mathcal{M}_g\,,\,\mathcal{M}_g\,,\,\mathcal{M}_g)\ , \tag{3.4}$$

where the 2×2 matrix $\mathcal{M}_g$ turns out to be

$$\mathcal{M}_g = -2\begin{pmatrix} \tilde{c}_1^2 + 2c_0c_2 + c_1^2 & \tilde{c}_1c_2 + c_1c_2 + c_0c_3 + c_1\tilde{c}_2 \\ \tilde{c}_1c_2 + c_1c_2 + c_0c_3 + c_1\tilde{c}_2 & \tilde{c}_2^2 + 2c_1c_3 + c_2^2 \end{pmatrix} \ . \tag{3.5}$$

Since $\mathcal{M}_g$ is symmetric, we conclude that $\mathcal{M}_Q$ can have up to two distinct real eigenvalues, each with multiplicity three.

The full 12-dimensional algebra $\mathfrak{g}$ in (2.46) incorporating also the isometry Z_a generators enjoys distinctive features alike. In the isotropic case the remaining isometry-isometry algebra commutators involving the NS-NS $\bar{H}_3$ fluxes are given by

$$\begin{aligned} [Z_{2I-1}, Z_{2J-1}] &= \epsilon_{IJK}\left(b_3\,X^{2K-1} + b_2\,X^{2K}\right)\ , \\ [Z_{2I-1}, Z_{2J}] &= \epsilon_{IJK}\left(b_2\,X^{2K-1} + b_1\,X^{2K}\right)\ , \\ [Z_{2I}, Z_{2J}] &= \epsilon_{IJK}\left(b_1\,X^{2K-1} + b_0\,X^{2K}\right)\ , \end{aligned} \tag{3.6}$$

whereas the mixed piece of the algebra involving the gauge-isometry brackets is determined by the co-adjoint action $Q^\star$ of Q as

$$\begin{aligned} \left[Z_{2I-1}, X^{2J-1}\right] &= \epsilon_{IJK}\left(\tilde{c}_1\,Z_{2K-1} + c_2\,Z_{2K}\right)\ , \\ \left[Z_{2I-1}, X^{2J}\right] &= \epsilon_{IJK}\left(c_2\,Z_{2K-1} + c_3\,Z_{2K}\right)\ , \\ \left[Z_{2I}, X^{2J-1}\right] &= \epsilon_{IJK}\left(c_0\,Z_{2K-1} + c_1\,Z_{2K}\right)\ , \\ \left[Z_{2I}, X^{2J}\right] &= \epsilon_{IJK}\left(c_1\,Z_{2K-1} + \tilde{c}_2\,Z_{2K}\right)\ . \end{aligned} \tag{3.7}$$

Besides the Jacobi identities purely involving the non-geometric Q-fluxes, there are the additional mixed constraints of (3.1). Computing the full Cartan-Killing metric, denoted $\mathcal{M}$, shows that there are no mixed XZ-terms. In fact, the matrix is again block-diagonal

$$\mathcal{M} = \text{diag}\,(\mathcal{M}_g\,,\,\mathcal{M}_g\,,\,\mathcal{M}_g\,,\,\mathcal{M}_{isom}\,,\,\mathcal{M}_{isom}\,,\,\mathcal{M}_{isom})\ , \tag{3.8}$$

with $\mathcal{M}_g$ shown above. The new 2×2 matrix $\mathcal{M}_{isom}$ is found to be

$$\mathcal{M}_{isom} = -4\begin{pmatrix} b_3\tilde{c}_1 + 2b_2c_2 + b_1c_3 & b_2(c_1+\tilde{c}_1) + b_1(c_2+\tilde{c}_2) \\ b_2(c_1+\tilde{c}_1) + b_1(c_2+\tilde{c}_2) & b_0\tilde{c}_2 + 2b_1c_1 + b_2c_0 \end{pmatrix} \ . \tag{3.9}$$

Here we have simplified $\mathcal{M}_{isom}$ using the Jacobi identities in (3.1). We conclude that the allowed 12-dimensional algebras $\mathfrak{g}$ are such that the Cartan-Killing matrix can have up to four distinct eigenvalues, each with multiplicity three.

Let us now return to the subalgebra $\mathfrak{g}_{gauge}$ spanned by the X^a generators and the task of solving the constraints in (2.74) that arise from the Jacobi identity $Q^2 = 0$. The idea is to fulfil these constraints by choosing the non-geometric Q-fluxes to be the structure constants of six-dimensional Lie algebras whose Cartan-Killing matrix has the simple block-diagonal form of (3.4). To proceed it is convenient to distinguish whether $\mathcal{M}_Q$ is non-degenerate or not, i.e. whether $\mathfrak{g}_{gauge}$ is semisimple or not. If $\det \mathcal{M}_Q \neq 0$, and $\mathcal{M}_Q$ is negative definite, the only possible Q-algebra is the compact $\mathfrak{so}(4) \sim \mathfrak{su}(2)^2$. On the other hand, the only non-compact semisimple Q-algebra with the required block structure is $\mathfrak{so}(3,1)$. When $\det \mathcal{M}_Q = 0$, the Q-algebra is non-semisimple. In this class to begin we find two compatible algebras, namely the direct sum $\mathfrak{su}(2) + \mathfrak{u}(1)^3$ and the semi-direct sum $\mathfrak{su}(2) \oplus_{\mathbb{Z}_3} \mathfrak{u}(1)^3$ that is isomorphic to the Euclidean algebra $\mathfrak{iso}(3)$. The $\oplus_{\mathbb{Z}_3}$ symbol denotes the semidirect sum of algebras endowed with the $\mathbb{Z}_3$ cyclic structure coming from isotropy. The remaining possibility is that the non-semisimple Q-algebra be completely solvable. One example is the nilpotent $\mathfrak{u}(1)^6$ that we disregard because the non-geometric Q-fluxes vanish identically. A second non-trivial solvable algebra, that is actually nilpotent, will be discussed shortly.

After classifying the allowed 6-dimensional subalgebras $\mathfrak{g}_{gauge}$ the next step is to find the set of corresponding non-geometric Q-fluxes. Except for the nilpotent example, all other cases have an $\mathfrak{su}(2)$ factor. This suggests to make a change of basis from (X^{2I-1}, X^{2I}) with $I = 1, 2, 3$, to new generators $(E^I, \widetilde{E}^I)$ such that basically one type, say E^I, spans $\mathfrak{su}(2)$. The $\mathbb{Z}_2^3$ symmetry of the fluxes require that we form combinations that transform in a definite way. For instance, E^I can only be a combination of X^{2I-1} and X^{2I} with the same I. Furthermore, for isotropic fluxes it is natural to make the same transformation for each I. We will then make the $\mathrm{SL}(2, \mathbb{R})$ transformation

$$\begin{pmatrix} E^I \\ \widetilde{E}^I \end{pmatrix} = \frac{1}{|\Gamma|^2} \begin{pmatrix} -\alpha & \beta \\ -\gamma & \delta \end{pmatrix} \begin{pmatrix} X^{2I-1} \\ X^{2I} \end{pmatrix}, \tag{3.10}$$

for all $I = 1, 2, 3$. Here $|\Gamma| = \alpha\delta - \beta\gamma$, and it must be that $|\Gamma| \neq 0$. In the following we will refer to $(\alpha, \beta, \gamma, \delta)$ as the Γ parameters.

Substituting into (3.2), it is straightforward to obtain the algebra satisfied by the new generators E^I and $\widetilde{E}^J$. This algebra will depend on the non-geometric Q-fluxes as well as on the parameters $(\alpha, \beta, \gamma, \delta)$. We can then prescribe the commutators to have the standard form for the allowed algebras found previously. For instance, in the direct product examples we impose $\left[E^I, \widetilde{E}^J\right] = 0$.

In the following sections we will discuss each compatible 6-dimensional subalgebra $\mathfrak{g}_{gauge}$ in more detail. As we explained, this subalgebra is spanned by the X^a generators coming from the reduction of the B-field. The goal is to parameterise the non-geometric

Q-fluxes in terms of $(\alpha, \beta, \gamma, \delta)$ for each type of B-field reduction, namely, semisimple and non-semisimple reductions. By construction these fluxes will satisfy the Jacobi identity $Q^2 = 0$ of the algebra $\mathfrak{g}$ in (2.46). We will then solve the mixed constraints in (3.1) involving the NS-NS $\bar{H}_3$ fluxes. The main result will be an explicit factorisation of the cubic polynomials $P_3(U)$ and $P_2(U)$ that dictate the couplings among the moduli fields.

3.1.1 Semisimple B-field reductions

The Q-algebra is semisimple when the Cartan-Killing metric is non-degenerate. This means $\det \mathcal{M}_Q \neq 0$ and hence $\det \mathcal{M}_g \neq 0$. Now, six-dimensional semisimple algebras are completely classified. If $\mathcal{M}_Q$ is negative definite the Q-algebra is compact so that it must be $\mathfrak{so}(4) \sim \mathfrak{su}(2) + \mathfrak{su}(2)$. When $\mathcal{M}_Q$ has positive eigenvalues the Q-algebra is non-compact and it could be $\mathfrak{so}(3,1)$ or $\mathfrak{so}(2,2)$ but the latter does not fit the required block-diagonal form in (3.4).

3.1.1.1 The $\mathfrak{so}(4) \sim \mathfrak{su}(2)^2$ case

The standard commutators of this algebra are

$$\left[E^I, E^J\right] = \epsilon_{IJK} E^K \quad , \quad \left[\widetilde{E}^I, \widetilde{E}^J\right] = \epsilon_{IJK} \widetilde{E}^K \quad , \quad \left[E^I, \widetilde{E}^J\right] = 0 \; . \tag{3.11}$$

After performing the change of basis in (3.2) we find that the non-geometric Q-fluxes needed to describe this algebra can be parameterise as

$$\begin{array}{llllll} c_0 = & \beta\,\delta\,(\beta+\delta) \quad , & c_1 = & \beta\,\delta\,(\alpha+\gamma) \quad , & \tilde{c}_2 = & \gamma^2\,\beta + \alpha^2\,\delta \quad , \\ c_3 = & -\,\alpha\,\gamma\,(\alpha+\gamma) \quad , & c_2 = & -\,\alpha\,\gamma\,(\beta+\delta) \quad , & \tilde{c}_1 = & -\,(\gamma\,\beta^2 + \alpha\,\delta^2) \quad , \end{array} \tag{3.12}$$

provided that $|\Gamma| = (\alpha\delta - \beta\gamma) \neq 0$. It is easy to show that these fluxes verify the Jacobi identities in (2.74). What we have done is to trade the six non-geometric Q-fluxes, constrained by two independent conditions, by the four independent parameters $(\alpha, \beta, \gamma, \delta)$. These parameters are real but the resulting non-geometric Q-fluxes in (3.12) must be integers.

For future purposes we need to determine the cubic polynomial $P_3(U)$ that corresponds to the parameterised non-geometric Q-fluxes. Substituting in (2.81) yields

$$P_3(U) = 3(\alpha U + \beta)(\gamma U + \delta)\left[(\alpha+\gamma)U + (\beta+\delta)\right] \; . \tag{3.13}$$

This clearly shows that in this case $P_3(U)$ has three real roots. Moreover, the roots are all different because $|\Gamma| \neq 0$. We will prove that for other algebras $P_3(U)$ has either complex roots or degenerate real roots. The remarkable conclusion is that $P_3(U)$ has three different real roots if and only if the algebra of the non-geometric Q-fluxes is the compact $\mathfrak{so}(4) \sim \mathfrak{su}(2) + \mathfrak{su}(2)$. Alternatively, we may start with the condition that the polynomial has three different real roots that we can choose to be at 0, -1 and ∞ without loss of generality. These roots can then be moved to arbitrary real locations by a linear fractional transformation

$$\mathcal{Z} = \frac{\alpha\, U + \beta}{\gamma\, U + \delta} \ , \tag{3.14}$$

with $(\alpha, \beta, \gamma, \delta) \in \mathbb{R}$ and $|\Gamma| \neq 0$. By comparing the roots of $P_3(U)$ in terms of the fluxes with those in terms of the transformation parameters we rediscover the map in (3.12) and the associated $\mathfrak{su}(2)^2$ algebra. In the next sections we will see that the variable $\mathcal{Z}$ introduced above plays a very important physical role.

We now turn to the Jacobi constraints in (3.1) involving the NS-NS $\bar{H}_3$ fluxes. Inserting the non-geometric Q-fluxes of (3.12) we find that the b's fluxes can be completely fixed by the Γ parameters plus two new real variables (ϵ_1, ϵ_2) as follows

$$\begin{array}{llll} b_0 = -\left(\epsilon_1\, \beta^3 + \epsilon_2\, \delta^3\right) & , & b_2 = -\left(\epsilon_1\, \alpha^2\, \beta + \epsilon_2\, \gamma^2\, \delta\right) & , \\ b_3 = \quad \epsilon_1\, \alpha^3 + \epsilon_2\, \gamma^3 & , & b_1 = \quad \epsilon_1\, \alpha\, \beta^2 + \epsilon_2\, \gamma\, \delta^2 & . \end{array} \tag{3.15}$$

We also need to compute the polynomial $P_2(U)$ that depends on the NS-NS $\bar{H}_3$ fluxes. Substituting the above b's in (3.15) yields

$$P_2(U) = \epsilon_1(\alpha U + \beta)^3 + \epsilon_2(\gamma U + \delta)^3 \ . \tag{3.16}$$

It is easy to show that because $|\Gamma| \neq 0$, the $P_2(U)$ flux-induced polynomial has complex roots whenever $\epsilon_1\, \epsilon_2 \neq 0$. Contrariwise, $P_2(U)$ has a triple real root if either ϵ_1 or ϵ_2 vanishes.

We may expect that the full 12-dimensional algebra $\mathfrak{g}$ has special properties when $P_2(U)$ has a triple root. Indeed, inserting the fluxes in (3.12) and (3.15) into (3.9) yields $\det \mathcal{M}_{isom} = 16\, \epsilon_1\, \epsilon_2\, |\Gamma|^6$. Hence, the full Cartan-Killing matrix $\mathcal{M}$ happens to be degenerate when $\epsilon_1\, \epsilon_2 = 0$. To learn more about the full algebra it is convenient to switch from the original Z_a isometry generators to a new basis $(D_I, \widetilde{D}_I)$ defined by

$$\begin{pmatrix} D_I \\ \widetilde{D}_I \end{pmatrix} = \frac{1}{|\Gamma|^2} \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} Z_{2I-1} \\ Z_{2I} \end{pmatrix} \ , \tag{3.17}$$

for $I = 1, 2, 3$. It is straightforward to compute the piece of the full algebra generated by the $(D_I, \widetilde{D}_I)$. Substituting the parameterised fluxes in (3.6) and (3.7) we obtain

$$\begin{array}{llll} \left[D_I, D_J\right] = -\epsilon_1\, \epsilon_{IJK} E^K & , & \left[\widetilde{D}_I, \widetilde{D}_J\right] = -\epsilon_2\, \epsilon_{IJK} \widetilde{E}^K & , \\ \left[E^I, D_J\right] = \epsilon_{IJK} D_K & , & \left[\widetilde{E}^I, \widetilde{D}_J\right] = \epsilon_{IJK} \widetilde{D}_K & , \end{array} \tag{3.18}$$

where all other commutators do vanish. A quick inspection of the whole algebra encoded in (3.11) and (3.18) shows that when either ϵ_1 or ϵ_2 is zero, the D_I or the $\widetilde{D}_I$, generate a 3-dimensional invariant abelian subalgebra. Moreover, when say $\epsilon_1 = 0$ and $\epsilon_2 \neq 0$, the

$\mathcal{M}_{isom}$ block of the full Cartan-Killing metric has one zero and one non-zero eigenvalue which is negative for $\epsilon_2 < 0$ and positive for $\epsilon_2 > 0$. The upshot is that when $\epsilon_1 \epsilon_2 = 0$, the 12-dimensional algebra $\mathfrak{g}$ is $\mathfrak{iso}(3) + \mathfrak{g}$, where $\mathfrak{g}$ is either $\mathfrak{so}(4)$ or $\mathfrak{so}(3,1)$. On the other hand, when $\epsilon_1 \epsilon_2 < 0$, the algebra is $\mathfrak{so}(4) + \mathfrak{so}(3,1)$, whereas for $\epsilon_1, \epsilon_2 < 0$ it is $\mathfrak{so}(4)^2$, and for $\epsilon_1, \epsilon_2 > 0$ it is $\mathfrak{so}(3,1)^2$.

The methods developed in this section will be applied shortly to other subalgebras. In summary, the non-geometric Q and the NS-NS $\bar{H}_3$ fluxes can be parameterised using auxiliary variables $(\alpha, \beta, \gamma, \delta)$ and (ϵ_1, ϵ_2) in such a way that the Jacobi identities are satisfied and flux-induced superpotential terms are explicitly factorised. The full 12-dimensional algebra $\mathfrak{g}$ can be simply characterised after the changes of basis (3.10) and (3.17) are performed. We will carry out this algebra classification in the next chapter.

The auxiliary variables are constrained by the condition that the resulting fluxes be integers. This issue deserves further explanation. There are two cases depending on whether the polynomial $P_2(U)$ has complex roots or not. If it does not, we can take $\epsilon_1 = 0$ to be concrete. From the structure of the NS-NS fluxes $\bar{H}_3$ in (3.15) it is then obvious that, for $\alpha \neq 0$, the quotient β/α is a rational number. Going back to the non-geometric Q-fluxes, it can be shown that the ratios γ/α and δ/α, as well as α^3 and ϵ_2 also belong to $\mathbb{Q}$. If $P_2(U)$ admits complex roots the generic result is that ϵ_2/ϵ_1, β/α, α^3, etc., involve square roots of rationals. However, it happens that when at least one of the non-geometric parameters $(\alpha, \beta, \gamma, \delta)$ is zero then all well defined quotients are again rational numbers.

3.1.1.2 The $\mathfrak{so}(3,1)$ case

This is the well known Lorentz algebra. We can take E^I to be the angular momentum generators and $\widetilde{E}^J$ to be the boost generators. Thus, the Q-algebra can be written as

$$\left[E^I, E^J\right] = \epsilon_{IJK} E^K \quad , \quad \left[\widetilde{E}^I, \widetilde{E}^J\right] = -\epsilon_{IJK} E^K \quad , \quad \left[E^I, \widetilde{E}^J\right] = \epsilon_{IJK} \widetilde{E}^K \ . \tag{3.19}$$

In this case the non-geometric Q-fluxes that produce the algebra are found to be

$$\begin{aligned} c_0 &= -\beta\left(\beta^2+\delta^2\right) \quad , \quad c_1 = -\alpha\left(\beta^2+\delta^2\right) \quad , \quad \tilde{c}_2 = -\beta\left(\alpha^2-\gamma^2\right) - 2\,\gamma\,\delta\,\alpha \quad , \\ c_3 &= \ \alpha\left(\alpha^2+\gamma^2\right) \quad , \quad c_2 = \ \beta\left(\alpha^2+\gamma^2\right) \quad , \quad \tilde{c}_1 = \ \alpha\left(\beta^2-\delta^2\right) + 2\,\beta\,\gamma\,\delta \quad , \end{aligned} \tag{3.20}$$

as long as $|\Gamma| \neq 0$. Substituting the resulting non-geometric Q-fluxes in (2.81) gives the $P_3(U)$ polynomial

$$P_3(U) = -3(\alpha U + \beta)\left[(\alpha U + \beta)^2 + (\gamma U + \delta)^2\right] \ . \tag{3.21}$$

Since $\Gamma \neq 0$, $P_3(U)$ always has complex roots. We will see that for non-semisimple algebras all roots of $P_3(U)$ are real, as for the compact $\mathfrak{so}(4)$. Hence, the important observation now is that $P_3(U)$ has complex roots if and only if the algebra of the non-geometric Q-fluxes is the non-compact $\mathfrak{so}(3,1)$.

The Jacobi constraints in (3.1) for the NS-NS $\bar{H}_3$ fluxes can again be solved in terms of the Γ parameters plus two real constants that we again denote by (ϵ_1, ϵ_2). Concretely,

$$\begin{aligned} b_0 &= -\beta\left(\beta^2 - 3\delta^2\right)\epsilon_1 - \delta\left(\delta^2 - 3\beta^2\right)\epsilon_2 \quad , \\ b_1 &= \left(\alpha\beta^2 - 2\beta\gamma\delta - \alpha\delta^2\right)\epsilon_1 + \left(\gamma\delta^2 - 2\alpha\delta\beta - \gamma\beta^2\right)\epsilon_2 \quad , \\ b_2 &= \left(\beta\gamma^2 + 2\gamma\delta\alpha - \beta\alpha^2\right)\epsilon_1 + \left(\delta\alpha^2 + 2\beta\gamma\alpha - \delta\gamma^2\right)\epsilon_2 \quad , \\ b_3 &= \alpha\left(\alpha^2 - 3\gamma^2\right)\epsilon_1 + \gamma\left(\gamma^2 - 3\alpha^2\right)\epsilon_2 \quad . \end{aligned} \tag{3.22}$$

These fluxes give rise to the NS-NS $\bar{H}_3$ flux-induced polynomial

$$P_2(U) = (\gamma U + \delta)^3(\epsilon_1 \mathcal{Z}^3 - 3\epsilon_2 \mathcal{Z}^2 - 3\epsilon_1 \mathcal{Z} + \epsilon_2) \ , \tag{3.23}$$

where $\mathcal{Z} = (\alpha U + \beta)/(\gamma U + \delta)$ as before. The discriminant of this cubic polynomial is always negative, so the $P_2(U)$ polynomial has three different real roots.

3.1.2 Non-semisimple B-field reductions

Accordingly to the Levi's decomposition theorem, the Q-algebra in this case is the semidirect sum of a semisimple algebra and a solvable invariant subalgebra. Lack of simplicity is detected imposing $\det \mathcal{M}_Q = 0$ which requires $\det \mathcal{M}_g = 0$, where $\mathcal{M}_g$ is shown in (3.5). Combining with the Jacobi identities in (2.74) we deduce that up to isomorphisms there are only two solutions in which the solvable invariant subalgebra has dimension less than six. In practice this means that $\mathcal{M}_g$ has only one zero eigenvalue. As expected from the underlying symmetries, this invariant subalgebra can only have dimension three and be $\mathfrak{u}(1)^3$. The semisimple piece can only be $\mathfrak{su}(2)$, so the two solutions are the direct and semidirect sum discussed below.

The remaining possibility consistent with the symmetries is for the solvable invariant subalgebra to have dimension six. The criterion for solvability is that the derived algebra $[\mathfrak{g}_{gauge}, \mathfrak{g}_{gauge}]$ be orthogonal to the whole algebra $\mathfrak{g}_{gauge}$ with respect to the Cartan-Killing metric. In our case this means $Q_c^{ab}\mathcal{M}_Q^{dc} = 0, \ \forall a, b, d$. The non-geometric Q-fluxes further satisfy the Jacobi identity $Q^2 = 0$. On the other hand, the stronger condition for nilpotency is $\mathcal{M}_Q^{dc} = 0$. For our Q-algebra of isotropic fluxes given in (3.2), we find that all solvable flux configurations are necessarily nilpotent. The proof can be carried out using the algebraic package *Singular* to manipulate the various ideals[1]. This result is consistent with the fact that in our model $\mathcal{M}_Q$ is block-diagonal so that when $\det \mathcal{M}_Q = 0$, it has three or six null eigenvalues and in the latter situation $\mathcal{M}_Q$ is identically zero. One

[1] A group theoretical derivation of the allowed $\mathfrak{g}_{gauge}$ in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ isotropic orbifold will be given in chapter 4.

obvious nilpotent algebra is $\mathfrak{u}(1)^6$, but it is uninteresting because the associated Q-fluxes vanish identically. There is a second solution which will be described later.

The allowed non-semisimple Q-algebras can all be obtained starting from $\mathfrak{su}(2)^2$ and performing contractions consistent with the underlying symmetries of the isotropic fluxes. For example, setting $E'^I = E^I$ and $\widetilde{E}'^I = \lambda\, \widetilde{E}^I$ in (3.11) and then letting $\lambda \to 0$ obviously gives the direct sum $\mathfrak{su}(2) + \mathfrak{u}(1)^3$. More generically we can take $E'^I = \lambda^a\,(E^I + \widetilde{E}^I)$ and $\widetilde{E}'^I = \lambda^b\,(E^I - \widetilde{E}^I)$ with $a, b \geq 0$. The limit $a = 0$, $b > 0$ and $\lambda \to 0$ yields the Euclidean algebra $\mathfrak{iso}(3)$. Letting instead $2b = a > 0$ and contracting gives the nilpotent algebra.

3.1.2.1 The $\mathfrak{su}(2) + \mathfrak{u}(1)^3$ case

Since this algebra is a direct sum and one factor is abelian, the brackets take the simple form

$$\left[E^I, E^J\right] = \epsilon_{IJK} E^K \quad , \quad \left[\widetilde{E}^I, \widetilde{E}^J\right] = 0 \quad , \quad \left[E^I, \widetilde{E}^J\right] = 0 \; . \tag{3.24}$$

Requiring that, after the change of basis in (3.10), the algebra in (3.2) is of this type returns the following non-geometric Q-fluxes

$$\begin{array}{llllll} c_0 = & \beta\,\delta^2 \quad , & c_1 = & \beta\,\delta\,\gamma \quad , & \tilde{c}_2 = & \gamma^2\,\beta \quad , \\ c_3 = & -\alpha\,\gamma^2 \quad , & c_2 = & -\alpha\,\gamma\,\delta \quad , & \tilde{c}_1 = & -\delta^2\,\alpha \quad , \end{array} \tag{3.25}$$

assuming $|\Gamma| \neq 0$. These fluxes automatically satisfy the Jacobi identities in (2.74). They also satisfy the additional condition $c_0\, c_2 = c_1\, \tilde{c}_1$ arising from $\det \mathcal{M}_g = 0$. The non-geometric Q-fluxes of the Q-algebra $\mathfrak{su}(2) + \mathfrak{u}(1)^3$ lead to the $P_3(U)$ polynomial

$$P_3(U) = 3(\alpha U + \beta)(\gamma U + \delta)^2 \; . \tag{3.26}$$

One observes that $P_3(U)$ has one single and one double real root. The Jacobi identities in (3.1) again fix the NS-NS $\bar{H}_3$ fluxes as in the previous cases. The solution in terms of the free parameters is given by

$$\begin{array}{llll} b_0 = -\left(\epsilon_1\,\beta^3 + \epsilon_2\,\delta^3\right) \quad , & b_2 = -\left(\epsilon_1\,\alpha^2\,\beta + \epsilon_2\,\gamma^2\,\delta\right) \quad , \\ b_3 = \quad \epsilon_1\,\alpha^3 + \epsilon_2\,\gamma^3 \quad , & b_1 = \quad \epsilon_1\,\alpha\,\beta^2 + \epsilon_2\,\gamma\,\delta^2 \quad . \end{array} \tag{3.27}$$

For the associated polynomial $P_2(U)$ we then find

$$P_2(U) = \epsilon_1(\alpha U + \beta)^3 + \epsilon_2(\gamma U + \delta)^3 \; , \tag{3.28}$$

which, as in the compact case, has complex roots whenever $\epsilon_1\,\epsilon_2 \neq 0$.

3.1.2.2 The $\mathfrak{su}(2) \oplus_{\mathbb{Z}_3} \mathfrak{u}(1)^3 \sim \mathfrak{iso}(3)$ case

The Levi's theorem states that in general the Q-algebra can be characterised as

$$\left[E^I, E^J\right] = \epsilon_{IJK}\left(E^K + \widetilde{E}^K\right) \quad , \quad \left[\widetilde{E}^I, \widetilde{E}^J\right] = 0 \quad , \quad \left[E^I, \widetilde{E}^J\right] = \epsilon_{IJK}\widetilde{E}^K \; . \tag{3.29}$$

The typical form of the Euclidean algebra in three dimensions is recognised after the isomorphism $(E^I - \widetilde{E}^I) \to \widehat{E}^I$. The non-geometric Q-fluxes needed to reproduce the above commutators turn out to be

$$\begin{array}{llllll} c_0 = -\delta^2\,(\beta-\delta) & , & c_1 = -\delta^2\,(\alpha-\gamma) & , & \tilde{c}_2 = \ \ \gamma^2\,(\beta+\delta) - 2\,\gamma\,\delta\,\alpha & , \\ c_3 = \ \ \gamma^2\,(\alpha-\gamma) & , & c_2 = \ \ \gamma^2\,(\beta-\delta) & , & \tilde{c}_1 = -\delta^2\,(\alpha+\gamma) + 2\,\gamma\,\delta\,\beta & , \end{array} \tag{3.30}$$

for $|\Gamma| \neq 0$. Besides the Jacobi identity $Q^2 = 0$, these Q-fluxes satisfy the additional condition $4c_0c_2 = -(c_1-\tilde{c}_1)^2$ by virtue of $\det \mathcal{M}_g = 0$. For the flux configuration of this Q-algebra the $P_3(U)$ polynomial becomes

$$P_3(U) = 3(\gamma U+\delta)^2\big[(\gamma-\alpha)U + (\delta-\beta)\big]\ . \tag{3.31}$$

As in the direct sum $\mathfrak{su}(2)+\mathfrak{u}(1)^3$, this $P_3(U)$ polynomial has one single and one double real root.

The NS-NS $\bar{H}_3$ fluxes can be determined from the Jacobi identities in (3.1). Introducing again parameters (ϵ_1, ϵ_2) leads to

$$\begin{array}{llll} b_0 = -\delta^2\,(\beta\,\epsilon_1 + \delta\,\epsilon_2) & , & b_2 = -\dfrac{\gamma}{3}(\beta\,\gamma + 2\,\alpha\,\delta)\,\epsilon_1 - \gamma^2\,\delta\,\epsilon_2 & , \\ b_3 = \ \ \gamma^2\,(\alpha\,\epsilon_1 + \gamma\,\epsilon_2) & , & b_1 = \ \ \dfrac{\delta}{3}(\alpha\,\delta + 2\,\beta\,\gamma)\,\epsilon_1 + \gamma\,\delta^2\,\epsilon_2 & . \end{array} \tag{3.32}$$

The companion polynomial $P_2(U)$ of the NS-NS $\bar{H}_3$ fluxes is fixed as

$$P_2(U) = (\gamma U+\delta)^2\,[\epsilon_1(\alpha U+\beta) + \epsilon_2(\gamma U+\delta)]\ . \tag{3.33}$$

Analogous to the non-compact case, this $P_2(U)$ has only real roots, but one of them is degenerate.

3.1.2.3 The $\mathfrak{u}(1)^3 \oplus_{\mathbb{Z}_3} \mathfrak{u}(1)^3 \sim \mathfrak{nil}$ case

To search for Q-flux configurations that generate a nilpotent Q-algebra we impose that the Cartan-Killing metric vanishes. Now, in our model $\mathcal{M}_Q = 0$ implies the much simpler conditions $\det \mathcal{M}_g = 0$ and $\operatorname{Tr}\mathcal{M}_g = 0$. Up to isomorphisms, we find only one non-trivial solution. This is the expected result based on the known classification of 6-dimensional nilpotent algebras[2].

From the 34 isomorphism classes of nilpotent algebras, besides $\mathfrak{u}(1)^6$, only one is compatible with isotropic fluxes invariant under the $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold symmetry. The algebra is 2-step nilpotent and its brackets can be written as

$$\big[E^I, E^J\big] = \epsilon_{IJK}\widetilde{E}^K \quad , \quad \big[\widetilde{E}^I, \widetilde{E}^J\big] = 0 \quad , \quad \big[E^I, \widetilde{E}^J\big] = 0\ , \tag{3.34}$$

[2] A table and references to the original literature are given in ref. [99].

so, up to isomorphisms, this is the Q-algebra $\mathfrak{nil} = \mathfrak{u}(1)^3 \oplus_{\mathbb{Z}_3} \mathfrak{u}(1)^3$ labelled $n3.5$ in Table 4 of ref. [99].

The change of basis from the original (X^{2I-1}, X^{2I}) generators to the new $(E^I, \widetilde{E}^I)$ ones is still given by (3.10). Starting from the X^a commutators in (3.2) we can then deduce Q-fluxes such that the nilpotent algebra in (3.34) is reproduced. In this way we obtain

$$\begin{array}{llllll} c_0 = & \delta^3 \quad , & c_1 = & \delta^2\,\gamma \quad , & \tilde{c}_2 = & \delta\,\gamma^2 \quad , \\ c_3 = & -\gamma^3 \quad , & c_2 = & -\delta\,\gamma^2 \quad , & \tilde{c}_1 = & -\delta^2\,\gamma \quad . \end{array} \tag{3.35}$$

Notice that these Q-fluxes only depend on two independent parameters. This occurs because besides the Jacobi constraints there are two more conditions $\det \mathcal{M}_g = 0$ and $\mathrm{Tr}\,\mathcal{M}_g = 0$. The non-geometric Q-fluxes of the nilpotent algebra generate the $P_3(U)$ polynomial

$$P_3(U) = 3(\gamma U + \delta)^3 \; , \tag{3.36}$$

which clearly has one triple real root.

In analogy with all previous examples, the $\bar{H}_3 Q = 0$ Jacobi identity determines the NS-NS fluxes $\bar{H}_3$ in terms of two additional parameters (ϵ_1, ϵ_2). Inserting the non-geometric Q-fluxes of the nilpotent algebra into (3.1) readily yields

$$\begin{array}{llll} b_0 = & -\delta^2\,(\delta\,\epsilon_2 + \gamma\,\epsilon_1) \quad , & b_2 = & -\gamma^2\,\delta\,\epsilon_2 + \dfrac{\gamma}{3}\left(2\,\delta^2 - \gamma^2\right)\epsilon_1 \quad , \\ b_3 = & \gamma^2\,(\gamma\,\epsilon_2 - \delta\,\epsilon_1) \quad , & b_1 = & \gamma\,\delta^2\,\epsilon_2 - \dfrac{\delta}{3}\left(\delta^2 - 2\,\gamma^2\right)\epsilon_1 \quad . \end{array} \tag{3.37}$$

Substituting in (2.69) we easily obtain the corresponding polynomial

$$P_2(U) = (\gamma U + \delta)^2\,[\epsilon_2(\gamma U + \delta) + \epsilon_1(\gamma - \delta U)] \; . \tag{3.38}$$

As in the $\mathfrak{su}(2) \oplus_{\mathbb{Z}_3} \mathfrak{u}(1)^3$ case, this $P_2(U)$ has one single and one double real root. Without loss of generality we can choose $\alpha = -\delta$ and $\beta = \gamma$ (so that $|\Gamma| < 0$) in order to write $P_2(U)$ in terms of the variable $\mathcal{Z} = (\alpha U + \beta)/(\gamma U + \delta)$ as

$$P_2(U) = (\gamma U + \delta)^3(\epsilon_1 \mathcal{Z} + \epsilon_2) \; . \tag{3.39}$$

The advantage of this choice of parameters will become evident when we perform a transformation from U to $\mathcal{Z}$ in the scalar potential.

3.2 New variables and R-R background fluxes

The T-duality invariant superpotential in (2.28) depends on the complex structure parameter U through the three cubic polynomials $P_1(U)$, $P_2(U)$ and $P_3(U)$ induced respectively by the R-R $\bar{F}_3$, the NS-NS $\bar{H}_3$ and the non-geometric Q-fluxes. Our results in last section show that the last two polynomials can be concisely written as

$$P_2(U) = (\gamma U + \delta)^3 \mathcal{P}_2(\mathcal{Z}) \qquad \text{and} \qquad P_3(U) = (\gamma U + \delta)^3 \mathcal{P}_3(\mathcal{Z}) \; , \tag{3.40}$$

where $\mathcal{Z} = (\alpha U + \beta)/(\gamma U + \delta)$. The real parameters $(\alpha, \beta, \gamma, \delta)$, with $|\Gamma| = (\alpha\delta - \beta\gamma) \neq 0$, encode the non-geometric Q-fluxes while for the NS-NS fluxes $\bar{H}_3$ two additional real constants (ϵ_1, ϵ_2) are needed. As summarised in table 3.1, the polynomials $\mathcal{P}_2(\mathcal{Z})$ and $\mathcal{P}_3(\mathcal{Z})$ take very specific forms according to the Q-algebra underlying the non-geometric Q flux background.

$\mathfrak{g}_{gauge}$	$\mathcal{P}_3(\mathcal{Z})/3$	$\mathcal{P}_2(\mathcal{Z})$	$\mathcal{P}_1(\mathcal{Z})$
$\mathfrak{so}(3,1)$	$-\mathcal{Z}(\mathcal{Z}^2+1)$	$\epsilon_1\mathcal{Z}^3 - 3\epsilon_2\mathcal{Z}^2 - 3\epsilon_1\mathcal{Z} + \epsilon_2$	$\xi_3\,(\epsilon_1 + 3\epsilon_2\mathcal{Z} - 3\epsilon_1\mathcal{Z}^2 - \epsilon_2\mathcal{Z}^3)$ $+\,3\,\xi_7\,(\mathcal{Z}^2+1)$
$\mathfrak{su}(2)^2$	$\mathcal{Z}(\mathcal{Z}+1)$	$\epsilon_1\mathcal{Z}^3 + \epsilon_2$	$\xi_3\,(\epsilon_1 - \epsilon_2\mathcal{Z}^3) + 3\,\xi_7\,\mathcal{Z}(1-\mathcal{Z})$
$\mathfrak{su}(2)+\mathfrak{u}(1)^3$	$\mathcal{Z}$	$\epsilon_1\mathcal{Z}^3 + \epsilon_2$	$\xi_3\,(\epsilon_1 - \epsilon_2\mathcal{Z}^3) - 3\,\xi_7\,\mathcal{Z}^2$
$\mathfrak{su}(2)\oplus_{\mathbb{Z}_3}\mathfrak{u}(1)^3$	$1-\mathcal{Z}$	$\epsilon_1\mathcal{Z} + \epsilon_2$	$3\lambda_1\mathcal{Z} + 3\lambda_2\mathcal{Z}^2 + \lambda_3\mathcal{Z}^3$
$\mathfrak{nil}$	1	$\epsilon_1\mathcal{Z} + \epsilon_2$	$3\lambda_1\mathcal{Z} + 3\lambda_2\mathcal{Z}^2 + \lambda_3\mathcal{Z}^3$

Table 3.1: Q-algebras and flux-induced polynomials.

A very nice property of the variable $\mathcal{Z}$ introduced in (3.14) is its invariance under $\mathrm{SL}(2,\mathbb{Z})_U$ modular transformations

$$U' = \frac{k\,U + \ell}{m\,U + n} \qquad \text{with} \quad k,\,\ell,\,m,\,n\,\in\mathbb{Z} \qquad \text{and} \quad kn - \ell m = 1\ . \tag{3.41}$$

Since this is a symmetry of the compactification, the effective action must be invariant. The Kähler potential, $K = -3\log[-i(U-\bar{U})] + \cdots$, clearly transforms as

$$K' = K + 3\log|mU+n|^2\ . \tag{3.42}$$

Therefore, the physically important quantity $e^K|W|^2$ is invariant as long as the superpotential satisfies

$$W' = \frac{W}{(mU+n)^3}\ . \tag{3.43}$$

In order for W to fulfil this property the fluxes must transform in definite patterns. In fact, it follows that (3.43) holds separately for each of the flux-induced polynomials in the superpotential.

We claim that the fluxes transform under $\mathrm{SL}(2,\mathbb{Z})_U$ precisely in such a manner that

$$\mathcal{Z}' \;=\; \mathcal{Z}\ . \tag{3.44}$$

The proof begins by first finding how the non-geometric Q-fluxes mix among themselves from the condition $P_3'(U) = P_3(U)/(mU+n)^3$. For example, under $U' = -1/U$, the Q-fluxes transform as

$$c_0' = -c_3 \quad , \quad c_1' = c_2 \quad , \quad c_2' = -c_1 \quad , \quad c_3' = c_0 \quad , \quad \tilde{c}_1' = \tilde{c}_2 \quad , \quad \tilde{c}_2' = -\tilde{c}_1\ . \tag{3.45}$$

Next we read off the corresponding transformation of the parameters $(\alpha, \beta, \gamma, \delta)$ that are better thought of as the elements of a matrix Γ. The result is

$$\Gamma' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} n & -\ell \\ -m & k \end{pmatrix} , \tag{3.46}$$

so it easily follows that $\mathcal{Z}' = \mathcal{Z}$. One additionally has that $|\Gamma'| = |\Gamma|$.

For the NS-NS flux $\bar{H}_3$ we can study the transformation of $P_2(U)$ with coefficients given by the b_A. Alternatively, we may start from $P_2(U)$ written as function of $\mathcal{Z}$ as in (3.40). The conclusion is that the transformation of the b_A is also determined by Γ' together with

$$(\epsilon_1', \epsilon_2') \quad = \quad (\epsilon_1, \epsilon_2) \ , \tag{3.47}$$

for all the Q-algebras.

At this point it must be evident that we want to change variables from U to $\mathcal{Z}$. It is also convenient to trade the axiodilaton S and the Kähler modulus T by new fields defined by

$$\mathcal{S} = S + \xi_s \qquad \text{and} \qquad \mathcal{T} = T + \xi_t \ , \tag{3.48}$$

where the shifts ξ_s and ξ_t are some real parameters. The motivation is that such shifts in the axions $\text{Re}\, S$ and $\text{Re}\, T$ can be reabsorbed into R-R fluxes as explained in the following.

3.2.1 Parameterisation of R-R fluxes

The systematic procedure is to express the R-R fluxes a_A in such a way that their contribution to the superpotential is of the form

$$P_1(U) = (\gamma U + \delta)^3 \widehat{\mathcal{P}}_1(\mathcal{Z}) \ , \tag{3.49}$$

in complete analogy with (3.40). To arrive at this factorisation we must relate the four R-R fluxes a_A to the parameters $(\alpha, \beta, \gamma, \delta)$ that define $\mathcal{Z} = (\alpha U + \beta)/(\gamma U + \delta)$, and to four additional independent variables. Obviously, $\widehat{\mathcal{P}}_1(\mathcal{Z})$ can be expanded in the basis of monomials $\{1\, , \, \mathcal{Z}\, , \, \mathcal{Z}^2\, , \, \mathcal{Z}^3\}$. However, a more convenient basis contains the already known polynomials $\mathcal{P}_3(\mathcal{Z})$ and $\mathcal{P}_2(\mathcal{Z})$ that are generically linearly independent. We still need two independent polynomials and these are taken to be the duals $\widetilde{\mathcal{P}}_3(\mathcal{Z})$ and $\widetilde{\mathcal{P}}_2(\mathcal{Z})$. The dual $\widetilde{\mathcal{P}}_i(\mathcal{Z})$ is such that $\mathcal{P}_i(\mathcal{Z}) \to \widetilde{\mathcal{P}}_i(\mathcal{Z})/\mathcal{Z}^3$ when $\mathcal{Z} \to -1/\mathcal{Z}$. The last two subalgebras in table 3.1 must be treated slightly different because linear independence of $\mathcal{P}_3(\mathcal{Z})$ and $\mathcal{P}_2(\mathcal{Z})$ fails for particular values of the NS-NS $\bar{H}_3$ flux parameters (ϵ_1, ϵ_2).

We concretely make the expansion

$$\widehat{\mathcal{P}}_1(\mathcal{Z}) = \xi_s\, \mathcal{P}_2(\mathcal{Z}) + \xi_t\, \mathcal{P}_3(\mathcal{Z}) + \mathcal{P}_1(\mathcal{Z}) \ . \tag{3.50}$$

In the full superpotential the first two terms in $\widehat{\mathcal{P}}_1(\mathcal{Z})$ will precisely offset the axionic shifts in the new variables $\mathcal{S}$ and $\mathcal{T}$ shown in (3.48). Let us now discuss the remaining piece $\mathcal{P}_1(\mathcal{Z})$ that also depends on the Q-algebra and is displayed in table 3.1. As explained before, for the first three subalgebras in the table we can further choose

$$\mathcal{P}_1(\mathcal{Z}) = \xi_7\,\widetilde{\mathcal{P}}_3(\mathcal{Z}) - \xi_3\,\widetilde{\mathcal{P}}_2(\mathcal{Z}) \ . \tag{3.51}$$

A motivation for this choice is that the R-R tadpoles turn out to depend on the R-R fluxes only through the real coefficients (ξ_3, ξ_7).

For the last two subalgebras in table 3.1, $\mathcal{P}_3(\mathcal{Z})$ and $\mathcal{P}_2(\mathcal{Z})$ are not independent when ϵ_1 takes a particular critical value. For $\mathfrak{su}(2) \oplus_{\mathbb{Z}_3} \mathfrak{u}(1)^3$ this happens when $\epsilon_1 = -\epsilon_2$, whereas for the nilpotent Q-algebra the critical value is $\epsilon_1 = 0$. To take into account these possibilities, compensating at the same time for the axionic shifts, we still make the decomposition in (3.50) but with

$$\mathcal{P}_1(\mathcal{Z}) = 3\lambda_1\,\mathcal{Z} + 3\,\lambda_2\,\mathcal{Z}^2 + \lambda_3\,\mathcal{Z}^3 \ . \tag{3.52}$$

Away from the critical values of the ϵ_1 parameter we can take $\lambda_1 = 0$ because ξ_s and ξ_t are independent parameters. At the critical value necessarily $\lambda_1 \neq 0$ but in this case ξ_s and ξ_t enter in the R-R fluxes in only one linearly independent combination. The R-R flux-induced tadpoles happen to depend just on the parameters (λ_2, λ_3).

The next step is to compare the expansion of $P_1(U)$ in U with its factorised form, c.f. (3.49) and (2.68). In this way we can obtain an explicit parameterisation of the R-R fluxes a_A in terms of the variables that determine $\widehat{\mathcal{P}}_1(\mathcal{Z})$, namely (ξ_s, ξ_t) together with (ξ_3, ξ_7) or $(\lambda_1, \lambda_2, \lambda_3)$, depending on the Q-algebra. These results are collected in the appendix B . We stress that the ξ's and λ's are real parameters but the emerging R-R fluxes must be integers.

$\mathfrak{g}_{gauge}$	$N_3/\lvert\Gamma\rvert^3$	$N_7/\lvert\Gamma\rvert^3$
$\mathfrak{so}(3,1)$	$4\,(\epsilon_1^2 + \epsilon_2^2)\,\xi_3$	$4\,\xi_7$
$\mathfrak{su}(2)^2$	$(\epsilon_1^2 + \epsilon_2^2)\,\xi_3$	$2\,\xi_7$
$\mathfrak{su}(2) + \mathfrak{u}(1)^3$	$(\epsilon_1^2 + \epsilon_2^2)\,\xi_3$	ξ_7
$\mathfrak{su}(2) \oplus_{\mathbb{Z}_3} \mathfrak{u}(1)^3$	$\lambda_2\,\epsilon_1 - \lambda_3\,\epsilon_2$	$\lambda_2 + \lambda_3$
$\mathfrak{nil}$	$\lambda_2\,\epsilon_1 - \lambda_3\,\epsilon_2$	λ_3

Table 3.2: Q-algebras and R-R flux-induced tadpoles.

A vacuum solution in which the moduli $(\mathcal{Z}, \mathcal{S}, \mathcal{T})$ are fixed generically requires specific values of the non-geometric Q, NS-NS $\bar{H}_3$ and R-R $\bar{F}_3$ fluxes. These fluxes also generate R-R flux-induced tadpoles that must be balanced by adding orientifold planes or D-branes. To determine the type of sources that must be included we need to evaluate the R-R tadpole cancellation conditions using all parameterised fluxes. Substituting into (2.77) and (2.78) we arrive at the very compact expressions for the number of sources N_3 and N_7 gathered in table 3.2. As advertised before, the R-R fluxes only enter either through the parameters (ξ_3, ξ_7) or (λ_2, λ_3). The non-geometric Q and NS-NS $\bar{H}_3$ fluxes only contribute through $|\Gamma|^3$ and (ϵ_1, ϵ_2). We will see that there is also a clear correlation between the tadpoles and the VEVs of the moduli fields.

Finally, let us remark that, just like (ϵ_1, ϵ_2), the ξ's and λ's variables are all invariant under $\mathrm{SL}(2,\mathbb{R})_U$ modular transformations on the complex structure U. Indeed, from the explicit parameterisation of the R-R fluxes a_A we deduce that their correct behaviour under $\mathrm{SL}(2,\mathbb{Z})_U$, analogous to (3.45), precisely follows from the transformation of $(\alpha, \beta, \gamma, \delta)$ in (3.46). This is of course consistent with the fact that the number of sources N_3 and N_7 in the tadpole cancellation conditions are physical quantities that must be modular invariant.

3.2.2 Moduli potential in the new variables

We have just seen how a systematic parameterisation of the fluxes has guided us to new moduli fields denoted $(\mathcal{Z}, \mathcal{S}, \mathcal{T})$. As we may expect, the effective action in the transformed variables also takes a form more suitable for finding vacua. The shifts in the axionic real parts of the axiodilaton and the Kähler field do not affect the Kähler potential K whereas in the superpotential W they can be reabsorbed in R-R fluxes. On the other hand, the change from the complex structure U to $\mathcal{Z}$ is the $\mathrm{SL}(2,\mathbb{R})$ transformation $U = (\beta - \delta\mathcal{Z})/(\gamma\mathcal{Z} - \alpha)$ whose effect on K and W is completely analogous to a modular transformation except for factors of $|\Gamma| = (\alpha\delta - \beta\gamma)$.

Combining previous results we obtain $e^K|W|^2 \to e^{\mathcal{K}}|\mathcal{W}|^2$, where the transformed Kähler potential $\mathcal{K}$ and superpotential $\mathcal{W}$ are given by

$$\begin{aligned} \mathcal{K} &= -3\,\log\left(-i\,(\mathcal{Z}-\bar{\mathcal{Z}})\right) - \log\left(-i\,(\mathcal{S}-\bar{\mathcal{S}})\right) - 3\,\log\left(-i\,(\mathcal{T}-\bar{\mathcal{T}})\right) \quad , \\ \mathcal{W} &= |\Gamma|^{3/2}\left[\,\mathcal{P}_1(\mathcal{Z}) + \mathcal{S}\,\mathcal{P}_2(\mathcal{Z}) + \mathcal{T}\,\mathcal{P}_3(\mathcal{Z})\,\right] \quad . \end{aligned} \tag{3.53}$$

The flux-induced polynomials $\mathcal{P}_i(\mathcal{Z})$ are displayed in table 3.1 for each Q-algebra. In the effective four-dimensional action with $\mathcal{N}=1$ supersymmetry, the functions $\mathcal{K}$ and $\mathcal{W}$ determine the scalar potential of the moduli according to

$$V = e^{\mathcal{K}}\left(\sum_{\Phi=\mathcal{Z},\mathcal{S},\mathcal{T}} \mathcal{K}^{\Phi\bar{\Phi}}|D_\Phi\mathcal{W}|^2 - 3|\mathcal{W}|^2\right) . \tag{3.54}$$

In this chapter we will be interested in supersymmetric minima for which the Kähler derivative $D_\Phi\mathcal{W} = \partial_\Phi\mathcal{W} + \mathcal{W}\,\partial_\Phi\mathcal{K} = 0$ for all moduli fields.

3.3 Supersymmetric vacua

This section is devoted to searching for supersymmetric vacua of the moduli potential induced by R-R $\bar{F}_3$, NS-NS $\bar{H}_3$ and non-geometric Q fluxes together. We will show that by using our new variables the problem simplifies substantially and analytic solutions are feasible.

Supersymmetric vacua are characterised by the vanishing of the F-terms. In our setup these conditions are written as

$$D_{\mathcal{Z}}\mathcal{W} = \frac{\partial \mathcal{W}}{\partial \mathcal{Z}} + \frac{3\,i\,\mathcal{W}}{2\,\mathrm{Im}\,\mathcal{Z}} = 0 \;\;,\;\; D_{\mathcal{S}}\mathcal{W} = \frac{\partial \mathcal{W}}{\partial \mathcal{S}} + \frac{i\,\mathcal{W}}{2\,\mathrm{Im}\,\mathcal{S}} = 0 \;\;,\;\; D_{\mathcal{T}}\mathcal{W} = \frac{\partial \mathcal{W}}{\partial \mathcal{T}} + \frac{3\,i\,\mathcal{W}}{2\,\mathrm{Im}\,\mathcal{T}} = 0, \tag{3.55}$$

so the task will be to determine whether there are solutions with the moduli fields completely stabilised at VEVs denoted

$$\mathcal{Z}_0 = x_0 + i\,y_0 \quad , \quad \mathcal{S}_0 = s_0 + i\,\sigma_0 \quad , \quad \mathcal{T}_0 = t_0 + i\,\mu_0 \;. \tag{3.56}$$

It is worth noticing that, at any supersymmetric moduli solution satisfying (3.55), the potential energy in (3.54) is given by

$$V_0 \;\; = \;\; -3\,e^{\mathcal{K}_0}|\mathcal{W}_0|^2 \leq 0 \;. \tag{3.57}$$

This translates into the well known fact that supersymmetric vacua are forced to be either Minkowski or AdS_4. Then obtaining dS solutions necessarily implies supersymmetry to be broken at the vacuum. We will carry out an exhaustive search of non-supersymmetric dS solutions within these T-duality invariant supergravity flux models in the chapter 4.

Besides stabilisation, there are further physical requirements. At the minimum the imaginary part of the axiodilaton in (2.12), $\sigma_0 = e^{-\varphi}$, must be positive for the reason it is the inverse of the string coupling constant g_s as shown in (15). It can be argued that the geometric moduli are subject to similar conditions. The main assumption is that they arise from the metric of the internal space, which is $\mathbb{T}^6$ in absence of fluxes. In particular, the Kähler modulus in (2.14) has $\mathrm{Im}\,T = e^{-\varphi}A^2$, where A^2 is the area of a 4-dimensional subtorus. Hence, it must be $\mu_0 > 0$. Notice also that the internal volume is measured by $V_{int} = (\mu_0/\sigma_0)^{3/2}$. For the transformed complex structure $\mathcal{Z}$ it happens that $\mathrm{Im}\,\mathcal{Z} = |\Gamma|\,\mathrm{Im}\,U/|\gamma U + \delta|^2$. Therefore, necessarily $\mathrm{Im}\,\mathcal{Z}_0 = y_0 \neq 0$ because for $\mathrm{Im}\,U_0 = 0$ the internal space is degenerate. Without loss of generality we choose that $\mathrm{Im}\,U_0$ is always positive.

Another physical issue is whether the moduli take values such that the effective supergravity action is a reliable approximation to String Theory. Specifically, the string coupling $g_s = 1/\sigma_0$ is expected to be small to justify the exclusion of non-perturbative string effects. Conventionally, there is also a requirement of large internal volume to disregard corrections in α'. However, in the presence of non-geometric Q-fluxes the internal space might be a T-fold in which there can exist cycles with sizes related by T-duality [43,164]. Thus, for large volume there could be tiny cycles whose associated winding modes would be light (see section 1.3). To date these effects are not well understood. At any rate, we limit ourselves to finding supersymmetric vacua of an effective field theory defined by a very precise Kähler potential and flux-induced superpotential. A more detailed discussion of the landscape of vacua is left for section 3.4. We will see that the moduli can be fixed at small string coupling and cosmological constant.

In the following we will first consider supersymmetric Minkowski vacua that have $\mathcal{W} = 0$ at the minimum. In our approach it is straightforward to show that for isotropic fluxes such vacua are disallowed. We then turn our attention to the richer class of AdS_4 vacua. Since superpotential terms adopt very specific forms depending on the particular subalgebra satisfied by the non-geometric Q-fluxes, we will study the corresponding vacua case by case. We will mostly focus on the model associated to the non-geometric Q-fluxes of the compact $\mathfrak{su}(2)^2$ but will also consider other allowed subalgebras to some extent.

3.3.1 Minkowski vacua

Minkowski solutions with zero cosmological constant require that the potential vanishes. Imposing supersymmetry further implies that the superpotential must be zero at the minimum $(\mathcal{Z}_0, \mathcal{S}_0, \mathcal{T}_0)$. A key property of the superpotential in (3.53) is its linearity in $\mathcal{S}$ and $\mathcal{T}$. This implies in particular that the F-flatness conditions $D_{\mathcal{S}}\mathcal{W} = 0$ and $D_{\mathcal{T}}\mathcal{W} = 0$, together with $\mathcal{W} = 0$, reduce just to

$$\mathcal{P}_3(\mathcal{Z}_0) \;=\; \mathcal{P}_2(\mathcal{Z}_0) \;=\; \mathcal{P}_1(\mathcal{Z}_0) = 0 \;. \tag{3.58}$$

The third condition $D_{\mathcal{Z}}\mathcal{W} = 0$ yields a linear relation between $\mathcal{S}_0$ and $\mathcal{T}_0$ so that not all moduli can be stabilised. The situation is actually worse because (3.58) cannot be fulfilled appropriately. Indeed, for the specific polynomials for each subalgebra shown in table 3.1, it is evident that $\mathcal{P}_3(\mathcal{Z})$ and $\mathcal{P}_2(\mathcal{Z})$ can only have a common real root $\mathcal{Z}_0$. But then $\text{Im}\, U_0 = \text{Im}\, \mathcal{Z}_0 = 0$ and this is inconsistent with a well defined internal space.

It must be emphasised that we are assuming that non-geometric Q-fluxes, and their induced $\mathcal{P}_3(\mathcal{Z})$, are non-trivial. Our motivation is to fix the Kähler modulus without invoking non-perturbative effects. If only R-R $\bar{F}_3$ and NS-NS $\bar{H}_3$ fluxes are turned on there do exist physical supersymmetric Minkowski vacua in which only the axiodilaton and the complex structure are stabilised [81,85]. In such solutions the gauge fluxes $(\bar{F}_3, \bar{H}_3)$ must still satisfy a non-linear constraint [85,174].

No-go results for supersymmetric Minkowski vacua in the presence of non-geometric Q-fluxes have been previously obtained [8,23,174][3]. In ref. [8] their existence was disproved

[3] In ref. [23] it is further shown that Minkowski vacua with all moduli stabilised can exist in more general

assuming special solutions for the Jacobi identities in (2.74). We are now extending the proof to all possible non-trivial *isotropic* non-geometric Q-fluxes solving these constraints.

3.3.2 AdS$_4$ vacua

We now want to solve the supersymmetry conditions when $\mathcal{W} \neq 0$. The three complex equations $D_\Phi \mathcal{W} = 0$ with $\Phi = \mathcal{Z}, \mathcal{S}, \mathcal{T}$, in principle admit solutions with all moduli fixed at values $\mathcal{Z}_0 = x_0 + iy_0$, $\mathcal{S}_0 = s_0 + i\sigma_0$, and $\mathcal{T}_0 = t_0 + i\mu_0$. We will also impose the physical requirements $\sigma_0 > 0$, $\mu_0 > 0$ and $\mathrm{Im}\, U_0 > 0$ which implies $|\Gamma|\, y_0 > 0$. In general existence of such solutions demands that the fluxes satisfy some specific properties.

In the AdS$_4$ vacua, the polynomials $\mathcal{P}_2(\mathcal{Z})$ and $\mathcal{P}_3(\mathcal{Z})$ are necessarily different from zero. Moreover, combining the equations $D_\mathcal{S}\mathcal{W} = 0$ and $\mathrm{D}_\mathcal{T}\mathcal{W} = 0$ shows that at the minimum $\mathrm{Im}\, (\mathcal{P}_3(\mathcal{Z})/\mathcal{P}_2(\mathcal{Z})) = 0$, or equivalently

$$\mathcal{P}_3(\mathcal{Z})\,\mathcal{P}_2(\mathcal{Z})^* \;-\; \mathcal{P}_3(\mathcal{Z})^*\,\mathcal{P}_2(\mathcal{Z})\Big|_0 = 0\ . \tag{3.59}$$

From this condition we can quickly extract useful information. For example, for the polynomials of the nilpotent subalgebra we find that $\epsilon_1 = 0$. Similarly, for the semidirect product $\mathfrak{su}(2) \oplus_{\mathbb{Z}_3} \mathfrak{u}(1)^3$, it follows that $\epsilon_1 = -\epsilon_2$. Thus, in these two cases $\mathcal{P}_2(\mathcal{Z})$ and $\mathcal{P}_3(\mathcal{Z})$ are forced to be parallel and the equation (3.59) is inconsequential for the moduli. Having one equation less means that all moduli cannot be fixed simultaneously. In fact, what happens is that only a linear combination of the axions s_0 and t_0 is determined [93].

Another instructive example is that of the $\mathfrak{su}(2) + \mathfrak{u}(1)^3$ Q-algebra. With the polynomials provided in table 3.1 the supersymmetry condition in (3.59) implies

$$\epsilon_2 \;-\; 2\,\epsilon_1\, x_0\, (x_0^2 \;+\; y_0^2) = 0\ , \tag{3.60}$$

where we already used that $y_0 \neq 0$. Now we see that forcefully $\epsilon_1 \neq 0$ because otherwise ϵ_2, and thus $\mathcal{P}_2(\mathcal{Z})$ itself, would vanish. However, it could be $\epsilon_2 = 0$ and then $x_0 = 0$. If $\epsilon_2 \neq 0$ we will just have one equation that gives y_0 in terms of x_0.

In other examples with $\mathcal{P}_2(\mathcal{Z})$ and $\mathcal{P}_3(\mathcal{Z})$ not parallel there are analogous results. It can happen that the supersymmetry condition in (3.59) already fixes x_0 or it gives y_0 as function of x_0. The remaining five equations can be used to obtain $\mathcal{S}_0$ and $\mathcal{T}_0$ in terms of y_0 or x_0, and to find a polynomial equation that determines y_0 or x_0. This procedure can be efficiently carried out using the algebraic package *Singular*. The results are described below in more detail.

The superpotential for each Q-algebra is constructed with the flux-induced polynomials listed in table 3.1. The number of sources needed to cancel tadpoles are given in

setups having more complex structure than Kähler moduli (in our type IIB language).

table 3.2. Recall that O3-planes (D3-branes) make a positive (negative) contribution to N_3, whereas O7-planes (D7-branes) yield negative (positive) values of N_7.

Each supersymmetric vacua can be distinguished by the modular invariant values of the string coupling constant g_s and the potential at the minimum V_0 that is equal to the cosmological constant up to normalisation. In the models at hand these quantities are given by

$$V_0 = -\frac{3|\mathcal{W}_0|^2}{128\, y_0^3\, \mu_0^3\, \sigma_0} \qquad \text{and} \qquad g_s = \frac{1}{\sigma_0} \; . \tag{3.61}$$

In all examples the VEVs of the moduli y_0, σ_0, μ_0, as well as the value $\mathcal{W}_0$ of the superpotential at the minimum, can be completely determined and will be given explicitly. It is then straightforward to evaluate the characteristic data (g_s, V_0).

3.3.2.1 The $\mathfrak{nil}$ case

When $\epsilon_1 = 0$, the model based on the non-geometric Q-fluxes of the nilpotent subalgebra is $U \leftrightarrow T$ dual to a type IIA orientifold with only R-R $\bar{F}_3$ and NS-NS $\bar{H}_3$ fluxes already considered in the literature [92, 93]. Supersymmetry actually requires $\epsilon_1 = 0$. There are some salient features that are easily reproduced in our setup. For instance, a solution exists only if $\lambda_3 \neq 0$ and $(\lambda_1\,\lambda_3 - \lambda_2^2) > 0$. The axions s_0 and t_0 can only be fixed in the linear combination

$$3\, t_0 \, + \, \epsilon_2\, s_0 = \frac{\lambda_2}{\lambda_3^2}\,(3\,\lambda_1\,\lambda_2 \, - \, 2\,\lambda_2^2) \; . \tag{3.62}$$

The rest of the moduli fields are determined as

$$x_0 = -\frac{\lambda_2}{\lambda_3} \quad , \quad y_0^2 = \frac{5(\lambda_1\lambda_3 - \lambda_2^2)}{3\lambda_3^2} \quad , \quad \sigma_0 = -\frac{2(\lambda_1\lambda_3 - \lambda_2^2)y_0}{3\epsilon_2\lambda_3} \quad , \quad \mu_0 = \epsilon_2\sigma_0 \; , \tag{3.63}$$

and the cosmological constant can be computed using $\mathcal{W}_0 = 2\,i\,\mu_0\,|\Gamma|^{3/2}$.

From the results we deduce that $\epsilon_2 > 0$, and $\lambda_3 > 0$ for $y_0 < 0$. Then $\operatorname{Im} U_0 > 0$ requires $|\Gamma| < 0$ as it happens for the nilpotent algebra. The tadpole conditions then verify $N_3 = -\lambda_3\,\epsilon_2\,|\Gamma|^3 > 0$ and $N_7 = \lambda_3\,|\Gamma|^3 < 0$. The relevant conclusion is that the model necessarily requires O3-planes and O7-planes.

3.3.2.2 The $\mathfrak{iso}(3)$ case

The non-geometric Q-fluxes of this subalgebra are $U \leftrightarrow T$ dual to NS-NS $\bar{H}_3$ plus *geometric* fluxes ω in a type IIA orientifold. Supergravity models of this type have been studied previously [84, 93, 175, 176]. For completeness we will briefly summarise our results that totally agree with the general solution presented in [93]. Existence of a supersymmetric minimum imposes the constraint $\epsilon_1 = -\epsilon_2$. In this case it occurs again that the axions s_0 and t_0 can only be determined in a linear combination given by

$$3\, t_0 \, + \, \epsilon_2\, s_0 = 3\,\lambda_1 \, + \, 3\,\lambda_2\,(9 - 7x_0) \, + \, 3\,\lambda_3\,x_0\,(9 - 8x_0) \; . \tag{3.64}$$

The imaginary parts of the Kähler and the axiodilaton fields are stabilised at values

$$\mu_0 = \epsilon_2\,\sigma_0 \qquad , \qquad \epsilon_2\,\sigma_0 = 6\,(\lambda_2 + \lambda_3\,x_0)\,y_0 \ , \tag{3.65}$$

so the NS-NS ϵ_2 parameter must be positive. It also follows that $\mathcal{W}_0 = 2\,i\,\mu_0\,(1 - x_0 - i\,y_0)\,|\Gamma|^{3/2}$ determines the AdS$_4$ vacuum energy.

The VEVs of the x_0 and y_0 real axions depend on whether the R-R flux parameter λ_3 is zero or not:

i) If $\lambda_3 = 0$ we obtain

$$x_0 = 1 \qquad \text{and} \qquad 3\,\lambda_2\,y_0^2 = -(\lambda_1 + \lambda_2)\ . \tag{3.66}$$

Notice that $\lambda_2 \neq 0$ to guarantee $\sigma_0 \neq 0$. In fact, choosing $y_0 > 0$ it must be $\lambda_2 > 0$. For the number of sources we find $N_3 = -\lambda_2\,\epsilon_2\,|\Gamma|^3 < 0$ and $N_7 = \lambda_2\,|\Gamma|^3 > 0$. Therefore, D3-branes and D7-branes must be included.

ii) If $\lambda_3 \neq 0$ we instead find

$$\lambda_3\,y_0^2 = 15\,(x_0 - 1)\,(\lambda_2 + \lambda_3\,x_0)\ , \tag{3.67}$$

whereas the axion x_0 must be a root of the cubic equation

$$160(x_0-1)^3 + 294\left(1 + \frac{\lambda_2}{\lambda_3}\right)(x_0-1)^2 + 135\left(1 + \frac{\lambda_2}{\lambda_3}\right)^2(x_0-1) + \frac{1}{\lambda_3}(\lambda_3 + 3\lambda_2 + 3\lambda_1) = 0. \tag{3.68}$$

The solution for x_0 must be real and such that $y_0^2 > 0$. For the tadpoles we now have $N_7 = |\Gamma|^3\,(\lambda_2 + \lambda_3)$ and $N_3 = -\epsilon_2\,N_7$. Thus, in general N_3 and N_7 have opposite signs. The remarkable feature is that now they can be zero simultaneously. This occurs when the R-R parameters satisfy $\lambda_2 = -\lambda_3$, in which case the cubic equation for x_0 can be solved exactly.

3.3.2.3 The $\mathfrak{su}(2) + \mathfrak{u}(1)^3$ case

As explained before, necessarily $\epsilon_1 \neq 0$. Let us consider $\epsilon_2 = 0$ which is the condition for $\mathcal{P}_2(\mathcal{Z})$ to have only real roots. Now it happens that all moduli fields can be determined. The axions are fixed at $x_0 = 0$, $s_0 = 0$ and $t_0 = 0$. The imaginary parts have VEVs

$$y_0^2 = \frac{\epsilon_1\,\xi_3}{\xi_7} \qquad , \qquad \sigma_0 = -\frac{2\,\xi_7^2\,y_0}{\epsilon_1^2\,\xi_3} \qquad , \qquad \mu_0 = 2\,\xi_7\,y_0\ , \tag{3.69}$$

and the cosmological constant is easily found substituting $\mathcal{W}_0 = -2\,\mu_0\,y_0\,|\Gamma|^{3/2}$.

Clearly, the solution exists only if $\xi_3 \neq 0$ and $\xi_7 \neq 0$. Moreover, $\epsilon_1\,\xi_3\,\xi_7 > 0$ and if we take $y_0 > 0$, $\xi_3 < 0$, $\xi_7 > 0$ and $\epsilon_1 < 0$. The number of sources satisfy $N_3 < 0$ and $N_7 > 0$, so that D3-branes and D7-branes are needed.

Taking $\epsilon_2 \neq 0$ we deduce that there are no solutions at all when $\xi_7 = 0$ and $\xi_3 \neq 0$. However, there are minima that require $\epsilon_1 < 0$ and $N_7 > 0$ when $\xi_3 = 0$.

3.3.2.4 The $\mathfrak{so}(3,1)$ case

This is the only flux configuration for which $\mathcal{P}_3(\mathcal{Z})$ has complex roots. It also happens that $\mathcal{P}_2(\mathcal{Z})$ always has three non-degenerate real roots. We will briefly discuss the vacua according to whether the NS-NS flux parameter ϵ_2 vanishes or not.

i) Taking $\epsilon_2 = 0$: In this setup the axions are determined to be $x_0 = 0$, $s_0 = 0$ and $t_0 = 0$. For the imaginary parts of the Kähler modulus and the axiodilaton we obtain

$$\mu_0 = \frac{\epsilon_1\, \sigma_0\, (3 + y_0^2)}{(1 - y_0^2)} \qquad \text{and} \qquad \epsilon_1\, \sigma_0 = \frac{3\, \xi_7\, (y_0^2 - 1) - \epsilon_1\, \xi_3\, (3y_0^2 + 1)}{2\, y_0\, (3 + y_0^2)} \ . \tag{3.70}$$

To evaluate the potential at the minimum we use $\mathcal{W}_0 = 2\, \mu_0\, y_0\, (1 - y_0^2)\, |\Gamma|^{3/2}$. Notice that ξ_3 and ξ_7 cannot be zero simultaneously and that $y_0^2 = 1$ is not allowed, so we are interested in real roots $y_0 \neq 0$ and $y_0 \neq \pm 1$. Actually, the imaginary part of the transformed complex structure satisfies a third order polynomial equation in y_0^2 given by

$$\epsilon_1\, \xi_3\, (5\, y_0^6 + 13\, y_0^4 + 15\, y_0^2 - 1) - \xi_7\, (y_0^2 - 1)\, (5\, y_0^4 + 6\, y_0^2 - 3) = 0 \ . \tag{3.71}$$

Although we have not made an exhaustive analysis, it is clear that the solutions of (3.71) depend on the range of the ratio $\xi_7 / \epsilon_1\, \xi_3$. For instance, there are values for which there is no real root at all, as it occurs e.g. for $2\, \xi_7 = -\epsilon_1\, \xi_3$. For other values there might be only one real positive solution for y_0^2. An special example happens when $\xi_3 = 0$ and the net O3/D3 charge N_3 is zero, while the net O7/D7 charge N_7 is negative as implied by the conditions $\mu_0 > 0$ and $|\Gamma|\, y_0 > 0$. Similarly, when $\xi_7 = 0$, there is only one solution in which $N_7 = 0$ while $N_3 < 0$.

The third possibility is to have two allowed solutions. For instance, taking $\xi_7 = 2\, \epsilon_1\, \xi_3$ gives roots $y_0^2 = 1/5$ and $y_0^2 = 1 + 2\sqrt{2}$. However, in principle the corresponding vacua cannot be realized simultaneously because the net charges would have to jump. In fact, for $y_0^2 < 1$, it happens that $N_3 N_7 > 0$, whereas for $y_0^2 > 1$, it must be $N_3 N_7 < 0$. It can also arise that both solutions have $y_0^2 < 1$. For example, when $\xi_7 = -30\, \epsilon_1\, \xi_3$ each of the two vacua has $N_3 > 0$ and $N_7 < 0$. We will explore the phenomenon of multiple AdS_4 vacua in more detail for the non-geometric Q-fluxes of the $\mathfrak{su}(2)^2$ algebra.

ii) Taking $\epsilon_2 \neq 0$: We have only studied the special cases when one of the flux-tadpoles N_3 or N_7 is zero. We find that when $\epsilon_1 = 0$ the F-flatness conditions can not be solved but for $\epsilon_1 > 0$ there are consistent solutions for a particular range of $|\epsilon_2/\epsilon_1|$. Vacua with $\xi_3 = 0$ exist provided that $\xi_7 < 0$. Vacua with no O7/D7 flux-tadpoles, i.e. with $\xi_7 = 0$, require $\xi_3 < 0$. One important conclusion is that, for the fluxes of the non-compact Q-algebra, solutions with $N_7 = 0$ must have $N_3 < 0$.

3.3.2.5 The $\mathfrak{su}(2)^2$ case

This is the only situation in which the polynomial $P_3(U)$ induced by the non-geometric Q-fluxes has three different real roots. The polynomial $P_2(U)$ generated by the NS-NS $\bar{H}_3$ fluxes has complex roots whenever $\epsilon_1\,\epsilon_2 \neq 0$ and one triple real root otherwise. We will study the vacua in both cases in some detail.

The full model based on the non-geometric Q-fluxes of the $\mathfrak{su}(2)^2$ subalgebra has an interesting residual symmetry that exchanges the NS-NS auxiliary parameters $\epsilon_{1,2}$. It can be shown that the effective action is invariant under $\epsilon_1 \leftrightarrow \epsilon_2$, $\xi_3 \to \xi_3$ and $\xi_7 \to \xi_7$, together with the field transformations

$$\mathcal{Z} \to 1/\mathcal{Z}^* \qquad , \qquad \mathcal{S} \to -\mathcal{S}^* \qquad , \qquad \mathcal{T} \to -\mathcal{T}^* \; . \tag{3.72}$$

This symmetry leaves one of the $\mathcal{P}_3(\mathcal{Z})$ roots invariant while exchanging the other two. We will make extensive use of it in the next chapter.

$P_2(U)$ polynomial with a triple real root

Due to the symmetry in (3.72) it is enough to consider $\epsilon_1 = 0$ and $\epsilon_2 \neq 0$. In this model the axions are stabilised at VEVs

$$x_0 = -\frac{1}{2} \qquad , \qquad \epsilon_2\, s_0 = 3\,\xi_7 - \frac{\epsilon_2\,\xi_3}{2} \qquad , \qquad t_0 = \xi_7 - \frac{\epsilon_2\,\xi_3}{2} \; . \tag{3.73}$$

The imaginary parts of the Kähler modulus and the axiodilaton are fixed in terms of y_0 according to

$$\mu_0 = -\frac{4\,\epsilon_2\,\sigma_0}{(1+4\,y_0^2)} \qquad \text{and} \qquad \epsilon_2\,\sigma_0 = -y_0\left[3\,\xi_7 + \frac{\epsilon_2\,\xi_3}{8}(4\,y_0^2-3)\right] \; . \tag{3.74}$$

At the minimum $\mathcal{W}_0 = 2\,i\,\epsilon_2\,\sigma_0\,|\Gamma|^{3/2}$. Clearly ξ_3 and ξ_7 cannot vanish simultaneously so that the model always requires additional sources to cancel tadpoles. One has that necessarily $\epsilon_2 < 0$.

The modulus y_0 is determined by the fourth order polynomial equation

$$\epsilon_2\,\xi_3\,(4\,y_0^2-1)\,(4\,y_0^2+5) - 8\,\xi_7\,(4\,y_0^2-5) = 0 \; . \tag{3.75}$$

In the two special cases $\xi_7 = 0$ and $\xi_3 = 0$ an exact solution is easily found. When $\xi_3\,\xi_7 \neq 0$ there can be two AdS$_4$ solutions. The corresponding vacua, which can be characterised by the net tadpoles N_3 and N_7, are described more extensively in the following.

i) Taking $N_7 = 0$: When $\xi_7 = 0$ the moduli VEVs and the cosmological constant have the very simple expressions

$$y_0^2 = \frac{1}{4} \quad , \quad \sigma_0 = \frac{\xi_3 y_0}{4} \quad , \quad \mu_0 = -2\epsilon_2\sigma_0 \quad , \quad V_0 = \frac{12|\Gamma|^3 y_0}{\epsilon_2 \xi_3^2} \; . \tag{3.76}$$

Since both μ_0 and σ_0 are positive, it must be $\epsilon_2 < 0$, and taking $y_0 > 0$ then $\xi_3 > 0$. Therefore $N_3 > 0$ and O3-planes must be included.

ii) Taking $N_3 = 0$: This is the case $\xi_3 = 0$. The moduli and the cosmological constant are fixed at values

$$y_0^2 = \frac{5}{4} \quad , \quad \epsilon_2\sigma_0 = -3\xi_7 y_0 \quad , \quad \mu_0 = -\frac{2}{3}\epsilon_2\sigma_0 \quad , \quad V_0 = \frac{9|\Gamma|^3\epsilon_2 y_0}{500\,\xi_7^2} \ . \tag{3.77}$$

Necessarily $\epsilon_2 < 0$, and choosing $y_0 > 0$ then $\xi_7 > 0$. Hence $N_7 > 0$ and D7-branes are required.

iii) Taking $N_3 N_7 \neq 0$: The solutions for y_0 depend on the ratio $\xi_7/\epsilon_2\,\xi_3$. A detailed analysis can be easily performed because the polynomial equation (3.75) is quadratic in y_0^2. We find that there are no real solutions in the interval $1/8 < \xi_7/\epsilon_2\,\xi_3 < (7 + 2\sqrt{10})/4$. On the other hand, when $0 < \xi_7/\epsilon_2\,\xi_3 < 1/8$, there is only one real positive solution for y_0^2 and it requires $N_3 > 0$ and $N_7 < 0$. For $\xi_7/\epsilon_2\,\xi_3 \leq 0$ there is only one acceptable root for y_0^2 and it leads to $N_3 > 0$ and $N_7 \geq 0$. A more interesting range of parameters is $\xi_7/\epsilon_2\,\xi_3 > (7 + 2\sqrt{10})/4$ because there are two allowed solutions for y_0^2 and for both it must be that $N_3 < 0$ and $N_7 > 0$. The upshot is that there can be metastable AdS_4 vacua in the presence of D3-branes and D7-branes.

$P_2(U)$ polynomial with complex roots

The F-flatness conditions can be unfolded to obtain analytic expressions for the VEVs of all the moduli. However, for generic range of parameters, a higher order polynomial equation has to be solved to determine y_0 in the end. The main interesting feature is the appearance of multiple vacua even when $N_3 N_7 = 0$, i.e. when there are either no O7/D7 or no O3/D3 net charges present. We will first describe the overall picture and then present examples. For definiteness we always choose $y_0 > 0$ so that $|\Gamma| > 0$ is required to have $\mathrm{Im}\, U_0 > 0$ for the complex structure.

To obtain and examine the results it is useful to make some redefinitions. The idea is to leave as few free parameters as possible in the F-flatness equations. Since ϵ_1 is different from zero we can work with the ratio

$$\rho = \frac{\epsilon_2}{\epsilon_1} \ . \tag{3.78}$$

By virtue of the residual symmetry in (3.72) there is an invariance under $\rho \to 1/\rho$. Therefore, we can restrict to the range $-1 \leq \rho \leq 1$, where the boundary corresponds to the fixed points of the inversion. Furthermore, as discussed at the end of section 3.1.1.1, the parameter ρ is either a rational number or involves at most square roots of rationals.

When the R-R flux parameter $\xi_3 \neq 0$ it is also convenient to introduce new variables as

$$\mathcal{T} = \epsilon_1\, \xi_3\, \hat{\mathcal{T}} \qquad , \qquad \mathcal{S} = \xi_3\, \hat{\mathcal{S}} \qquad , \qquad \xi_7 = \epsilon_1\, \xi_3\, (\rho^2+1)\, \eta\ . \tag{3.79}$$

Even though the definition of the parameter η seems awkward, it simplifies the results since $\eta \to \eta\rho$ under the transformations in (3.72). In the new variables of (3.79) the superpotential becomes

$$\mathcal{W} = |\Gamma|^{3/2}\, \epsilon_1\, \xi_3 \left[3\, \hat{\mathcal{T}}\, \mathcal{Z}(\mathcal{Z}+1) + \hat{\mathcal{S}}\, (\mathcal{Z}^3+\rho) + (1-\rho\, \mathcal{Z}^3) + 3\, \eta\, (1+\rho^2)\, \mathcal{Z}\, (1-\mathcal{Z})\right]\ . \tag{3.80}$$

Since the F-flatness conditions are homogeneous in $\mathcal{W}$ the resulting equations will only depend on the parameters ρ and η. When $\xi_3 = 0$ we just make different field redefinitions, i.e. $\mathcal{T} = \epsilon_1\, \xi_7\, \hat{\mathcal{T}}$ and $\mathcal{S} = \xi_7\, \hat{\mathcal{S}}$, so that the free parameters will be ρ and ξ_7/ϵ_1.

Manipulating the F-flatness conditions enables us to find the VEVs $\mathcal{T}_0$ and $\mathcal{S}_0$ as functions of (x_0, y_0). The expressions are tractable but bulky so that we refrain from presenting them. The exception is the handy relation between the size and string coupling moduli fields

$$\mu_0 = \frac{\epsilon_1\, \sigma_0\, (3\, x_0^2 - y_0^2)}{1 + 2\, x_0}\ , \tag{3.81}$$

which is valid when $x_0 \neq -\frac{1}{2}$ and $y_0^2 \neq \frac{3}{4}$. There is a solution with $x_0 = -\frac{1}{2}$ and $y_0^2 = \frac{3}{4}$ but it has $\mu_0 = -\epsilon_1\, (1+\rho)\, \sigma_0$, $\mu_0 = 3\, \xi_7\, y_0$ and it requires $\eta = -(1+\rho)/(\rho^2 - 7\, \rho + 1)$. There is another vacuum with $x_0 = -\frac{1}{2}$ that occurs when $\rho \to \infty$ ($\epsilon_1 = 0$) and has already been discussed. The case $x_0^2 = y_0^2$, which is better treated separately, requires $\xi_7 \neq 0$ unless $\rho = 0$.

The residual unknowns (x_0, y_0) are determined from the coupled system of high degree algebraic equations

$$\begin{aligned}
0 &= y_0^4 + 2\, x_0\, (1+x_0)\, y_0^2 - \rho\, (2\, x_0 + 1) + x_0^3\, (x_0+2)\ , \\
0 &= \left(2\, \rho + 4\, \rho\, x_0 + 11\, x_0^3 + 13\, x_0^4\right) \left(2\, \rho\, \eta + 2\, \eta\, x_0^3 + x_0^2 + x_0\right) \\
&+ y_0^6\, (1 + 2\, \eta x_0 - 2\, \eta) + \left(1 + 30\, \eta x_0^3 - x_0^2 + 18\, \eta\, x_0^2 - 6\, \rho\, \eta\right)\, y_0^4 \\
&+ x_0\, \left(54\, \eta\, x_0^4 + 11\, x_0^3 + 42\, \eta\, x_0^3 + 8\, x_0^2 + 12\, \rho\, \eta\, x_0 - 4\, x_0 - 6\, \rho\, \eta\right)\, y_0^2\ .
\end{aligned} \tag{3.82}$$

The corresponding equations when $\xi_3 = 0$ can be obtained taking the limit $\eta \to \infty$. Eliminating y_0 for generic parameters gives a ninth-order polynomial equation for x_0.

For some range of parameters the above equations can admit several solutions for $\mathcal{Z}_0 = x_0 + i\, y_0$, which in turn yield consistent values for the remaining moduli fields. The existence of multiple vacua is most easily detected in the limiting cases in which one of the net tadpoles N_7 or N_3 vanishes, equivalently when $\xi_7 = 0$ ($\eta = 0$) or $\xi_3 = 0$ ($\eta \to \infty$). In either limit the new NS-NS flux parameter ρ can still be adjusted. We expect the results to be invariant under $\rho \to 1/\rho$ and this is indeed what happens.

We have mostly looked at models having no O7/D7 net charge, namely with $\eta = 0$. It turns out that the solutions require $\xi_3 > 0$ so that $N_3 > 0$ and O3-planes must be present. Below we list the main results:

i) Taking $\rho = 1$ there are no minima with the moduli fields stabilised.

ii) Taking $\rho = -1$ there is only one distinct vacuum with data

$$\mathcal{Z}_0 = -0.876 + 1.158\, i \quad , \quad \mathcal{S}_0 = \xi_3\,(-0.381 + 0.238\, i) \quad , \quad \mathcal{T}_0 = \epsilon_1\, \xi_3 (0.602 - 0.305\, i)\ ,$$

and $V_0 = \frac{2.38\,|\Gamma|^3}{\xi_3^2\,\epsilon_1}$. Notice that necessarily $\xi_3 > 0$ and $\epsilon_1 < 0$. Actually, for $\rho = -1$, there is a second consistent solution but it is related to the above by the residual symmetry in (3.72).

iii) There can be only one solution when $\rho_c \leq \rho < 1$, where $\rho_c = -0.7267361874$. The critical value ρ_c is such that the discriminant of the polynomial equation that determines x_0 is zero. Consistency requires $\epsilon_1 < 0$ and $\xi_3 > 0$ so that O3-planes are needed. For instance, when $\rho = 0$ the solution is exact and has

$$\mathcal{Z}_0 = -1 + i \quad , \quad \mathcal{S}_0 = \frac{\xi_3}{8}(4 + i) \quad , \quad \mathcal{T}_0 = \frac{\epsilon_1\, \xi_3}{4}(2 - i) \quad , \quad V_0 = \frac{6\,|\Gamma|^3}{\xi_3^2\, \epsilon_1}\ . \tag{3.83}$$

As expected, when applying the transformation in (3.72) this vacuum coincides with that having $\xi_7 = 0$ and $\epsilon_1 = 0$, given in (3.76). For other values of the parameter ρ the solution is numerical. For example, taking $\rho = \frac{1}{2}$ leads to the VEVs

$$\mathcal{Z}_0 = -1.036 + 0.834\, i \quad , \quad \mathcal{S}_0 = \xi_3\,(1.561 + 0.192) \quad , \quad \mathcal{T}_0 = \xi_3 \epsilon_1\,(1.055 - 0.453\, i)\ ,$$

and the vacuum energy $V_0 = \frac{2.283\,|\Gamma|^3}{\xi_3^2\,\epsilon_1}$.

$\mathcal{Z}_0$	$\mathcal{S}_0/\xi_3$	$\mathcal{T}_0/\xi_3\,\epsilon_1$	$V_0\,\xi_3^2\,\epsilon_1/\lvert\Gamma\rvert^3$
$-0.91105442 + 1.14050441\, i$	$-0.26002362 + 0.19059447\, i$	$0.53128071 - 0.27572497\, i$	3.353
$-0.43550654 + 0.73478523\, i$	$0.28605555 + 0.55017649\, i$	$0.60410811 + 0.12407321\, i$	-2.168
$-0.40368586 + 0.57866160\, i$	$0.49215445 + 0.33255331\, i$	$0.57101568 + 0.26593032\, i$	-1.880

Table 3.3: Degenerate moduli flux vacua for $\xi_7 = 0$ and $\rho = -\frac{4}{5}$.

iv) The important upshot is that in the interval $-1 < \rho < \rho_c$ there can be two distinct solutions for the same set of fluxes. An example with $\rho = -\frac{4}{5}$ is shown in table 3.3. Notice that the last two solutions can exist for $\xi_3 > 0$ and $\epsilon_1 > 0$. The first solution can also occur but for $\xi_3 > 0$ and $\epsilon_1 < 0$.

For models having no O3/D3 net charge, namely with $\eta \to \infty$, a detailed analysis is clearly feasible but we have only sampled narrow ranges of the adjustable parameter ρ. Consistent solutions must have $\epsilon_1 < 0$ and $\xi_7 > 0$. Hence $N_7 > 0$ and D7-branes must be included. There are values of ρ, e.g. $\rho = -1$, for which there are no vacua with the moduli stabilised. For $\rho = 1$ there is only one minimum which can be computed exactly. More interestingly, the supergravity models of this type can also exhibit multiple vacua. In table 3.4 we show an example with $\rho = \frac{3}{4}$. Observe that both solutions exist for $\epsilon_1 < 0$ and $\xi_7 > 0$.

$\mathcal{Z}_0$	$\epsilon_1 \mathcal{S}_0/\xi_7$	$\mathcal{T}_0/\xi_7$	$V_0\, \xi_7^2/\epsilon_1 \lvert\Gamma\rvert^3$
$-0.88312113 + 0.74580943\, i$	$-6.1818994 - 1.6867660\, i$	$-4.20643209 + 3.92605399\, i$	0.026
$0.20646056 + 0.89488895\, i$	$0.03039439 - 2.49813344\, i$	$-0.06455485 + 1.18981502\, i$	0.084

Table 3.4: Degenerate moduli flux vacua for $\xi_3 = 0$ and $\rho = \frac{3}{4}$.

3.4 Aspects of the non-geometric landscape

In this section we review and discuss the main aspects of the AdS$_4$ vacua in our models that are standard examples of type IIB toroidal orientifolds with O3/O7-planes. Besides the axiodilaton S, after an isotropic ansatz the massless scalars reduce to the overall complex structure U and the size modulus T. Fluxes of the R-R and NS-NS 3-forms $(\bar{F}_3, \bar{H}_3)$ generate a potential that gives masses only to S and U. The new ingredient here are the non-geometric Q-fluxes, which are required to restore T-duality between type IIA and type IIB, and that induce a superpotential for the Kähler field T. The various fluxes must satisfy certain constraints arising from Jacobi or Bianchi identities. The problem is then to minimise the scalar potential while solving the constraints. The question is whether there are solutions with all moduli stabilised. We have seen that the answer is affirmative and now we intend to analyse it in more detail.

It is instructive to begin by recounting the findings of the previous sections. The initial step is to classify the subalgebras $\mathfrak{g}_{gauge}$ whose structure constants are the Q-fluxes. With the isotropic ansatz there are only five classes. For each type, the non-geometric Q-fluxes can be written in terms of four auxiliary parameters $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) = \Gamma$, in such a way that the Jacobi identities are automatically satisfied. Other fluxes can also be parameterised using Γ plus additional variables: (ϵ_1, ϵ_2) for the NS-NS $\bar{H}_3$ fluxes, and $(\xi_3\,, \xi_7\,, \xi_s\,, \xi_t)$ or $(\lambda_1\,, \lambda_2\,, \lambda_3\,, \xi_s\,, \xi_t)$ for the R-R $\bar{F}_3$. The significance of the Γ matrix is that it defines a transformed complex structure $\mathcal{Z} = (\alpha U + \beta)/(\gamma U + \delta)$ that is invariant under the modular group $\mathrm{SL}(2,\mathbb{Z})_U$. The effective action can be expressed in terms of $\mathcal{Z}$ according to the Q-algebra. Once the subalgebra is chosen the vacua will depend only on the variables Γ, (ϵ_1, ϵ_2), and (ξ_3, ξ_7) or $(\lambda_1, \lambda_2, \lambda_3)$, that in turn determine the values of the string coupling and the cosmological constant (g_s, V_0), as well as the net tadpoles (N_3, N_7). In many examples, the VEVs of the moduli fields can be determined in a closed form.

Our approach to analyse the vacua in the presence of non-geometric Q-fluxes has the great advantage that the degeneracy due to modular transformations of the complex structure is already taken into account. Inequivalent vacua are just labelled by the VEVs $(\mathcal{Z}_0\,, S_0\,, T_0)$ that are modular invariant. In practice this means that we can study families of modular invariant vacua by choosing a particular structure for Γ. In section 3.4.2 we will give concrete examples.

There is an additional vacuum degeneracy because the characteristic data (g_s, V_0) happen to be independent of the parameters (ξ_s, ξ_t). The explanation is that they correspond to shifts of the axions $\mathrm{Re}\, S$ and $\mathrm{Re}\, T$ which can be reabsorbed in the R-R $\bar{F}_3$ fluxes. The flux-induced R-R tadpoles (N_3, N_7) are blind to (ξ_s, ξ_t) as well. Apparently, generic shifts in $\mathrm{Re}\, S$ and $\mathrm{Re}\, T$ are not symmetries of the compactification, so that two vacua differing only in the R-R flux parameters (ξ_s, ξ_t) would be truly distinct. We argue below that the vacua are equivalent because the full background is symmetric under $S \to S - \xi_s$, and $T \to T - \xi_t$.

In the absence of non-geometric Q-fluxes, the R-R field strength $\widetilde{F}_3$ entering the piece (1.9) of the ten-dimensional type IIB supergravity action is given in (2.19). The natural generalisation to include non-geometric Q-fluxes is

$$\widetilde{F}_3 = F_3 - H_3 \wedge C_0 + Q \cdot C_4 + \bar{F}_3 \ , \tag{3.84}$$

where $Q \cdot C_4$ is a 3-form that we can extract from (2.29) because $\mathrm{Re}\, \mathcal{J} = C_4$. It is straightforward to see from (2.12) that $C_4 = \mathrm{Re}\, T \sum_I \tilde{\omega}^I$, where $\tilde{\omega}^I$ are the basis 4-forms, and that $C_0 = \mathrm{Re}\, S$. Notice then that $\widetilde{F}_3$ involves the two axions in question. The relevant result is that $\widetilde{F}_3$ is invariant under the shifts $S \to S - \xi_s$ and $T \to T - \xi_t$. To show this we first compute the variation of $\bar{F}_3$ using the universal terms in (B.1) for the parameterisation of the R-R fluxes and then substitute into (3.84). In the effective four-dimensional action the result is simply that the superpotential in (2.28) is invariant under these axionic shifts and the corresponding transformation of the R-R fluxes.

3.4.1 Overview

We now describe in order some prominent features of the AdS$_4$ vacua found in this chapter with non-geometric Q-fluxes switched on.

1. The explicit results of section 3.3.2 indicate that in all models the VEVs $\sigma_0 = \mathrm{Im}\, S_0$ and $\mu_0 = \mathrm{Im}\, T_0$ are correlated. This generic property follows from the F-flatness conditions simply because the superpotential is linear in the axiodilaton and the

Kähler modulus. Recall that the VEVs in question determine physically important quantities, namely the string coupling $g_s = 1/\sigma_0$, and the overall internal volume $V_{int} = (\mu_0/\sigma_0)^{3/2}$. To trust the perturbative string approximation g_s must be small and we will shortly explain, as already shown in ref. [95], that generically there are regions in flux space in which both g_s and the cosmological constant are small, while V_{int} is large. We stress again the caveat that even at large overall volume there could still exist light winding string states when non-geometric Q-fluxes are in play. These effects are certainly important in trying to lift the solutions to full string vacua. In this chapter we only claim to have found vacua of the effective field theory with a precise set of massless fields and interactions due to generalised fluxes.

2. Another common feature of all models is the relation between moduli VEVs and net R-R charges. In type IIB toroidal orientifolds it is known that in Minkowski supersymmetric vacua the contribution of R-R $\bar{F}_3$ and NS-NS $\bar{H}_3$ fluxes to the C_4 tadpole is positive ($N_3 > 0$) and this occurs if and only if $\text{Im}\, S_0 > 0$ [81]. The interpretation is that to cancel the tadpole due to $\bar{F}_3$ and $\bar{H}_3$ it is mandatory to include O3-planes, whereas D3-branes can be added only as long as N_3 stays positive. This is also true for no-scale Minkowski vacua in which supersymmetry is broken by the F-term of the Kähler field. Turning on non-geometric Q-fluxes enables to stabilise all moduli at a supersymmetric AdS_4 minimum. At the same time, the Q-fluxes induce a C_8 tadpole of magnitude N_7 that can be cancelled by adding O7-planes and/or D7-branes. We find in general that the VEVs $\text{Im}\, S_0$ and $\text{Im}\, T_0$, that must be positive, are correlated to the tadpoles (N_3, N_7). According to the Q-algebra there are several possibilities for the type of sources that have to be included. For example, the models considered in ref. [95], having $N_3 > 0$ and $N_7 = 0$, proceed only with the Q-fluxes of the compact $\mathfrak{su}(2)^2$ algebra.

   For the Q-fluxes of the $\mathfrak{nil}$ and the $\mathfrak{su}(2) \oplus_{\mathbb{Z}_3} \mathfrak{u}(1)^3$ algebras, there is a relation $N_3 = -\epsilon_2\, N_7$, with $\epsilon_2 > 0$. Only in the latter case it is allowed to have $N_3 = N_7 = 0$, and the sources can be avoided altogether. For the Q-fluxes of $\mathfrak{su}(2) + \mathfrak{u}(1)^3$ algebra it turns out that orientifold planes are unnecessary to cancel tadpoles, but both D3-branes and D7-branes must be added ($N_3 < 0$ and $N_7 > 0$).

   The fluxes of the semisimple Q-algebras are more flexible. In particular, it can happen that one flux-tadpole vanishes while the other must have a definite sign. Moreover, the sign is opposite for the compact and non-compact cases. For instance, when $N_7 = 0$, $N_3 > 0$ and O3-planes are obligatory for the $\mathfrak{su}(2)^2$ Q-fluxes, while for the $\mathfrak{so}(3,1)$ case, $N_3 < 0$ and D3-branes are required.

   The magnitudes of the VEVs are also proportional to the net tadpoles. This then implies that the string coupling typically decreases when N_3 and/or N_7 increase.

However, the number of D-branes cannot be increased arbitrarily without taking into account their backreaction.

3. Consistency of the vacua can in fact be related to the full 12-dimensional algebra $\mathfrak{g}$ in which the $\bar{H}_3$ and the Q-fluxes are the structure constants. The reason is that the conditions $\text{Im}\, S_0 > 0$ and $\text{Im}\, T_0 > 0$ also impose restrictions on the signs of the NS-NS flux parameters (ϵ_1, ϵ_2). For instance, in section 3.3.2.5 we have seen that for Q-fluxes of the compact $\mathfrak{so}(4) \sim \mathfrak{su}(2)^2$ algebra, the solutions with $\epsilon_1 = 0$ require $\epsilon_2 < 0$. This in turn implies, as explained in section 3.1.1.1, that the full algebra is $\mathfrak{so}(4) + \mathfrak{iso}(3)$. Another simple example is the model based on the $\mathfrak{su}(2) + \mathfrak{u}(1)^3$ Q-algebra. The vacua found in the section 3.3.2.3 with $\epsilon_2 = 0$ require $\epsilon_1 < 0$ and it can then be shown that the full algebra $\mathfrak{g}$ is $\mathfrak{so}(4) + \mathfrak{u}(1)^6$. A more detailed study of the 12-dimensional algebras is left for the next chapter 4.

4. We defer to section 3.4.2 a more thorough discussion of the landscape of values attained by the string coupling g_s and the cosmological constant V_0 for the Q-fluxes of the compact $\mathfrak{su}(2)^2$ algebra. The situation for the $\mathfrak{so}(3,1)$ case is similar and can be analysed using the results in section 3.3.2.4. The model based on the direct sum $\mathfrak{su}(2) + \mathfrak{u}(1)^3$ is different because both N_3 and N_7 must be non-zero, but it can still be shown that there exist vacua with small g_s and V_0. The models built using the $\mathfrak{nil}$ and the $\mathfrak{su}(2) \oplus_{\mathbb{Z}_3} \mathfrak{u}(1)^3$ Q-algebras have been studied in their T-dual type IIA formulation in refs [92,93], where it was found that there are infinite families of vacua within the perturbative region.

5. A peculiar result is the appearance of multiple vacua for certain combination of fluxes. These events occur only in models based on the semisimple Q-algebras. They can have $N_3\, N_7 = 0$ or $N_3\, N_7 \neq 0$, but in the former case both NS-NS flux parameters (ϵ_1, ϵ_2) must be non-zero. Reaching small string coupling and cosmological constant typically requires that N_3 and/or N_7 be sufficiently large.

6. To cancel R-R flux-induced tadpoles it might be necessary to add stacks of D3-branes and/or D7-branes. These additional D-branes could also generate a charged chiral spectrum but more generally a different sector of D-branes will serve this purpose. In any case, the D-branes that can be included are constrained by cancellation of Freed-Witten anomalies [93,98]. In absence of non-geometric Q-fluxes the condition amounts to the vanishing of $\bar{H}_3$ when integrated over any internal 3-cycle wrapped by the D-branes. For unmagnetised D7-branes in $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ with $\bar{H}_3$ given in (2.20), it is easy to see that the condition is met, whereas for D3-branes it is trivial. When the Q-fluxes are switched on the modified condition [98] is still satisfied basically because the 3-form $Q \cdot \mathcal{J}$ defined in (2.29) can be expanded in the same basis as $\bar{H}_3$.

D3-branes and unmagnetised D7-branes in $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ do not give rise to charged chiral matter. Therefore the models will not have $U(1)$ chiral anomalies. This is consistent with the fact that the axions $\mathrm{Re}\, S$ and $\mathrm{Re}\, T$ are generically stabilised by the fluxes and having acquired a mass they could not participate in the Green-Schwarz mechanism to cancel the chiral anomalies.

To construct a more phenomenologically viable scenario one could introduce magnetised D9-branes as in the $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ type IIB orientifolds with NS-NS and R-R fluxes that were considered some time ago [73, 83, 86–88, 177]. Now, care has to be taken because magnetised D9-branes suffer from Freed-Witten anomalies. They are actually forbidden in the absence of non-geometric Q-fluxes when $\bar{H}_3 \neq 0$.

The effect of the Q-fluxes can be studied as explained in [98]. Cancellation of Freed-Witten anomalies translates into invariance of the superpotential under shifts $S \to S + q_s\,\nu$ and $T \to T + q_t\,\nu$ where the real charges (q_s, q_t) depend on the $U(1)$ gauged by the D-brane. Applying this prescription we conclude that in our setup with isotropic fluxes magnetised D9-branes could be introduced only in models based on the $\mathfrak{nil}$ and the $\mathfrak{su}(2) \oplus_{\mathbb{Z}_3} \mathfrak{u}(1)^3$ Q-algebras. The reason is that only in these cases the flux-induced polynomials $P_2(U)$ and $P_3(U)$ can be chosen parallel and then W can remain invariant under the axionic shifts. Equivalently, only in these cases the axions $\mathrm{Re}\, S$ and $\mathrm{Re}\, T$ are not fully determined and the residual massless linear combination can give mass to an anomalous $U(1)$. For other Q-algebras the polynomials $P_2(U)$ and $P_3(U)$ are linearly independent and both axions are completely stabilised.

It would be interesting to study the consistency conditions on magnetised D9-branes in models with non-isotropic fluxes. In principle there could exist configuration of the fluxes such that the general superpotential in (2.28) is invariant under axionic shifts of the axiodilaton S and the Kähler T_I moduli fields.

3.4.2 Families of modular invariant vacua

To generate specific families of vacua we first choose the Q-algebra and then select the parameters in the Γ matrix. In general Γ can be chosen so that the non-geometric Q-fluxes are even integers. The NS-NS fluxes $\bar{H}_3$ turn out to be even integers by picking (ϵ_1, ϵ_2) appropriately. One can also start from given non-geometric Q and NS-NS $\bar{H}_3$ even integer fluxes and deduce the corresponding Γ and (ϵ_1, ϵ_2) parameters. Similar remarks apply to the R-R fluxes $\bar{F}_3$. We will illustrate the procedure for the $\mathfrak{su}(2)^2$ Q-algebra.

If one of the parameters vanishes, let us say $\gamma = 0$, it can be shown from (3.12) that the ratios δ/α and β/α are rational numbers (recall that $|\Gamma| \neq 0$ so that $\alpha, \delta \neq 0$). It then follows that by a modular transformation, c.f. (3.46), we can go to a canonical gauge in which also $\beta = 0$.

- When $\epsilon_2 = 0$ and $\epsilon_1 \neq 0$, the canonical diagonal gauge $\gamma = \beta = 0$ is completely generic. In this case we find that β/α and γ/δ are rational because they are given respectively by quotients of the original NS-NS $\bar{H}_3$ and non-geometric Q fluxes. Therefore β and γ can be gauged away by modular transformations. If instead $\epsilon_1 = 0$ but $\epsilon_2 \neq 0$, we can take $\alpha = \delta = 0$.

- When $\epsilon_1 \epsilon_2 \neq 0$ we can still use the canonical gauge but it will not give the most general results that are obtained simply by considering $\alpha, \beta, \gamma, \delta \neq 0$.

Canonical families for $\mathfrak{su}(2)^2$ fluxes

For each Q-algebra we can obtain families of vacua starting from the canonical gauge defined by $\gamma = \beta = 0$. In the $\mathfrak{su}(2)^2$ case only the non-geometric Q-fluxes $\tilde{c}_1$ and $\tilde{c}_2$ are different from zero and can be written as

$$\tilde{c}_1 = -2m \quad , \quad \tilde{c}_2 = 2n \qquad \text{with} \qquad m,\, n \in \mathbb{Z} \; . \tag{3.85}$$

From the non-geometric Q-fluxes in (3.12) we easily find $\alpha/\delta = n/m$ and $\delta^3 = 2m^2/n$, so that

$$|\Gamma|^3 = 4\, n\, m \; . \tag{3.86}$$

In addition, the non-zero NS-NS fluxes $\bar{H}_3$ in (3.15) and the R-R fluxes $\bar{F}_3$ in appendix B are found to be

$$b_0 = -\frac{2m^2}{n}\epsilon_2 \quad , \quad b_3 = \frac{2n^2}{m}\epsilon_1 \; . \tag{3.87}$$

$$a_0 = \frac{2m^2}{n}(\epsilon_1\xi_3 + \epsilon_2\xi_s) \;\; , \;\; a_1 = -2m(\xi_t + \xi_7) \;\; , \;\; a_2 = 2n(\xi_t - \xi_7) \;\; , \;\; a_3 = -\frac{2n^2}{m}(\epsilon_1\xi_s - \epsilon_2\xi_3) \; , \tag{3.88}$$

Since the above fluxes are (even) integers, it is obvious that (ϵ_1, ϵ_2) and $(\xi_3, \xi_7, \xi_s, \xi_t)$ are all rational numbers.

The moduli VEVs depend on (ξ_3, ξ_7) and (ϵ_1, ϵ_2). For concreteness, and to compare with the results in ref. [95], we focus on the case $\xi_7 = 0$. Other cases can be studied using the results of section 3.3.2.5. When $\xi_7 = 0$, the R-R fluxes a_1 and a_2 are spurious and they can be eliminated by setting $\xi_t = 0$, i.e. by a shift in the $\mathrm{Re}\, T$ modulus.

To continue we have to distinguish whether one of the NS-NS flux parameters ϵ_1 or ϵ_2 is zero. Recall that in this case the flux induced polynomial $P_2(U)$ does not have complex roots.

i) Taking $\epsilon_1 \epsilon_2 = 0$: Let us consider $\epsilon_2 = 0$. Then, also a_3 (or ξ_s) is irrelevant and can be set to zero by a shift in the $\mathrm{Re}\, S$ modulus. The important physical

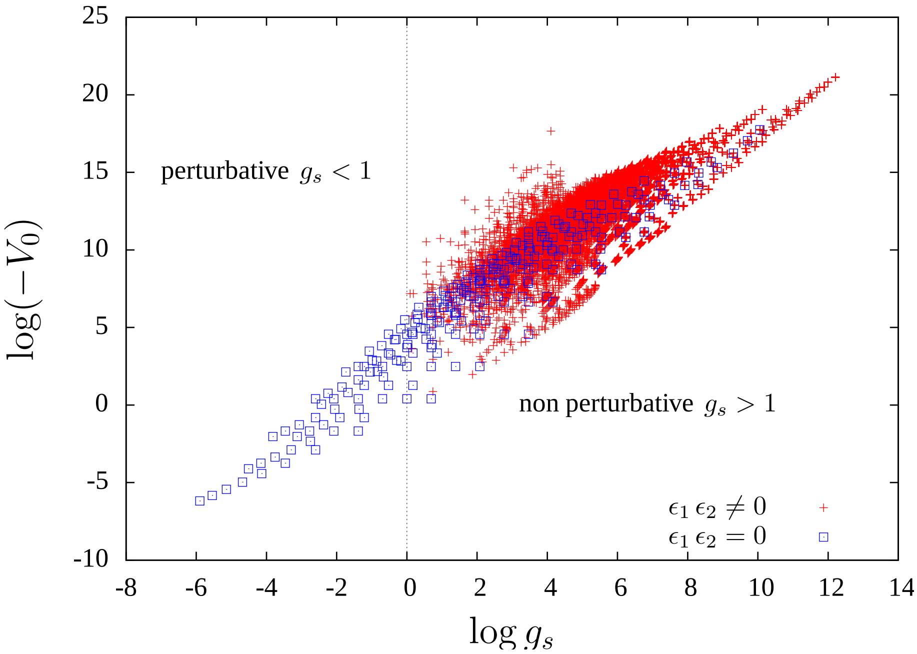

Figure 3.1: The (g_s, V_0) landscape of canonical flux vacua with even integer fluxes up to 20 and where we have set $N_3 = 16$ and $N_7 = 0$ as in ref. [95]. The blue points (⊡ marks) are vacua with $\epsilon_1\,\epsilon_2 = 0$ while the red points (+ marks) are those with $\epsilon_1\,\epsilon_2 \neq 0$.

parameters are ϵ_1 and ξ_3 and they can be deduced from b_3 and a_0. Notice also that at this point $N_3 = a_0\, b_3$. Using the results in (3.83) we obtain the values of the cosmological constant and the string coupling

$$V_0 = \frac{48\, m^6\, b_3^3}{n^3\, N_3^2} \quad , \quad g_s = \frac{8\, m^3\, b_3^2}{n^3\, N_3} \ . \tag{3.89}$$

Consistency requires $\epsilon_1 < 0$ and $\xi_3 > 0$, or equivalently $V_0 < 0$ and $g_s > 0$. For the purpose of counting distinct vacua we can safely assume $b_3 > 0$ and then $m,\, n < 0$.

As noticed in ref. [95], the important outcome is that g_s and V_0 can be made arbitrarily small by keeping b_3 and m fixed while letting $n \to \infty$.

In our approach it is also easy to see that (g_s, V_0) always take values of the form in (3.89) whenever $P_2(U)$ has only real roots. This follows because all vacua are related by modular transformations plus axionic shifts. However, if as in ref. [95] we want to count the vacua with fluxes bounded by an upper limit L, it does not suffice to just consider the canonical gauge. The reason is that by performing modular transformations and axionic shifts we can reach larger effective values of b_3 that seem to violate the tadpole condition. Rather than an elaborate argument we will just provide a simple example. We can go to a non-canonical gauge with $\gamma = 0$ but $\beta \neq 0$ and also take $\xi_t = 0$ but $\xi_s \neq 0$. With these choices it is straightforward to show that $N_3 = a_0\, b_3 - a_3\, b_0$, which would allow to take e.g. $b_3 = N_3$ that is forbidden when $b_0 = 0$ $(\beta = 0)$, or $a_3 = 0$ $(\xi_s = 0)$, because a_0 must be even. To do detailed vacua statistics it is necessary to use generic gauge and axionic shifts.

ii) Taking $\epsilon_1\, \epsilon_2 \neq 0$: As in section 3.3.2.5 we set $\epsilon_2 = \rho\, \epsilon_1$. In the canonical gauge the parameter ρ is a rational number that we assume to be given. We choose to vary the NS-NS flux b_3 that determines

$$\epsilon_1 = \frac{m\, b_3}{2\, n^2} \quad , \quad b_0 = -\frac{\rho\, m^3\, b_3}{n^3} \, , \tag{3.90}$$

where (m, n) are the integers coming from the non-geometric Q-fluxes. The vacuum data have been found to be

$$V_0 = \frac{4 F_V\, n\, m}{\epsilon_1\, \xi_3^2} \quad , \quad g_s = \frac{1}{F_g\, \xi_3} \, , \tag{3.91}$$

where we have used (3.86). The numerical factors F_V and F_g depend on ρ. For instance, for $\rho = 0$, $F_V = 6$ and $F_g = 1/8$. Other examples are given in section 3.3.2.5. We remark that for ρ in a particular range there can be multiple vacua, meaning that for some ρ the above numerical factors might take different values (e.g. table 3.3).

It is most convenient to extract ξ_3 from the tadpole relation $N_3 = 4\, m\, n\, \epsilon_1^2\, (1{+}\rho^2)\, \xi_3$, which in terms of the integer fluxes reads $N_3 = a_0\, b_3 - a_3\, b_0$. Combining all the information we readily find

$$V_0 = \frac{8\, F_V\, m^6\, b_3^3\, (1+\rho^2)^2}{n^3\, N_3^2} \quad , \quad g_s = \frac{m^3\, b_3^2\, (1+\rho^2)}{F_g\, n^3\, N_3} \, . \tag{3.92}$$

Unlike the case when $\rho = 0$, in general we cannot keep m and b_3 fixed while letting $n \to \infty$. The reason is that the NS-NS flux b_0 in (3.90) must be an integer.

The main conclusion is that it is not always possible to obtain small string coupling and cosmological constant. In fact, when $\rho \neq 0$, there are no vacua with $g_s < 1$ unless the tadpole N_3 is sufficiently big. To prove this, notice first that the string coupling can be rewritten as $g_s = -b_3\, b_0\, (1+\rho^2)/(F_s\, \rho\, N_3)$. The most favourable situation occurs when $\rho = -1$ for which $F_s = 0.238$. The smallest allowed NS-NS fluxes are $b_0 = b_3 = 2$ (compatible with $\rho = -1$). Hence, the minimum value of the coupling is $g_s^{min} = 8/(F_s\, N_3)$ and $g_s^{min} < 1$ would require $N_3 > 33$. The situation is worse for values of ρ such that multiple vacua can appear. The problem is that since such ρ's are rational, b_3 must be largish for b_0 to be integer. Going to a more general gauge does not change the conclusion.

We have just provided a quantitative, almost analytic, explanation of why there are no perturbative vacua when the flux polynomial $P_2(U)$ has complex roots and N_3

is not large enough (see figure 3.1). This observation was first made in ref. [95] based on a purely numerical analysis.

Chapter 4

De Sitter Vacua in T-duality Invariant Flux Models

In chapter 3 we have explored supersymmetric AdS_4 moduli vacua in a simple IIB orientifold theory that arises when including, apart from ordinary $(\bar{F}_3, \bar{H}_3)$ gauge fluxes, a non-geometric Q-flux which restores T-duality invariance at the effective level. After this, we now focus our attention on the existence of de Sitter (dS) and Minkowski (Mkw) vacua which are interesting for phenomenology, i.e. that break supersymmetry, in these supergravity flux models.

We should also point out that, since fluxes are relevant for the moduli dynamics, the cosmological implications of these are strongly related to the geometrical properties of the internal space [13, 14, 137, 178]. For instance, in the absence of non-geometric fluxes, the existence of de Sitter vacua as required by the observations needs of a (positive) source of potential energy directly coming from the (negative) curvature of the internal manifold induced by metric fluxes [16, 17, 179]. However, the concept of internal space is distorted or even lost once we include non-geometric fluxes. Therefore, the interplay between generalised fluxes and moduli stabilisation (or dynamics) has to be decoded from the whole 12-dimensional algebra in (2.26).

This chapter investigates the *algebra-moduli* interplay in the case of the $\mathcal{N} = 1$ four-dimensional effective flux models of the previous chapter 3. More concretely, we perform a systematic search of de Sitter vacua with all moduli stabilised at reasonable values. The result is that de Sitter vacua exist and, with a certain tuning of one of the parameters, such vacua can be made Minkowski. The process of searching for these solutions is systematic and could be easily extended to other supergravity models.

4.1 Classification of 12-dimensional isotropic flux algebras

In the previous chapter we carried out a classification of the allowed gauge subalgebras $\mathfrak{g}_{gauge}$ arising from B-field reductions on the $\mathbb{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ isotropic orbifold. Specifically we focused on type IIB orientifold flux models including O3/O7-planes and being invariant under T-duality transformations, so that these subalgebras turn out to be totally induced by non-geometric Q-fluxes as structure constants. However an exhaustive identification of the full twelve-dimensional supergravity algebras $\mathfrak{g}$ underlying such T-duality invariant isotropic flux backgrounds remains undone, and that is what we present in this section. Since $\mathfrak{g}$ is invariant under T-duality transformations, this classification of algebras is valid in any duality frame although we are performing it in the IIB orientifold case allowing for O3/O7-planes.

An exploration of their $\mathcal{N}=4$ origin, if any, after removing the orbifold projection, is beyond the scope of this chapter. Nevertheless, recent progress on this bottom-up approach has been made for the set of the geometric type IIA flux compactifications [180], complementing the previous work [155] that focused on non-geometric type IIB flux compactifications. The $\mathcal{N}=4$ lifting of these non-geometric type IIB flux models and their flux algebra has been further explained in ref. [161].

4.1.1 The set of gauge subalgebras

The discrete $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold symmetry together with the cyclic $\mathbb{Z}_3$ symmetry (isotropy) of the fluxes under the exchange $1 \to 2 \to 3$ in the factorisation

$$\mathbb{T}^6 = \mathbb{T}_1^2 \times \mathbb{T}_2^2 \times \mathbb{T}_3^2 \,, \tag{4.1}$$

select the simple $\mathfrak{so}(3) \sim \mathfrak{su}(2)$ algebra [84] as the fundamental block for building the set of compatible $\mathfrak{g}_{gauge}$ subalgebras within the $\mathcal{N}=1$ algebra in (2.46).

The two maximal $\mathfrak{g}_{gauge}$ subalgebras that our orbifold admits are the semisimple $\mathfrak{so}(4) \sim \mathfrak{su}(2)^2$ and $\mathfrak{so}(3,1)$ Lie algebras. Both possibilities come up with a $\mathbb{Z}_2$-graded structure differing in the way in which the two $\mathfrak{su}(2)$ factors are glued together when it comes to realising the grading.

Since there is no additional restriction over $\mathfrak{g}_{gauge}$, apart from that of respecting the isotropic orbifold symmetries, any $\mathbb{Z}_2$-graded contraction[1] of the previous maximal subalgebras is also a valid $\mathfrak{g}_{gauge}$. The set of such contractions comprises the non-semisimple subalgebras of $\mathfrak{iso}(3) \sim \mathfrak{su}(2) \oplus_{\mathbb{Z}_3} \mathfrak{u}(1)^3$ and $\mathfrak{nil} \sim \mathfrak{u}(1)^3 \oplus_{\mathbb{Z}_3} \mathfrak{u}(1)^3$ arising from continuous contractions[2], together with the direct sum $\mathfrak{su}(2) + \mathfrak{u}(1)^3$ coming from a discrete

[1] We refer the reader interested in the topic of G-graded Lie algebras and their contractions to refs [181, 182].

[2] As it was stated in the previous chapter, the $\oplus_{\mathbb{Z}_3}$ symbol denotes the semidirect sum of algebras endowed with the $\mathbb{Z}_3$ cyclic structure coming from isotropy.

contraction [181]. These are exactly the gauge subalgebras already identified in chapter 3.

Denoting $(E^I, \widetilde{E}^I)_{I=1,2,3}$ a basis for $\mathfrak{g}_{gauge}$, the entire set of gauge subalgebras previously found is gathered in the brackets

$$[E^I, E^J] = \kappa_1\, \epsilon_{IJK}\, E^K \quad , \quad [E^I, \widetilde{E}^J] = \kappa_{12}\, \epsilon_{IJK}\, \widetilde{E}^K \quad , \quad [\widetilde{E}^I, \widetilde{E}^J] = \kappa_2\, \epsilon_{IJK}\, E^K \ , \tag{4.2}$$

with an antisymmetric ϵ_{IJK} structure imposed by the isotropy $\mathbb{Z}_3$ symmetry. The structure constants $\mathcal{Q}$ given by

$$\mathcal{Q}_{E^I,E^J}^{E^K} = \kappa_1 \quad , \quad \mathcal{Q}_{E^I,\widetilde{E}^J}^{\widetilde{E}^K} = \kappa_{12} \quad \text{and} \quad \mathcal{Q}_{\widetilde{E}^I,\widetilde{E}^J}^{E^K} = \kappa_2 \ , \tag{4.3}$$

are restricted by the Jacobi identity $\mathcal{Q}^2 = 0$ to either

$$\kappa_1 = \kappa_{12} \qquad \text{or} \qquad \kappa_{12} = \kappa_2 = 0 \ . \tag{4.4}$$

The first solution in (4.4) gives rise to the maximal gauge subalgebras and their continuous contractions, whereas the second generates the discrete contraction. The intersection between both spaces of solutions contains just the trivial point $\kappa_1 = \kappa_{12} = \kappa_2 = 0$. The structure constants in (4.3) can always be normalised to 1, 0 or -1 by a rescaling of the generators in (4.2). These normalised κ-parameters are presented in table 4.1.

$\mathfrak{g}_{gauge}$	$\mathfrak{so}(3,1)$	$\mathfrak{so}(4)$	$\mathfrak{iso}(3)$	$\mathfrak{nil}$	$\mathfrak{su}(2)+\mathfrak{u}(1)^3$	$\mathfrak{u}(1)^6$
κ_1	1	1	1	0	1	0
κ_{12}	1	1	1	0	0	0
κ_2	-1	1	0	1	0	0

Table 4.1: The set of normalised gauge subalgebras satisfying (4.4).

In the following, we will refer to the $(E^I, \widetilde{E}^I)$ generator basis equipped with the $\mathcal{Q}$ structure constants shown in (4.3), as the *canonical basis* for $\mathfrak{g}_{gauge}$.

4.1.2 The extension to a full supergravity algebra

Thus far, we have explored the set of $\mathfrak{g}_{gauge}$ compatible with the isotropic $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold symmetries finding that there exists a gauge $\mathbb{Z}_2$-graded inner structure modding out all of them. Specifically, this set consists of the two maximal semisimple $\mathfrak{g}_{gauge}$, those of $\mathfrak{so}(3,1)$ and $\mathfrak{so}(4)$, and their non-semisimple $\mathbb{Z}_2$-graded contractions.

Two questions that arise at this point are the following

1. How does $\mathfrak{g}_{gauge}$ in (4.2) extend to a twelve-dimensional supergravity algebra $\mathfrak{g}$? Since we are dealing with an orientifold of the isotropic $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold, the structure

constants of $\mathfrak{g}$ can be classified according to the group $\text{SO}(2,2)\times\text{SO}(3)\subset\text{SO}(6,6)$ with the embedding $(\mathbf{4},\mathbf{3})=\mathbf{12}$ [84]. The $\text{SO}(3)$ factor accounts for the cyclic $\mathbb{Z}_3$ isotropy symmetry and imposes a ϵ_{IJK} structure, not only in the gauge brackets of (4.2), but also in the extended brackets involving the isometry generators. The $\text{SO}(2,2)$ factor reflects on a splitting of $\mathfrak{g}$ into four $(E,\widetilde{E},D,\widetilde{D})$ algebra subspaces expanded by the gauge generators $(E^I,\widetilde{E}^I)_{I=1,2,3}$ of (4.2) and a new set of isometry generators $(D_I,\widetilde{D}_I)_{I=1,2,3}$. In addition to the gauge brackets specified by $\mathcal{Q}$ in (4.3), the algebra $\mathfrak{g}$ will involve an enlarged set of structure constants

$$\mathcal{Q}^{*\,(D_K,\widetilde{D}_K)}_{\ (E^I,\widetilde{E}^I)\,,\,(D_J,\widetilde{D}_J)} \qquad \text{and} \qquad \mathcal{H}^{(E^K,\widetilde{E}^K)}_{(D_I,\widetilde{D}_I)\,,\,(D_J,\widetilde{D}_J)}\ , \tag{4.5}$$

such that the mixed gauge-isometry brackets in (4.5) are given by the co-adjoint action $\mathcal{Q}^*$ of $\mathcal{Q}$ and $(D_I,\widetilde{D}_I)_{I=1,2,3}$ become the generators of the reductive and symmetric coset space $\mathcal{G}/\mathcal{G}_{gauge}$ [171].

2. Does such an extension result in a G-graded structure? If it does, the G-grading of $\mathfrak{g}$ has to accommodate for both gauge-inner and gauge-isometry $\mathbb{Z}_2$-graded structures of (4.3) and (4.5). This reduces the candidates to the $G=\mathbb{Z}_2\oplus\mathbb{Z}_2$, $G=\mathbb{Z}_2\otimes\mathbb{Z}_2$ and $G=\mathbb{Z}_4$ groups.

Let us start by deriving the extension of the $\mathfrak{g}_{gauge}$ based on the $\kappa_1=\kappa_{12}$ solution in (4.4) to a full supergravity algebra $\mathfrak{g}$. For the extended Jacobi identity $\mathcal{H}\mathcal{Q}=0$ to be fulfilled, the most general twelve-dimensional supergravity algebra is given (up to redefinitions of the algebra basis) in table 4.2, where the (ϵ_1,ϵ_2) real quantities determine the new entries in the extended structure constants $\mathcal{H}$, involving the brackets between the isometry generators. Therefore, the ϵ-parameters determine the coset space $\mathcal{G}/\mathcal{G}_{gauge}$ and the supergravity algebra $\mathfrak{g}$ built from a specific $\mathfrak{g}_{gauge}$ [172]. Observe that $\mathfrak{g}$ has a non manifest graded structure. Although we omit the tedious proof, it can always be transformed into a G-graded form with $G=\mathbb{Z}_2\oplus\mathbb{Z}_2$, $G=\mathbb{Z}_2\otimes\mathbb{Z}_2$ and $G=\mathbb{Z}_4$, by an appropriate rotation of the $(E,\widetilde{E},D,\widetilde{D})$ vector of $\text{SO}(2,2)$ without mixing the gauge and isometry subspaces[3].

Working out the extension of the $\mathfrak{g}_{gauge}$ for $\kappa_{12}=\kappa_2=0$ solution in (4.4), the most general supergravity algebra verifying $\mathcal{H}\mathcal{Q}=0$ is written (again up to redefinitions of the algebra basis) in table 4.3, resulting in an explicit $G=\mathbb{Z}_2\oplus\mathbb{Z}_2$ graded structure. It factorises into the direct sum of two six-dimensional pieces spanned by (E^I,D_I) and $(\widetilde{E}^I,\widetilde{D}_I)$ respectively.

We will refer to the $(E^I,\widetilde{E}^I\,;\,D_I,\widetilde{D}_I)_{I=1,2,3}$ generators, satisfying the commutation relations either in table 4.2 or table 4.3, as the *canonical basis* of $\mathfrak{g}$. The structure constants

[3] $\mathbb{Z}_2\otimes\mathbb{Z}_2$ acts naturally on the bi-complex numbers. The maximal supergravity algebra with this grading is $\mathfrak{g}=\mathfrak{so}(3,1)^2$, namely, the $\mathfrak{su}(2)$ bicomplexification.

$\kappa_1 = \kappa_{12}$	E^J	$\widetilde{E}^J$	D_J	$\widetilde{D}_J$
E^I	$\kappa_1\, E^K$	$\kappa_1\, \widetilde{E}^K$	$\kappa_1\, D_K$	$\kappa_1\, \widetilde{D}_K$
$\widetilde{E}^I$	$\kappa_1\, \widetilde{E}^K$	$\kappa_2\, E^K$	$-\kappa_2\, \widetilde{D}_K$	$-\kappa_1\, D_K$
D_I	$\kappa_1\, D_K$	$-\kappa_2\, \widetilde{D}_K$	$-\epsilon_1\, \kappa_2\, E^K - \epsilon_2\, \kappa_2\, \widetilde{E}^K$	$\epsilon_2\, \kappa_2\, E^K + \epsilon_1\, \kappa_1\, \widetilde{E}^K$
$\widetilde{D}_I$	$\kappa_1\, \widetilde{D}_K$	$-\kappa_1\, D_K$	$\epsilon_2\, \kappa_2\, E^K + \epsilon_1\, \kappa_1\, \widetilde{E}^K$	$-\epsilon_1\, \kappa_1\, E^K - \epsilon_2\, \kappa_1\, \widetilde{E}^K$

Table 4.2: Commutation relations for the algebra $\mathfrak{g}$ based on the $\kappa_1 = \kappa_{12}$ solution in (4.4).

$\kappa_{12} = \kappa_2 = 0$	E^J	$\widetilde{E}^J$	D_J	$\widetilde{D}_J$
E^I	$\kappa_1\, E^K$	0	$\kappa_1\, D_K$	0
$\widetilde{E}^I$	0	0	0	0
D_I	$\kappa_1\, D_K$	0	$-\epsilon_1\, \kappa_1\, E^K$	0
$\widetilde{D}_I$	0	0	0	$-\epsilon_2\, \kappa_1\, \widetilde{E}^K$

Table 4.3: Commutation relations for the algebra $\mathfrak{g}$ based on the $\kappa_{12} = \kappa_2 = 0$ solution in (4.4).

$\mathcal{Q}$ and $\mathcal{H}$ in this basis depend on the $(\kappa_1, \kappa_2, \epsilon_1, \epsilon_2)$ parameters and can be directly read from there.

A powerful clue to identifying the set of supergravity algebras that can be realized within the brackets in tables 4.2 and 4.3 comes from the study of their associated Cartan-Killing matrix, denoted $\mathcal{M}$. It has a block-diagonal structure

$$\mathcal{M} = \text{Diag}\left(\mathcal{M}_g, \mathcal{M}_g, \mathcal{M}_g, \mathcal{M}_{isom}, \mathcal{M}_{isom}, \mathcal{M}_{isom}\right) \ , \tag{4.6}$$

where $\mathcal{M}_g$ and $\mathcal{M}_{isom}$ are 2×2 matrices referring to the pairs $(E^I, \widetilde{E}^I)$ and $(D_I, \widetilde{D}_I)$ of generator subspaces, respectively. Let us study the diagonalisation of $\mathcal{M}$. The two eigenvalues of the $\mathcal{M}_g$ matrix

$$\begin{array}{llll} \kappa_1 = \kappa_{12}: & \lambda_{gauge}^{(1)} = -2^3\,\kappa_1^2 & \text{and} & \lambda_{gauge}^{(2)} = -2^3\,\kappa_1\,\kappa_2 \ , \\ \kappa_{12} = \kappa_2 = 0: & \lambda_{gauge}^{(1)} = -2^2\,\kappa_1^2 & \text{and} & \lambda_{gauge}^{(2)} = 0 \ , \end{array} \tag{4.7}$$

are obtained by substituting the normalised κ-configurations in table 4.1. For the $\mathcal{M}_{isom}$ matrix, they are computed by solving the characteristic polynomial

$$\lambda_{isom}^2 - \text{T}\,\lambda_{isom} + \text{D} = 0 \ , \tag{4.8}$$

determined by its trace T and its determinant D. Those are given by

$$\begin{aligned} \kappa_1 = \kappa_{12}: &\quad \mathrm{T} = 2^3\,\epsilon_1\,\kappa_1\,(\kappa_1+\kappa_2) &\quad \text{and} &\quad \mathrm{D} = 2^6\,\kappa_1^2\,\kappa_2\,(\epsilon_1^2\,\kappa_1 - \epsilon_2^2\,\kappa_2)\ , \\ \kappa_{12} = \kappa_2 = 0: &\quad \mathrm{T} = 12\,\epsilon_1\,\kappa_1^2 &\quad \text{and} &\quad \mathrm{D} = 0\ . \end{aligned} \tag{4.9}$$

Provided the $\kappa_1 = \kappa_{12}$ solution in (4.4), the supergravity algebra $\mathfrak{g}$ becomes semisimple if and only if

$$\kappa_1\,\kappa_2\,(\kappa_1\,\epsilon_1^2 - \kappa_2\,\epsilon_2^2) \neq 0\ , \tag{4.10}$$

whereas for the $\kappa_{12} = \kappa_2 = 0$ solution $\mathfrak{g}$ is always non-semisimple since a null $\lambda_{isom} = 0$ eigenvalue comes out from (4.8).

Some of the generators in the adjoint representation of $\mathfrak{g}$ may vanish when we are dealing with a non-semisimple algebra. If so, this representation is no longer faithful and the supergravity algebra $\mathfrak{g}_{emb}$ realised on the curvatures and embeddable within the $\mathfrak{o}(6,6)$ duality algebra becomes smaller than the algebra $\mathfrak{g}$ involving the vector fields [166].

After a detailed exploration, the set of $\mathfrak{g}$ allowed by the $\mathcal{N} = 1$ orientifolds of the isotropic $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold are listed in table 4.4. The spectrum includes from the flux-vanishing $\mathfrak{g} = \mathfrak{u}(1)^{12}$ case up to the most involved $\mathfrak{g} = \mathfrak{so}(3,1)^2$ algebra. All of them are G-graded contractions of those supergravity algebras built from the maximal $\mathfrak{g}_{gauge} = \mathfrak{so}(4)$ and $\mathfrak{g}_{gauge} = \mathfrak{so}(3,1)$ subalgebras. Specifically, contractions based on the abelian $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $G = \mathbb{Z}_2 \otimes \mathbb{Z}_2$ and $G = \mathbb{Z}_4$ finite groups, compatible with the isotropic orbifold symmetries.

It can be observed that the cases of $\mathfrak{g} = \mathfrak{so}(3,1)^2$ and $\mathfrak{g} = \mathfrak{so}(3,1) \oplus_{\mathbb{Z}_3} \mathfrak{u}(1)^6$ arising as the extensions of $\mathfrak{g}_{gauge} = \mathfrak{so}(3,1)$, also appear as extensions of $\mathfrak{g}_{gauge} = \mathfrak{so}(4)$ and $\mathfrak{g}_{gauge} = \mathfrak{iso}(3)$ respectively. At this point, we have to go back to section 2.4.2 and emphasise that, in the type IIB with O3/O7-planes duality frame (the *T-fold* description), a family of 4d effective models is determined not only by the supergravity algebra $\mathfrak{g}$, but also by specifying the subalgebra $\mathfrak{g}_{gauge}$ associated to the isotropy subgroup of the coset space $\mathcal{G}/\mathcal{G}_{gauge}$. In this sense, effective models based on the same $\mathfrak{g}$, but containing different $\mathfrak{g}_{gauge}$, result in non equivalent models.

4.2 Type IIA supergravity flux models and no-go theorems

So far, we have been mostly centered on the T-fold description of the type II orientifold flux models on the isotropic $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. This is mainly due to its suitability to classify the supergravity algebras underlying the generalised fluxes. Any effective model in this description becomes an apparently[4] non-geometric model once we switch on a non

[4]By apparently we mean that it may result in a type IIA geometric flux model when changing the duality frame.

#	$\mathfrak{g}_{gauge}$	$\mathfrak{g}$	$\mathfrak{g}_{emb}$	$\mathcal{H}$ EXTENSION	
1	$\mathfrak{so}(3,1)$	$\mathfrak{so}(3,1)^2$	$\mathfrak{g}$	$\epsilon_1^2+\epsilon_2^2\neq 0$	
2		$\mathfrak{so}(3,1)\oplus_{\mathbb{Z}_3}\mathfrak{u}(1)^6$		$\epsilon_1^2+\epsilon_2^2=0$	
3	$\mathfrak{so}(4)$	$\mathfrak{so}(3,1)^2$	$\mathfrak{g}$	$(\epsilon_1+\epsilon_2)>0$, $(\epsilon_1-\epsilon_2)>0$	
4		$\mathfrak{iso}(3)^2$		$(\epsilon_1+\epsilon_2)=0$, $(\epsilon_1-\epsilon_2)=0$	
5		$\mathfrak{so}(4)^2$		$(\epsilon_1+\epsilon_2)<0$, $(\epsilon_1-\epsilon_2)<0$	
6		$\mathfrak{so}(3,1)+\mathfrak{iso}(3)$		$(\epsilon_1+\epsilon_2)\geqslant 0$, $(\epsilon_1-\epsilon_2)\geqslant 0$	
7		$\mathfrak{so}(3,1)+\mathfrak{so}(4)$		$(\epsilon_1+\epsilon_2)\gtrless 0$, $(\epsilon_1-\epsilon_2)\lessgtr 0$	
8		$\mathfrak{iso}(3)+\mathfrak{so}(4)$		$(\epsilon_1+\epsilon_2)\leqslant 0$, $(\epsilon_1-\epsilon_2)\leqslant 0$	
9	$\mathfrak{iso}(3)$	$\mathfrak{so}(3,1)\oplus_{\mathbb{Z}_3}\mathfrak{u}(1)^6$	$\mathfrak{g}$	$\epsilon_1>0$	$\epsilon_2=$ *free*
10		$\mathfrak{iso}(3)\oplus_{\mathbb{Z}_3}\mathfrak{u}(1)^6$		$\epsilon_1=0$	
11		$\mathfrak{so}(4)\oplus_{\mathbb{Z}_3}\mathfrak{u}(1)^6$		$\epsilon_1<0$	
12	$\mathfrak{nil}$	$\mathfrak{nil}_{12}(4)$	$\mathfrak{nil}_{12}(3)$	$\epsilon_1=$ *free*	$\epsilon_2\neq 0$
13		$\mathfrak{nil}_{12}(2)$	$\mathfrak{u}(1)^{12}$		$\epsilon_2=0$
14	$\mathfrak{su}(2)+\mathfrak{u}(1)^3$	$\mathfrak{so}(3,1)+\mathfrak{nil}$	$\mathfrak{so}(3,1)+\mathfrak{u}(1)^6$	$\epsilon_1>0$	$\epsilon_2\neq 0$
15		$\mathfrak{so}(3,1)+\mathfrak{u}(1)^6$			$\epsilon_2=0$
16		$\mathfrak{iso}(3)+\mathfrak{nil}$	$\mathfrak{iso}(3)+\mathfrak{u}(1)^6$	$\epsilon_1=0$	$\epsilon_2\neq 0$
17		$\mathfrak{iso}(3)+\mathfrak{u}(1)^6$			$\epsilon_2=0$
18		$\mathfrak{so}(4)+\mathfrak{nil}$	$\mathfrak{so}(4)+\mathfrak{u}(1)^6$	$\epsilon_1<0$	$\epsilon_2\neq 0$
19		$\mathfrak{so}(4)+\mathfrak{u}(1)^6$			$\epsilon_2=0$
20	$\mathfrak{u}(1)^6$	$\mathfrak{nil}_{12}(2)$	$\mathfrak{u}(1)^{12}$	UNCONSTRAINED	

Table 4.4: List of the $\mathcal{N}=1$ supergravity algebras $\mathfrak{g}$ allowed by the isotropic $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold. The κ-parameters fixing $\mathfrak{g}_{gauge}$ are taken to their normalised values shown in table 4.1. The number p within the parenthesis in the twelve-dimensional $\mathfrak{nil}_{12}(p)$ nilpotent algebras denotes the nilpotency order.

vanishing Q-flux background. By changing the duality frame it can always be mapped to a $\mathcal{N}=1$ type IIA string compactification with O6/D6 sources including the entire set of generalised fluxes, i.e. $\bar{H}_3$, ω , Q and R, together with a set of R-R p-form fluxes $\bar{F}_p$ with $p=0,2,4,6$.

4.2.1 Power law dependence of IIA scalar potential

Starting from the ten-dimensional type IIA massive supergravity action and performing dimensional reduction to four dimensions, the $\bar{H}_3$ and $\bar{F}_p$ flux-induced terms in the scalar potential were worked out in ref. [13]. More concretely, their power law dependence on the the volume modulus y and the four-dimensional dilaton modulus $\tilde{\sigma}$,

$$y \equiv (\text{Vol}_6)^{\frac{1}{3}} \sim A \qquad , \qquad \tilde{\sigma} \equiv e^{-\varphi} \sqrt{\text{Vol}_6} \sim e^{-\varphi} A^{3/2} \ , \tag{4.11}$$

where φ is the ten-dimensional dilaton field and $\text{Vol}_6 \sim A^3$ denotes the volume of the internal space $\mathbb{T}^6 = (\mathbb{T}_2)^3$ with A the area of $\mathbb{T}_2$. Further terms induced by generalised ω, Q and R fluxes were also introduced by applying T-dualities on the $\bar{H}_3$ one due to the non existence of a ten-dimensional supergravity formulation of the theory in such generalised flux backgrounds.

The $\mathcal{N}=1$ scalar potential coming from these generalised type IIA flux compactifications can be split into three main contributions

$$V_{\text{IIA}} = V_{\text{Gen}} + V_{\text{loc}} + \sum_{p=0\,(even)}^{6} V_{\bar{F}_p} \ , \tag{4.12}$$

with

$$V_{\text{Gen}} = V_{\bar{H}_3} + V_{\omega} + V_Q + V_R \ . \tag{4.13}$$

The latter is the potential energy induced by the set of generalised fluxes. The term V_{loc} accounts for the potential localised sources as O6-planes and D6-branes. $V_{\bar{H}_3}$ and $V_{\bar{F}_p}$ account for the $\bar{H}_3$ and $\bar{F}_p$ flux-induced terms in the scalar potential. These are non negative because they come from quadratic terms in the ten-dimensional action. V_ω accounts for the potential energy induced by the geometric ω flux. Finally, V_Q and V_R account for the contributions generated by the non-geometric Q and R fluxes.

Working with the two $(y, \tilde{\sigma})$ moduli fields (a.k.a in the volume-dilaton plane limit), the power law dependence of all the terms in (4.12) on those fields, was found to be

$$V_{\bar{H}_3} \propto \tilde{\sigma}^{-2}\, y^{-3} \quad , \quad V_\omega \propto \tilde{\sigma}^{-2}\, y^{-1} \quad , \quad V_Q \propto \tilde{\sigma}^{-2}\, y \quad , \quad V_R \propto \tilde{\sigma}^{-2}\, y^{3} \ , \tag{4.14}$$

for the set of generalised flux-induced contributions, together with

$$V_{\text{loc}} \propto \tilde{\sigma}^{-3} \ , \tag{4.15}$$

for the potential energy induced by the localised sources and

$$V_{\bar{F}_p} \propto \tilde{\sigma}^{-4}\, y^{3-p} \ , \tag{4.16}$$

for the R-R flux-induced terms [13]. The contributions to the scalar potential of eq. (4.12) can be arranged as

$$V_{\text{IIA}} = A(y, M)\, \tilde{\sigma}^{-2} + B(M)\, \tilde{\sigma}^{-3} + C(y, M)\, \tilde{\sigma}^{-4} \ , \tag{4.17}$$

where M denotes the set of additional moduli fields in the model. $A(y, M)$ contains the contributions to the scalar potential resulting from the generalised fluxes. $B(M)$ accounts for the O6-planes and D6-branes contributions to the potential energy. Finally, $C(y, M)$ incorporates the terms in the scalar potential induced by the set of R-R fluxes. The explicit form of these functions depends on the features of the specific model under consideration. In contrast with previous works, our initial setup does not contain KK5-branes [16, 17] generating a contribution to the scalar potential that scales as the geometric ω flux-induced term of eq. (4.14), nor NS5-branes [13, 17] that would induce a $V_{\text{NS5}} \propto \tilde{\sigma}^{-2} y^{-2}$ term. We work within the framework of ref. [14], extended to include the set of generalised fluxes needed for restoring T-duality invariance at the 4d effective level.

4.2.2 Simple no-go theorems in the volume-dilaton plane limit

Using the general scaling properties of eqs (4.14)-(4.16), it can be shown that the potential of eq. (4.12) verifies

$$-\left(y\,\frac{\partial V_{\text{IIA}}}{\partial y} + 3\,\tilde{\sigma}\,\frac{\partial V_{\text{IIA}}}{\partial \tilde{\sigma}}\right) = 9\,V_{\text{IIA}} + \sum_{p=0\,(\text{even})}^{6} p\,V_{\bar{F}_p} - 2\,V_{\omega} - 4\,V_Q - 6\,V_R\ . \qquad (4.18)$$

Notice that the l.h.s of eq. (4.18) vanishes identically at any extremum of the scalar potential yielding

$$V_{\text{IIA}} = \frac{1}{9}\left(\Delta V - \sum_{p=0\,(even)}^{6} p\,V_{\bar{F}_p}\right)\ , \qquad (4.19)$$

with $\Delta V \equiv 2\,V_{\omega} + 4\,V_Q + 6\,V_R$. Whenever $V_{\bar{F}_p} > 0$ for some $p = 0, 2, 4, 6$, there can not exist dS/Mkw solutions (i.e. $V_{\text{IIA}} \geq 0$) unless

$$\Delta V \geq \sum_{p=0\,(even)}^{6} p\,V_{\bar{F}_p}\ . \qquad (4.20)$$

This was the line followed in refs [16, 17] where certain type IIA flux compactifications on curved internal spaces generating a contribution $\Delta V = 2\,V_{\omega} \neq 0$ were presented. Additionally, if building the linear combination

$$-\left(y\,\frac{\partial V_{\text{IIA}}}{\partial y} + \tilde{\sigma}\,\frac{\partial V_{\text{IIA}}}{\partial \tilde{\sigma}}\right) = 3\,V_{\text{IIA}} + 2\,\left(V_{\bar{H}_3} + V_{\bar{F}_4} + 2\,V_{\bar{F}_6}\right) - 2\,\left(V_{\bar{F}_0} + V_Q + 2\,V_R\right)\ , \quad (4.21)$$

it shows that $V_{\bar{F}_0} \neq 0$ for dS vacua to exist in any geometric model, i.e. $V_Q = V_R = 0$. This implies having a Romans massive supergravity, as it was stated in ref. [17].

4.3 Type IIA scalar potential from type IIB flux models

In this section we are deriving the characteristic flux-induced $\mathcal{N} = 1$ isotropic scalar potential associated to each of the 12-dimensional supergravity algebras found in section 4.1

and displayed in table 4.4. Our main goal here is to do it in a unified manner that will allow us to work with all the T-duality invariant supergravity flux models simultaneously.

To start with, we consider the type IIB with O3/O7-planes duality frame used in chapter 3. In this duality frame only the Q and $\bar{H}_3$ fluxes are turned on in the supergravity models and $\mathfrak{g}$ takes the form given in (2.46). Next, and by performing three T-dualities along the internal space coordinates, these effective models are mapped to type IIA compactifications in the presence of $\bar{H}_3$, ω , Q and R fluxes and O6-planes.

couplings	IIA with O6	IIB with O3/O7	fluxes
1	$\bar{F}_{\alpha i\beta j\gamma k}$	$\bar{F}_{ijk}$	a_0
U	$\bar{F}_{\alpha i\beta j}$	$\bar{F}_{ij\gamma}$	a_1
U^2	$\bar{F}_{\alpha i}$	$\bar{F}_{i\beta\gamma}$	a_2
U^3	$\bar{F}_0$	$\bar{F}_{\alpha\beta\gamma}$	a_3
S	$\bar{H}_{ijk}$	$\bar{H}_{ijk}$	b_0
$S\,U$	ω^{α}_{jk}	$\bar{H}_{\alpha jk}$	b_1
$S\,U^2$	$Q^{\alpha\beta}_{k}$	$\bar{H}_{i\beta\gamma}$	b_2
$S\,U^3$	$R^{\alpha\beta\gamma}$	$\bar{H}_{\alpha\beta\gamma}$	b_3
T	$\bar{H}_{\alpha\beta k}$	$Q^{\alpha\beta}_{k}$	c_0
$T\,U$	$\omega^{j}_{k\alpha}\,,\,\omega^{i}_{\beta k}\,,\,\omega^{\alpha}_{\beta\gamma}$	$Q^{\alpha j}_{k}\,,\,Q^{i\beta}_{k}\,,\,Q^{\beta\gamma}_{\alpha}$	$\check{c}_1\,,\,\hat{c}_1\,,\,\tilde{c}_1$
$T\,U^2$	$Q^{\gamma i}_{\beta}\,,\,Q^{i\beta}_{\gamma}\,,\,Q^{ij}_{k}$	$Q^{i\beta}_{\gamma}\,,\,Q^{\gamma i}_{\beta}\,,\,Q^{ij}_{k}$	$\check{c}_2\,,\,\hat{c}_2\,,\,\tilde{c}_2$
$T\,U^3$	$R^{ij\gamma}$	Q^{ij}_{γ}	c_3

Table 4.5: IIA/IIB correspondence between isotropic flux coefficients in the $\mathbb{Z}_2\times\mathbb{Z}_2$ orientifolds.

Switching the duality frame, the Kähler modulus and the complex structure modulus are swapped. The resulting type IIA effective theory is still given by (2.67) when setting the non-geometric P-fluxes $d_A = 0$ so that $P_4(U) = 0$. However, the coefficients in the remaining flux-induced polynomials (2.68), (2.69) and (2.70) have to be reinterpreted in terms of the type IIA flux entries, namely, the set of $\bar{F}_p$ R-R fluxes with $p = 0, 2, 4, 6$ together with the entire set $\bar{H}_3$, ω , Q and R of fluxes [7,8]. The correspondence between IIA and IIB fluxes [7] is shown in table 4.5.

4.3.1 Unified description of type IIB models

Let us consider the T-duality invariant effective flux models described in chapter 3. Performing a non linear transformation on the complex structure U modulus

$$U \to \mathcal{Z} \equiv \Gamma\, U = \frac{\alpha\, U + \beta}{\gamma\, U + \delta} \ , \tag{4.22}$$

via the general $\Gamma \in \mathrm{GL}(2,\mathbb{R})$ matrix

$$\Gamma \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \ , \tag{4.23}$$

was found to be equivalent to applying the $\mathrm{SL}(2,\mathbb{R})$ rotations of (3.10) and (3.17) upon the generators of the supergravity algebra in (2.46). In terms of the Γ matrix in (4.23), this rotation of the algebra basis is given by

$$\begin{pmatrix} E^I \\ \widetilde{E}^I \end{pmatrix} = \frac{\Gamma}{|\Gamma|^2} \begin{pmatrix} -X^{2I-1} \\ X^{2I} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} -D_I \\ \widetilde{D}_I \end{pmatrix} = \frac{\mathrm{Adj}(\Gamma)}{|\Gamma|^2} \begin{pmatrix} -Z_{2I-1} \\ Z_{2I} \end{pmatrix} \ , \tag{4.24}$$

for $I = 1, 2, 3$. Thus, any non-geometric Q-flux background consistent with (2.46) can always be transformed into the canonical $\mathcal{Q}$ form in (4.3) satisfying (4.4), by means of an appropriate choice of the Γ matrix[5]. The new gauge-isometry mixed brackets are still given by the co-adjoint action $\mathcal{Q}^*$ of $\mathcal{Q}$, and $\mathfrak{g}$ is forced by the $\mathcal{H}\mathcal{Q} = 0$ Jacobi identity to be that of table 4.2 or 4.3.

Reading the canonical $\mathcal{Q}$ and $\mathcal{H}$ fluxes from there, and undoing the change of basis in (4.24), we obtain the non canonical embedding of $\mathfrak{g}$ within the original Q and $\bar{H}_3$ fluxes, respectively. Substituting them into the original flux-induced polynomials in (2.69) and (2.70), they result with the form of (3.40),

$$P_2(U) = (\gamma U + \delta)^3 \mathcal{P}_2(\mathcal{Z}) \qquad , \qquad P_3(U) = (\gamma U + \delta)^3 \mathcal{P}_3(\mathcal{Z}) \ , \tag{4.25}$$

where the $\mathcal{P}_2(\mathcal{Z})$ and $\mathcal{P}_3(\mathcal{Z})$ flux-induced polynomials are shown in table 4.6. Notice that all the supergravity flux models introduced case by case in section 3 are now described jointly.

In terms of the redefined complex structure modulus $\mathcal{Z}$, and following the philosophy of the previous chapter 3, we decide to make the R-R flux-induced polynomial expansion of

$$P_1(U) = (\gamma U + \delta)^3 \left[\xi_s \, \mathcal{P}_2(\mathcal{Z}) \,+\, \xi_t \, \mathcal{P}_3(\mathcal{Z}) \,-\, \xi_3 \widetilde{\mathcal{P}}_2(\mathcal{Z}) \,+\, \xi_7 \, \widetilde{\mathcal{P}}_3(\mathcal{Z}) \right] \ , \tag{4.26}$$

[5] At this point it becomes clear that a rescaling of the gauge generators in (4.2) is equivalent to a rescaling of the diagonal entries within the Γ matrix in (4.24). Therefore, κ_1 and κ_2 can always be expressed as their normalised values, shown in table 4.1, without lost of generality.

	$\mathcal{P}_3(\mathcal{Z})/3$	$\mathcal{P}_2(\mathcal{Z})$
$\kappa_1 = \kappa_{12}$	$\kappa_2\,\mathcal{Z}^3 - \kappa_1\,\mathcal{Z}$	$\kappa_2\,\left(\epsilon_1\,\mathcal{Z}^3 + 3\,\epsilon_2\,\mathcal{Z}^2\right) + \kappa_1\,\left(\epsilon_2 + 3\,\epsilon_1\,\mathcal{Z}\right)$
$\kappa_{12} = \kappa_2 = 0$	$\kappa_1\,\mathcal{Z}$	$\kappa_1\,\left(\epsilon_1\,\mathcal{Z}^3 + \epsilon_2\right)$

Table 4.6: The unified flux-induced $\mathcal{P}_2(\mathcal{Z})$ and $\mathcal{P}_3(\mathcal{Z})$ polynomials.

where $\widetilde{\mathcal{P}}_i(\mathcal{Z})$ denotes the dual of $\mathcal{P}_i(\mathcal{Z})$ such that $\mathcal{P}_i(\mathcal{Z}) \rightarrow \frac{\widetilde{\mathcal{P}}_i(\mathcal{Z})}{\mathcal{Z}^3}$ when $\mathcal{Z} \rightarrow -\frac{1}{\mathcal{Z}}$. As noticed in section 3.2.1, this parameterisation allows us to remove the R-R flux degrees of freedom (ξ_s, ξ_t) from the effective theory through the field shifts in (3.48). However, the previous argument for reabsorbing parameters fails when $\mathcal{P}_3(\mathcal{Z})$ and $\mathcal{P}_2(\mathcal{Z})$ are proportional to each other, i.e. $\mathcal{P}_3(\mathcal{Z}) = \lambda\,\mathcal{P}_2(\mathcal{Z})$. In this case only the linear combination, $\text{Re}(S) + \lambda\,\text{Re}(T)$, of axions enters the superpotential, and its orthogonal direction can not be stabilised due to the form of the Kähler potential in (2.67).

The effective theory built in this way corresponds to a transformation $e^K|W|^2 \rightarrow e^{\mathcal{K}}|\mathcal{W}|^2$ of the original $(\bar{F}_3, \bar{H}_3)$ and Q flux-induced model into an equivalent one described by the Kähler $\mathcal{K}$ and the superpotential $\mathcal{W}$

$$\begin{aligned} \mathcal{K} &= -3\,\log\left(-i\,(\mathcal{Z}-\bar{\mathcal{Z}})\right) - \log\left(-i\,(\mathcal{S}-\bar{\mathcal{S}})\right) - 3\,\log\left(-i\,(\mathcal{T}-\bar{\mathcal{T}})\right)\ , \\ \mathcal{W} &= |\Gamma|^{3/2}\left[\mathcal{T}\,\mathcal{P}_3(\mathcal{Z}) + \mathcal{S}\,\mathcal{P}_2(\mathcal{Z}) - \xi_3\widetilde{\mathcal{P}}_2(\mathcal{Z}) + \xi_7\,\widetilde{\mathcal{P}}_3(\mathcal{Z})\right]\ , \end{aligned} \tag{4.27}$$

with the $\mathcal{P}_{2,3}(\mathcal{Z})$ polynomials shown in table 4.6. This parameterisation makes more evident the discrete symmetries of the theory. In particular:

i) $\mathcal{W}$ is invariant under

$$\mathcal{S} \rightarrow -\mathcal{S} \quad , \quad (\epsilon_1\,,\,\epsilon_2\,,\,\xi_3\,,\,\xi_7) \rightarrow (-\epsilon_1\,,\,-\epsilon_2\,,\,-\xi_3\,,\,\xi_7)\ . \tag{4.28}$$

ii) $\mathcal{W}$ goes to $-\mathcal{W}$ under these two transformations:

$$\mathcal{T} \rightarrow -\mathcal{T} \quad , \quad (\epsilon_1\,,\,\epsilon_2\,,\,\xi_3\,,\,\xi_7) \rightarrow (-\epsilon_1\,,\,-\epsilon_2\,,\,\xi_3\,,\,-\xi_7)\ . \tag{4.29}$$

$$\mathcal{Z} \rightarrow -\mathcal{Z} \quad , \quad (\epsilon_1\,,\,\epsilon_2\,,\,\xi_3\,,\,\xi_7) \rightarrow (\epsilon_1\,,\,-\epsilon_2\,,\,-\xi_3\,,\,-\xi_7)\ . \tag{4.30}$$

iii) The dynamics of the moduli fields $(\mathcal{Z}, \mathcal{S}, \mathcal{T})$ is determined by the standard $\mathcal{N}=1$ scalar potential in (3.54) built from (4.27). Since the superpotential parameters are real, the potential is invariant under field conjugation. We can combine this action with the above transformations, namely

$$(\,\mathcal{Z}\,,\,\mathcal{S}\,,\,\mathcal{T}\,) \rightarrow -(\,\mathcal{Z}\,,\,\mathcal{S}\,,\,\mathcal{T}\,)^* \quad , \quad (\epsilon_1\,,\,\epsilon_2\,,\,\xi_3\,,\,\xi_7) \rightarrow (\epsilon_1\,,\,-\epsilon_2\,,\,\xi_3\,,\,\xi_7)\ , \tag{4.31}$$

to relate physical vacua at $\pm\epsilon_2$. Notice that this transformation keeps the supergravity algebra $\mathfrak{g}$ invariant.

The transformations in $i)$ and $ii)$ map physical vacua into non-physical ones. Using them it is possible to turn any non-physical vacuum into a physical one in a related model by flipping the signs of some parameters. It is interesting to notice that the supergravity algebra $\mathfrak{g}$ of these two models may be different (see table 4.4).

The original flux entries appearing in (2.68)-(2.70) can be read from (3.40) and (4.26) after substituting the $\mathcal{Z}$ redefined modulus of (4.22) into the flux-induced polynomials given in table 4.6. Then, the tadpole cancellation conditions in (2.77) and (2.78) result in a few simple expressions shown in table 4.7. As can be seen from it, the discrete transformations in (4.28), (4.29), (4.30) and (4.31) imply $(N_3, N_7) \to (-N_3, N_7)$, $(N_3, N_7) \to (N_3, -N_7)$, $(N_3, N_7) \to (-N_3, -N_7)$ and $(N_3, N_7) \to (N_3, N_7)$, respectively.

	$N_3/\lvert\Gamma\rvert^3$	$N_7/\lvert\Gamma\rvert^3$
$\kappa_1 = \kappa_{12}$	$3\epsilon_1\left(\kappa_1^2 - \kappa_2^2\right)\xi_7 + \left(\epsilon_1^2\left(3\kappa_1^2 + \kappa_2^2\right) + \epsilon_2^2\left(\kappa_1^2 + 3\kappa_2^2\right)\right)\xi_3$	$\epsilon_1(\kappa_1^2 - \kappa_2^2)\xi_3 + (\kappa_1^2 + 3\kappa_2^2)\xi_7$
$\kappa_{12} = \kappa_2 = 0$	$\kappa_1^2\left(\epsilon_1^2 + \epsilon_2^2\right)\xi_3$	$\kappa_1^2\,\xi_7$

Table 4.7: The unified R-R flux-induced tadpoles.

Summarising, all the 4d effective models can be jointly described by the Kähler potential and the superpotential in eqs. (4.27) with the flux-induced polynomials presented in table 4.6. They are totally defined in terms of the new $(\mathcal{Z}, \mathcal{S}, \mathcal{T})$ moduli fields, together with a small set of parameters

i) $(\kappa_1,\, \kappa_2\,;\, \epsilon_1,\, \epsilon_2)$, that determine the generalised fluxes and hence the twelve-dimensional supergravity algebra $\mathfrak{g}$. As it was previously explained, see footnote 5, κ_1 and κ_2 can always be taken to their normalised values shown in table 4.1 without lost of generality.

ii) $(\xi_3,\, \xi_7)$, related to the localised O3/D3 and O7/D7 sources through the tadpole cancellation conditions displayed in table 4.7.

Finally, the use of this parameterisation for the effective models allows us to extract an interesting result based on the following argument: it is well known (see table 4.5) that only the U^2 and U^3 couplings in the flux-induced polynomials $P_{2,3}(U)$ of (2.69) and (2.70) come from the non-geometric Q and R fluxes, respectively. This is in the type IIA description of the effective models [7, 8].
On the other hand, provided consistent Q and $\bar{H}_3$ fluxes in the type IIB description, their flux-induced polynomials can always be transformed to the form given in table 4.6 via the modular transformation $U \to \mathcal{Z}$ of eq. (4.22). Substituting the value of the κ-parameters given in table 4.1 into the flux-induced polynomials of table 4.6, we can conclude that such quadratic and cubic couplings can be removed from the superpotential

via a modular transformation for the models based on $\mathfrak{g}_{gauge} = \mathfrak{nil}$, $\mathfrak{iso}(3)$ and $\mathfrak{su}(2) + \mathfrak{u}(1)^3$ (if taking $\epsilon_1 = 0$). Therefore, these models can be described as geometric type IIA flux compactifications. In the $\mathfrak{nil}$ case with $\kappa_1 = \kappa_{12} = 0$, a further $\mathcal{Z} \to \frac{1}{-\mathcal{Z}}$ inversion is needed in order to remove the quadratic and cubic couplings from $\mathcal{P}_3(\mathcal{Z})$ and $\mathcal{P}_2(\mathcal{Z})$. As we showed in section 3, these geometric flux models do not possess supersymmetric AdS$_4$ vacua with all the moduli (including axionic fields) stabilised.

4.3.2 From type IIB with O3/O7 to type IIA with O6

In going from the IIB with O3/O7 duality frame to the IIA with O6 one by means of applying three T-duality transformations, the generalised NS-NS flux entries are shuffled. This is because T-duality raises and lowers indices according to the chain of transformations in (2.25)

$$\bar{H}_{abc} \xrightarrow{T_a} \omega^a_{bc} \xrightarrow{T_b} Q^{ab}_c \xrightarrow{T_c} R^{abc} \,. \tag{4.32}$$

This flux mixing also happens for the R-R fluxes, although this time the effect of a T-duality transformation T_p is that of creates/annihilates an index

$$T_p : \begin{cases} \bar{F}_{a_1 \ldots a_n} & \rightarrow \quad \bar{F}_{a_1 \ldots a_n p} \\ \bar{F}_{a_1 \ldots a_n p} & \rightarrow \quad \bar{F}_{a_1 \ldots a_n} \end{cases} \,. \tag{4.33}$$

After applying three T-dualities upon the $\bar{F}_3$ flux in the T-fold description, its components map to the different $\bar{F}_p$ flux entries with $p = 0, 2, 4, 6$ in the type IIA picture [7, 8]. The correspondence between flux components is displayed in table 4.5. Consequently, since the flux-induced terms in the scalar potential map between duality frames, the remaining contributions, namely, those coming from localised sources in both descriptions, will also do.

In the T-fold description of the effective models, the scalar potential in (3.54) can be entirely computed from (4.27) using the flux-induced polynomials in table 4.6. The contributions coming from the O3/D3 and O7/D7 localised sources are given by

$$V_{\text{O3/D3}} = -\frac{N_3}{16\,\mu^3} \qquad \text{and} \qquad V_{\text{O7/D7}} = \frac{3\,N_7}{16\,\mu^2\,\sigma} \,, \tag{4.34}$$

where we have made use of the tadpole cancellation conditions shown in table 4.7. The $\mathcal{N} = 1$ scalar potential computed from (3.54) contains the localised sources needed to cancel the flux-induced tadpoles [175]. We will not consider additional localised sources whose effect would have to be included directly in the scalar potential [12]. Therefore, all the contributions to $V_{\text{T-fold}}$ (the scalar potential computed in type IIB with O3/O7-planes) coming from localised sources are those in (4.34).

By applying the relation between the IIB/IIA moduli fields for the supergravity models based on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ isotropic orbifold [175],

$$\begin{array}{ccc} \text{T-fold description} & \leftrightarrow & \text{Type IIA description} \\ \sigma = \dfrac{\tilde{\sigma}}{\tilde{\mu}^3} & \leftrightarrow & \tilde{\sigma} \\ \mu = \tilde{\sigma}\,\tilde{\mu} & \leftrightarrow & \tilde{\mu} \end{array} \tag{4.35}$$

the $V_{\text{O3/D3}}$ and $V_{\text{O7/D7}}$ contributions in (4.34) turn out to depend on the $\tilde{\sigma}$ modulus in the same way as V_{loc} in (4.15).

Computing the IIB with O3/O7-planes scalar potential from eqs (4.27) with the flux-induced polynomials given in table 4.6, and using the moduli relation of eq. (4.35), we end up with the standard form in (4.17) of the scalar potential in the IIA with O6-planes language,

$$V_{\text{IIA}} = A(y,\tilde{\mu},x)\,\tilde{\sigma}^{-2} + B(\tilde{\mu})\,\tilde{\sigma}^{-3} + C(y,\phi)\,\tilde{\sigma}^{-4}\ , \tag{4.36}$$

where ϕ denotes the set of axions $(x,s,t) = (\text{Re}\,\mathcal{Z}, \text{Re}\,\mathcal{S}, \text{Re}\,\mathcal{T})$ and $y = \text{Im}\,\mathcal{Z}$. The functions A, B and C account for the sixteen different sources of potential energy which we list below:

- $A(y,\tilde{\mu},x)$ contains the contributions coming from the set of R, Q, ω and $\bar{H}_3$ fluxes,

$$\begin{aligned} A(y,\tilde{\mu},x) &= y^3\left(\frac{r_1^2}{\tilde{\mu}^6} + r_2^2\,\tilde{\mu}^2\right) + y\left(\frac{q_1^2}{\tilde{\mu}^6} + \frac{q_2}{\tilde{\mu}^2} + q_3\,\tilde{\mu}^2\right) + \\ &+ \frac{1}{y}\left(\frac{\omega_1^2}{\tilde{\mu}^6} + \frac{\omega_2}{\tilde{\mu}^2} + \omega_3\,\tilde{\mu}^2\right) + \frac{1}{y^3}\left(\frac{h_1^2}{\tilde{\mu}^6} + h_2^2\,\tilde{\mu}^2\right)\ . \end{aligned} \tag{4.37}$$

- $B(\tilde{\mu})$ accounts for the potential energy stored within the O6-planes and D6-branes localised sources,

$$B(\tilde{\mu}) = \frac{-1}{16}\left(\frac{N_3}{\tilde{\mu}^3} - 3\,N_7\,\tilde{\mu}\right)\ . \tag{4.38}$$

  - The O3/D3 sources in the T-fold description are interpreted in the type IIA language as O6/D6 sources wrapping a 3-cycle of the internal space, which is invariant under the type IIA with O6-planes orientifold action $\sigma_{ii)}$ in (2.9)[6]. In the following, we will refer to these O6/D6 sources as type 1, see figure 4.1.

[6] In the type IIA language, only the O6/D6 sources wrapping this invariant three cycle preserve $\mathcal{N} = 4$ supersymmetry [180]. Since these sources are reinterpreted as O3/D3 sources in the type IIB language, the Jacobi identities descending from a truncation of a $\mathcal{N} = 4$ supergravity algebra would have nothing to say about their number [155].

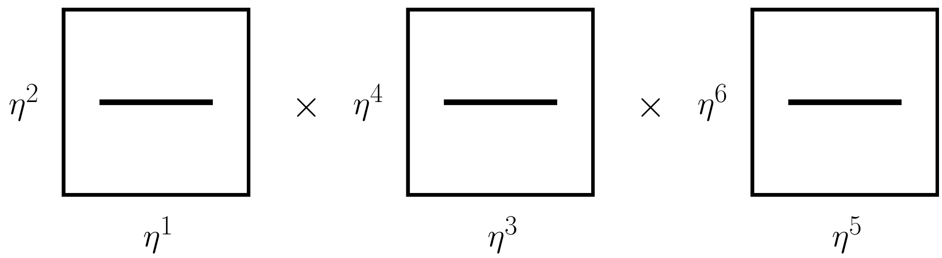

Figure 4.1: A type 1 D6-brane originated from an initial D3-brane by applying three T-dualities along the η^1, η^3 and η^5 directions.

- The O7/D7 sources in the T-fold description correspond in the type IIA picture as O6/D6 sources wrapping three cycles which are invariant under the composition of both the IIA orientifold together with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold actions [8]. We will refer to these O6/D6 sources as type 2, see figure 4.2.

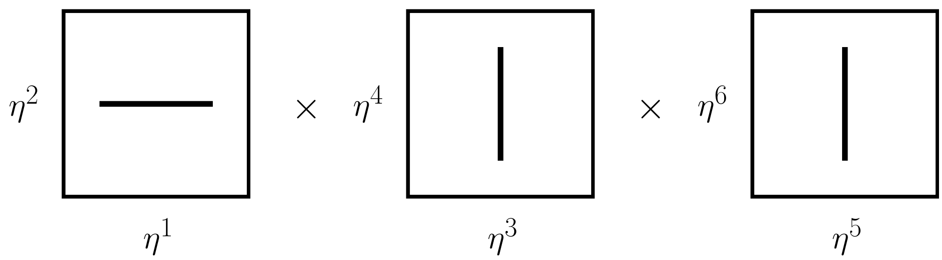

Figure 4.2: Example of type 2 D6-brane coming from an initial D7-brane wrapping the 4-torus $\left(\frac{1}{2}, \frac{1}{2}\right) \times \mathbb{T}_2 \times \mathbb{T}_3$ in the internal space. The D6-brane appears after performing three T-dualities along the η^1, η^3 and η^5 directions.

- $C(y, \phi)$ contains the terms in the scalar potential induced by the $\bar{F}_p$ R-R p-form fluxes with $p = 0, 2, 4$ and 6,

$$C(y, \phi) = y^3 f_0^2 + y f_2^2 + \frac{f_4^2}{y} + \frac{f_6^2}{y^3} \ . \tag{4.39}$$

Taking a look at the $(\tilde{\sigma}, y)$-scaling properties of the different terms appearing in these functions, they are easily identified in the type IIA picture of eqs (4.14)-(4.16), resulting in a dictionary between both descriptions at the level of the scalar potential. In fact,

$$V_{\text{T-fold}} \leftrightarrow V_{\text{IIA}} \ , \tag{4.40}$$

after applying (4.35) and reinterpreting the different scalar potential contributions in the T-fold description with respect to the type IIA with O6-planes duality frame.

All the terms in the scalar potential V_{IIA} of (4.12) and (4.13) are reproduced. Making their dependence on the axions explicit, they are given by

$$V_\omega = \tilde{\sigma}^{-2}y^{-1}\tilde{\mu}^{-6}\Big(\omega_1^2(x) + \omega_2(x)\tilde{\mu}^4 + \omega_3(x)\,\tilde{\mu}^8\Big) \quad , \quad V_{\bar{H}_3} = \tilde{\sigma}^{-2}y^{-3}\tilde{\mu}^{-6}\Big(h_1^2(x) + h_2^2(x)\,\tilde{\mu}^8\Big)$$

$$V_Q = \tilde{\sigma}^{-2}y\,\tilde{\mu}^{-6}\Big(q_1^2(x) + q_2(x)\tilde{\mu}^4 + q_3(x)\,\tilde{\mu}^8\Big) \qquad , \quad V_R = \tilde{\sigma}^{-2}y^3\,\tilde{\mu}^{-6}\Big(r_1^2(x) + r_2^2(x)\,\tilde{\mu}^8\Big) \tag{4.41}$$

for the set of generalised flux-induced terms,

$$V_{\text{loc}} = -\frac{1}{16}\,\tilde{\sigma}^{-3}\,\tilde{\mu}^{-3}\,\left(N_3 - 3\,N_7\,\tilde{\mu}^4\right) \ , \tag{4.42}$$

for the potential energy within the O6/D6 localised sources and

$$V_{\bar{F}_p} = \tilde{\sigma}^{-4}\,y^{(3-p)}\,f_p^2(x,s,t) \qquad , \qquad p = 0,2,4,6 \ , \tag{4.43}$$

for the R-R flux-induced contributions. The above decomposition of the scalar potential holds after the non linear action of an arbitrary matrix $\Theta \in \text{GL}(2,\mathbb{R})_{\mathcal{Z}}$ on the redefined complex structure modulus $\mathcal{Z} \to \Theta^{-1}\,\mathcal{Z}$.

The contributions $V_{\bar{H}_3}$, V_ω, V_Q and V_R involve only the axion $x = \text{Re}\mathcal{Z}$ unlike the set of $V_{\bar{F}_p}$ that depends on the entire set of them. Specifically, the functions $f_p(x,s,t)$ have a linear dependence on the axions s and t. It is clear from (4.41) and (4.43) that $V_{\bar{H}_3}, V_R, V_{\bar{F}_p}$ are positive definite, as well as the V_{ω_1} and V_{q_1} terms induced by ω_1 and q_1 respectively.

At this point we would like to make a rough comparison of the scalar potential in (4.36), involving the entire set of moduli fields, with that of ref. [17] obtained in the volume-dilaton two (non-axionic) moduli limit. First of all, the compactifications studied there do not include non-geometric fluxes, i.e. $V_Q = V_R = 0$, so that $V_\omega \geq 0$ at any dS/Mkw vacuum. The setup in ref. [17] also reduces the contributions in (4.42), accounting for localised sources, to the piece involving N_3 (with $N_3 > 0$). Finally, another difference is that the functions $r_{1,2}\,,\, q_{1,2,3}\,,\, \omega_{1,2,3}\,,\, h_{1,2}$ and f_p in (4.41) and (4.43) can not be taken to be constant as in ref. [17], but they do depend on the set of axions, ϕ. Hence these are dynamical quantities to be determined by the moduli VEVs.

4.4 Discarding type IIB flux models

Armed with the mapping between the T-fold and the type IIA descriptions of the effective models presented in the previous section, we investigate now how the no-go theorem of eq. (4.20), on the existence of dS/Mkw vacua, can be used in this context. We restrict ourselves to vacua with *all* moduli (including axions) stabilised by fluxes. We do not consider the limiting cases defined by:

1. $\epsilon_1 = \epsilon_2 = 0$, for which $\mathcal{P}_2(\mathcal{Z}) = 0$ and the shifted dilaton $\mathcal{S}$ can not be stabilised by the fluxes. This excludes algebras 2, 4 and 17 in table 4.4.

2. $\kappa_1 = \kappa_2 = 0$, yielding $\mathcal{P}_3(\mathcal{Z}) = 0$ and leaving the shifted Kähler modulus $\mathcal{T}$ not stabilised. This case results in no-scale supergravity models [152] previously found [84, 93], since the $\mathcal{T}$ modulus does not enter the superpotential in (4.27). This discards algebra 20 in table 4.4.

Furthermore, we will also assume that $\bar{F}_p \neq 0$ for some $p = 0, 2, 4, 6$. However, we will consider a much weaker version of the no-go theorem of eq. (4.20), given by

$$\Delta V > 0 \,. \tag{4.44}$$

The reason for doing this is that our classification of the supergravity algebras, which is the building block for finding vacua, has nothing to do with R-R fluxes[7]. These will not be used in the process of excluding algebras through the no-go theorem, and, therefore, we will use eq. (4.44) (rather than eq. (4.20)) in what follows. Note that, in any case, the R-R fluxes defining the $V_{\bar{F}_p}$ contributions in (4.43) will play a crucial role in the stabilisation of the axions.

Working with the flux-induced polynomials $\mathcal{P}_2(\mathcal{Z})$ and $\mathcal{P}_3(\mathcal{Z})$ from table 4.6, corresponds to defining the supergravity algebra $\mathfrak{g}$ in the canonical basis of tables 4.2 and 4.3. In this basis, ΔV reads

$$\Delta V = \frac{3\,|\Gamma|^3}{16\, y\, \tilde{\sigma}^2\, \tilde{\mu}^6} \left(l_2\, \tilde{\mu}^8 + l_1\, \tilde{\mu}^4 + l_0 \right) \qquad \text{where} \qquad |\Gamma|\,,\, l_0 > 0 \,. \tag{4.45}$$

The functions l_2, l_1 and l_0 in the polynomial of (4.45) may depend on the $\mathcal{Z}$ modulus and determine whether or not ΔV can be positive (provided that $y_0, \tilde{\sigma}_0, \tilde{\mu}_0 > 0$ at any physical vacuum).

In some cases, moving to a different algebra basis may simplify the flux-induced polynomials in the superpotential, since they are built from the structure constants of $\mathfrak{g}$. Then, a higher number of zero entries in the structure constants translates into simpler effective models. This also simplifies the l_2, l_1 and l_0 functions in (4.45), which determine whether the necessary condition in (4.44) can be fulfilled.

Starting with the effective theory derived in the canonical basis of $\mathfrak{g}$, and by applying a non linear $\Theta \in \mathrm{GL}(2,\mathbb{R})_{\mathcal{Z}}$ transformation upon the $\mathcal{Z}$ modulus, $\mathcal{Z} \rightarrow \Theta^{-1}\,\mathcal{Z}$, we end up

[7] In this work we have used the IIB with O3/O7 supergravity algebra given in eq. (2.46) and proposed in ref. [7]. It is totally specified by the non-geometric Q and the NS-NS $\bar{H}_3$ fluxes. In the most recent articles of refs [155, 161], the origin of these IIB generalised flux models as gaugings of $\mathcal{N} = 4$ supergravity was explored. The R-R $\bar{F}_3$ flux was found to also enter the $\mathcal{N} = 4$ supergravity algebra, written this time in terms of both electric and magnetic gauge/isometry generators.

with an equivalent effective theory formulated in a non canonical basis. The generators in this new basis are related to the original X^a and Z_a through the same rotation of (4.24), by simply replacing

$$\Gamma \rightarrow \Theta\,\Gamma\,. \tag{4.46}$$

Since the scalar potential decomposition introduced in section 4.3.2 holds after the $\mathcal{Z} \rightarrow \Theta^{-1}\,\mathcal{Z}$ transformation, the form of the ΔV in (4.45) also does. Therefore, restricting the Θ transformations to those with $|\Theta| > 0$, i.e. $\text{Im}\mathcal{Z}_0 > 0 \rightarrow \text{Im}(\Theta^{-1}\,\mathcal{Z}_0) > 0$, guarantees that (4.44) still holds as a necessary condition for dS/Mkw vacua to exist.

The usefulness of moving from the canonical basis to a non-canonical one can be illustrated in the two particular cases determined by the transformations

$$\Theta_1 \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \Theta_2 \equiv \frac{1}{2^{2/3}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\,. \tag{4.47}$$

Θ_1 exchanges the gauge generators $E^I \leftrightarrow \widetilde{E}^I$ as well as the isometry ones $D_I \leftrightarrow \widetilde{D}_I$ (up to signs). On the other hand, Θ_2 induces the well known rotation needed for turning the $\mathfrak{so}(4)$ algebra into the direct sum of $\mathfrak{su}(2)^2$ in both the gauge and isometry subspaces.

4.4.1 Using the canonical basis

Let us start by exploring the existence of dS/Mkw vacua in two sets of effective models computed in the canonical basis of $\mathfrak{g}$:

1. Taking the $\kappa_1 = \kappa_{12}$ solution in (4.4) and fixing $\kappa_2 = 0$, results in $V_Q = V_R = 0$. These models are based on $\mathfrak{g}_{gauge} = \mathfrak{iso}(3)$ and admit a geometric type IIA description. The coefficients determining the quadratic polynomial in (4.45) are given by

$$l_2 = -\kappa_1^2 \quad , \quad l_1 = 4\,\epsilon_1\,\kappa_1^2 \quad \text{and} \quad l_0 = \epsilon_1^2\,\kappa_1^2\,. \tag{4.48}$$

The case with $\epsilon_1 = 0$ translates into $\Delta V < 0$, so dS/Mkw solutions are forbidden for algebra 10 in table 4.4. This supergravity algebra has received special attention in ref. [180], where it has been identified as $\mathfrak{g} = \mathfrak{su}(2) \otimes_{\mathbb{Z}_3} \mathfrak{n}_{9,3}$. In fact, fixing $\kappa_2 = \epsilon_1 = 0$ in the commutation relations of table 4.2, the algebra is given by

$$\begin{aligned} [E^I, E^J] &= \epsilon_{IJK} E^K \quad , \quad [E^I, A_n^J] = \epsilon_{IJK} A_n^K \,, \\ [A_1^I, A_1^J] &= \epsilon_2\;\epsilon_{IJK} A_2^K \quad , \quad [A_1^I, A_2^J] = \epsilon_{IJK} A_3^K \,, \end{aligned} \tag{4.49}$$

with $n = 1, 2, 3$, after identifying $A_1 \equiv \widetilde{D}$, $A_2 \equiv -\widetilde{E}$ and $A_3 \equiv D$. It coincides with that of ref. [180][8], and we can now exclude that it has any dS/Mkw vacua[9]. Moreover, if $\epsilon_1 \neq 0$, the case $\epsilon_2 = 0$ cannot have all the axions stabilised, since only the linear combination $\text{Re}S - \epsilon_1^{-1}\,\text{Re}T$ enters the superpotential.

2. Taking $\kappa_{12} = \kappa_2 = 0$ in (4.4) induces non-geometric $V_Q \neq 0$ and $V_R \neq 0$ contributions in the scalar potential. These models are built from $\mathfrak{g}_{gauge} = \mathfrak{su}(2) + \mathfrak{u}(1)^3$ and the quadratic polynomial in (4.45) results in

$$l_2 = -\kappa_1^2 \quad , \quad l_1 = 2\,\epsilon_1\,\kappa_1^2\,\left(|\mathcal{Z}|^2 + (\text{Im}\mathcal{Z})^2\right) \quad \text{and} \quad l_0 = \epsilon_1^2\,\kappa_1^2\,|\mathcal{Z}|^4\,. \tag{4.50}$$

As in the previous case, the limit $\epsilon_1 = 0$ yields effective models with $\Delta V < 0$ as well as $V_Q = V_R = 0$. They also admit to be described as geometric type IIA flux compactifications where dS/Mkw solutions are again forbidden. Hence the exclusion of algebras 16 and 17 in table 4.4.

4.4.2 Using the Θ_1-transformed basis

Leaving the canonical basis via applying the Θ_1 transformation in (4.47), additional effective models can be excluded from having vacua with non-negative energy:

1. Taking $\kappa_1 = \kappa_{12}$ in (4.4), specifically $\kappa_1 = \kappa_{12} = 0$, the effective models are those based on $\mathfrak{g}_{gauge} = \mathfrak{nil}$. The condition in (4.44) is not efficient in excluding the existence of dS/Mkw vacua in any region of the parameter space when working in the canonical basis of $\mathfrak{g}$.

   Applying the Θ_1^{-1} transformation of $\mathcal{Z} \to \frac{1}{-\mathcal{Z}}$, the flux-induced polynomials get simplified to

$$\mathcal{P}_3(\mathcal{Z}) = 3\,\kappa_2 \quad , \quad \mathcal{P}_2(\mathcal{Z}) = \kappa_2\,\left(\epsilon_1 - 3\,\epsilon_2\,\mathcal{Z}\right)\,, \tag{4.51}$$

   having lower degree than their canonical version shown in table 4.6. In this new basis, the non-geometric contributions to the scalar potential identically vanish, $V_Q = V_R = 0$, so these effective models can eventually be described as geometric type IIA flux compactifications. The coefficients determining the quadratic polynomial in (4.45) are now given by

$$l_2 = 0 \quad , \quad l_1 = 0 \quad \text{and} \quad l_0 = \epsilon_2^2\,\kappa_2^2\,. \tag{4.52}$$

   Therefore, the condition in (4.44) excludes the existence of dS/Mkw vacua in the limit case of $\epsilon_2 = 0$ since $\Delta V = 0$. This is algebra 13 in table 4.4.

[8] Observe that the $(A_2^I, A_3^I)_{I=1,2,3}$ generators expand a $\mathfrak{u}(1)^6$ abelian ideal in the algebra (4.49). After taking the quotient by this abelian ideal, the resulting algebra involving the $(E^I, A_1^I)_{I=1,2,3}$ generators becomes $\mathfrak{iso}(3)$, so (4.49) is equivalent to $\mathfrak{g} = \mathfrak{iso}(3) \oplus_{\mathbb{Z}_3} \mathfrak{u}(1)^6$ as it was identified in table 4.4.

[9] We are always under the assumption of isotropy on both flux backgrounds and moduli VEVs.

2. Taking $\kappa_{12} = \kappa_2 = 0$ in (4.4), $\mathfrak{g}_{gauge} = \mathfrak{su}(2) + \mathfrak{u}(1)^3$. The resulting effective models were previously explored in the canonical basis of $\mathfrak{g}$, discarding the existence of dS/Mkw solutions if $\epsilon_1 = 0$. Applying again the Θ_1^{-1} inversion of $\mathcal{Z} \rightarrow \frac{1}{-\mathcal{Z}}$, the flux-induced polynomials result in

$$\mathcal{P}_3(\mathcal{Z}) = 3\,\kappa_1\,\mathcal{Z}^2 \quad , \quad \mathcal{P}_2(\mathcal{Z}) = \kappa_1\,\left(\epsilon_1 - \epsilon_2\,\mathcal{Z}^3\right) \ , \tag{4.53}$$

and the coefficients in the quadratic polynomial in (4.45) are modified to

$$l_2 = -2\,\kappa_1^2\,(\mathrm{Im}\mathcal{Z})^2 \quad , \quad l_1 = 2\,\kappa_1^2\,\epsilon_1 \quad \text{and} \quad l_0 = \epsilon_2^2\,\kappa_1^2\,|\mathcal{Z}|^4 \ . \tag{4.54}$$

As a result, dS/Mkw vacua are automatically forbidden for $\epsilon_2 = 0$ as long as $\epsilon_1 \leq 0$, corresponding to algebras 17 and 19 in table 4.4.

4.4.3 Using the Θ_2-transformed basis

The last family of effective models to which eq. (4.44) applies in a useful way is that coming from fixing $\kappa_1 = \kappa_{12}$, specifically $\kappa_1 = \kappa_{12} = \kappa_2 = \kappa$ in (4.4). It implies $\mathfrak{g}_{gauge} = \mathfrak{so}(4)$. Performing this time the Θ_2^{-1} transformation of $\mathcal{Z} \rightarrow \frac{1}{2^{1/3}}\left(\frac{\mathcal{Z}+1}{-\mathcal{Z}+1}\right)$, the flux-induced polynomials simplify to

$$\mathcal{P}_3(\mathcal{Z}) = 3\,\kappa\,\mathcal{Z}\,(\mathcal{Z}+1) \quad , \quad \mathcal{P}_2(\mathcal{Z}) = \kappa\,\left(\epsilon_-\,\mathcal{Z}^3 + \epsilon_+\right) \ , \tag{4.55}$$

where $\epsilon_\pm = \epsilon_1 \pm \epsilon_2$. This Θ_2-induced transformation splits $\mathfrak{g}$ into the direct sum of two six-dimensional pieces, $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$, determined by the sign of the $\epsilon_\pm$ parameters, respectively. $\mathfrak{g}$ acquires a manifest $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ graded structure. These effective models result with a symmetry under the exchange of ϵ_- and ϵ_+. In fact, the effective action becomes invariant under this swap, together with the moduli redefinition

$$\mathcal{Z} \rightarrow 1/\mathcal{Z}^* \quad , \quad \mathcal{S} \rightarrow -\mathcal{S}^* \quad , \quad \mathcal{T} \rightarrow -\mathcal{T}^* \ , \tag{4.56}$$

which arises from combining eq. (4.31) with Θ_2^{-1}. This symmetry was already found in the section 3.3.2.5 of chapter 3 where we precisely used this algebra basis when computing the effective theory based on the $\mathfrak{su}(2)^2$ Q-algebra. Notice that the (ϵ_-, ϵ_+) flux parameters in (4.55) when considering the $\mathfrak{so}(4)$ embedding are identified to the (ϵ_1, ϵ_2) parameters in the flux-induced polynomials of table 3.1 when using the $\mathfrak{su}(2)^2$ embedding.

Working out the scalar potential from the polynomials in (4.55), the coefficients in the quadratic polynomial of (4.45) take the form

$$l_2 = -\kappa^2\,\left(1 + 2\,(\mathrm{Im}\mathcal{Z})^2\right) \ , \ l_1 = 2\,\kappa^2\,\left(\epsilon_-\,\left(|\mathcal{Z}|^2 + (\mathrm{Im}\mathcal{Z})^2\right) + \epsilon_+\right) \quad \text{and} \quad l_0 = \epsilon_-^2\,\kappa^2\,|\mathcal{Z}|^4 \ , \tag{4.57}$$

and physically viable dS/Mkw vacua are excluded in the limiting case $\epsilon_- = 0$ as long as $\epsilon_+ \leq 0$. The invariance of the effective action under the exchange of ϵ_- and ϵ_+ together with the moduli redefinitions of (4.56), implies that $(\epsilon_+ = 0\,,\ \epsilon_- \leq 0)$ is also excluded. These are algebras 4 and 8 in table 4.4.

4.4.4 Collecting the results

Finally, the effective models with $\kappa_1 = \kappa_{12} = -\kappa_2 = \kappa$ built from $\mathfrak{g}_{gauge} = \mathfrak{so}(3,1)$ can not be ruled out and may have dS/Mkw vacua at any point in the parameter space. Three main results can be highlighted for our isotropic orbifold, also assuming isotropic VEVs for the moduli:

- Eight of the twenty algebra-based effective models admit a geometrical description as a type IIA flux compactifications, whereas the remaining twelve are forced to be non-geometric flux compactifications in any duality frame.
- The four effective models based on the semisimple supergravity algebras $1, 3, 5$ and 7 are non-geometric flux compactifications in any duality frame.
- No effective model based on a semisimple $\mathfrak{g}$ satisfying the condition in (4.10), can be excluded from having dS/Mkw vacua using (4.44). On the other hand, more than half of the effective models based on non-semisimple supergravity algebras can be discarded.

These results are presented in table 4.8 which complements the previous table 4.4 in characterising the set of non equivalent effective models.

#	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
CLASS	NG	NG	NG	NG	NG	NG	NG	NG	G	G	G	G	G	NG	NG	G	G	NG	NG	G
$V_0 \geq 0$	✓	×	✓	×	✓	✓	✓	×	✓	×	✓	✓	×	✓	✓	×	×	✓	×	×

Table 4.8: Splitting of the algebra-based effective models into two classes: those admitting to be described as geometric (G) flux backgrounds by changing the duality frame, and those being non-geometric (NG) flux backgrounds in any duality frame. The mark × indicates that the model is excluded by the necessary condition in (4.44) from having dS/Mkw vacua ($V_0 \geq 0$), whereas if it is not, we use the label ✓.

Let us analyse a concrete example to illustrate the usefulness of this table. We consider an effective model in terms of the original (U, S, T) moduli fields with the standard Kähler potential in (2.67) and with superpotential

$$\begin{aligned} W(U,S,T) &= 6\,T\,\left(U^3 + U^2 - U - 1\right) + 2\,S\,\left(U^3 + 3\,U^2 + 3\,U + 1\right) \\ &+ 2\,\left(-U^3 + 3\,U^2 - 3\,U - 3\right)\,, \end{aligned} \tag{4.58}$$

where the flux entries are given by $\tilde{c}_1 = \tilde{c}_2 = c_i = -2\,,\ i = 0,..,3$ for the non-geometric Q-flux and $b_0 = b_2 = -b_1 = -b_3 = -2,\ a_0 = -3\,a_1 = -3\,a_2 = -3\,a_3 = -6$ for the NS-NS $\bar{H}_3$ and the R-R $\bar{F}_3$ fluxes. This set of flux coefficients are even and satisfy the Jacobi identities in (2.74) and (3.1). The superpotential in (4.58) looks quite involved in terms of determining whether it can have dS/Mkw vacua.

By applying the GL(2,ℝ) transformation $\mathcal{Z} = \Gamma\, U$ with

$$\Gamma \equiv \frac{1}{16^{\frac{1}{3}}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad , \quad \text{so} \quad |\Gamma|^{3/2} = \frac{1}{4\sqrt{2}} \,, \tag{4.59}$$

the algebra underlying such a flux background results in that of 16 in table 4.4. The commutators are given by those of table 4.3 with $\kappa_1 = 1\,,\ \kappa_{12} = \kappa_2 = 0,\ \epsilon_1 = 0\,,\ \epsilon_2 = 1\,.$

Applying a shift on the moduli that enter the superpotential linearly, i.e. $\xi_s = -\frac{1}{2}$, $\xi_t = \frac{1}{2}$,

$$\mathcal{S} = S - \frac{1}{2} \quad , \quad \mathcal{T} = T + \frac{1}{2}\,, \tag{4.60}$$

we end up with the superpotential in (4.27)

$$\mathcal{W} = \frac{1}{4\sqrt{2}} \left[3\,\mathcal{T}\,\mathcal{Z} + \mathcal{S} - \frac{3}{2}\,\mathcal{Z}^2 - \frac{1}{2}\,\mathcal{Z}^3 \right] \,, \tag{4.61}$$

where $\xi_3 = \xi_7 = \frac{1}{2}$. The tadpole cancellation conditions read $N_3 = N_7 = 16$. In this very much simplified version, compare eq. (4.58) to eq. (4.61), it is easy to see, as was shown above, that the no-go theorem in (4.44) applies,

$$\Delta V \;=\; -\frac{3}{512\,y\,\sigma\,\mu} \;<\; 0 \;, \tag{4.62}$$

and this model does not possess dS/Mkw vacua.

Finally we present an example of non-supersymmetric dS/Mkw vacua. Let us start with the superpotential $\mathcal{W}$ in (4.27) and impose $\mathfrak{g}_{gauge} = \mathfrak{so}(3,1)$ by fixing $\kappa_1 = \kappa_{12} = -\kappa_2 = 1$. To make the model simpler, we will also take $\Gamma = \mathbb{I}_{2\times 2}$ as well as $\xi_s = \xi_t = 0\,.$ Hence,

$$\mathcal{Z} = U \quad , \quad \mathcal{S} = S \quad , \quad \mathcal{T} = T\,, \tag{4.63}$$

and the superpotential reads

$$\begin{aligned} W(U,S,T) \;&=\; -\,3\left(U^3 + U\right)\,T + \Big(3\,\epsilon_1\,U - \epsilon_1\,U^3 + \epsilon_2 - 3\,\epsilon_2\,U^2\Big)\,S \\ &-\; \xi_3\,\Big(\epsilon_1 - 3\,\epsilon_1\,U^2 + \epsilon_2\,U^3 - 3\,\epsilon_2\,U\Big) + 3\,\xi_7\,\left(U^2+1\right)\;. \end{aligned} \tag{4.64}$$

The original fluxes are given by $c_0 = c_2 = \tilde{c}_2 = 0\,,\, c_1 = \tilde{c}_1 = -c_3 = -1$ for the non-geometric Q-flux ; $-b_0 = b_2 = \epsilon_2\,, -b_3 = b_1 = \epsilon_1$ for the NS-NS $\bar{H}_3$ flux ; $a_0 = -\epsilon_1\,\xi_3 + 3\,\xi_7\,,\, -a_1 = a_3 = \epsilon_2\,\xi_3,\ a_2 = \epsilon_1\,\xi_3 + \xi_7$ for the R-R $\bar{F}_3$ flux, and satisfy the Jacobi identities in (2.74) and (3.1).

Further taking $\epsilon_1 = \xi_3 = 1$ and $\xi_7 = 16$, the model is totally defined in terms of a unique ϵ_2 parameter and the underlying supergravity algebra is that of 1 in table 4.4.

minimum	ϵ_2	U_0	S_0	T_0	V_0
dS	44	$0.435 + 0.481\, i$	$-1.152 + 2.008\, i$	$54.628 + 48.684\, i$	3.983×10^{-5}
Mkw	44.3086352	$0.444 + 0.467\, i$	$-1.159 + 1.567\, i$	$55.084 + 34.647\, i$	0
AdS_4	45	$0.454 + 0.444\, i$	$-1.160 + 1.184\, i$	$55.897 + 23.237\, i$	-2.295×10^{-4}

Table 4.9: Extrema of the scalar potential with positive mass for *all* the moduli fields.

Using the minimisation procedure that will be presented in the following section, a non-supersymmetric minimum can be easily found. Moreover, as long as the ϵ_2 parameter varies, this minimum changes from AdS_4 to dS crossing a Minkowski point, as it is shown in table 4.9 and also plotted in figure 4.3.

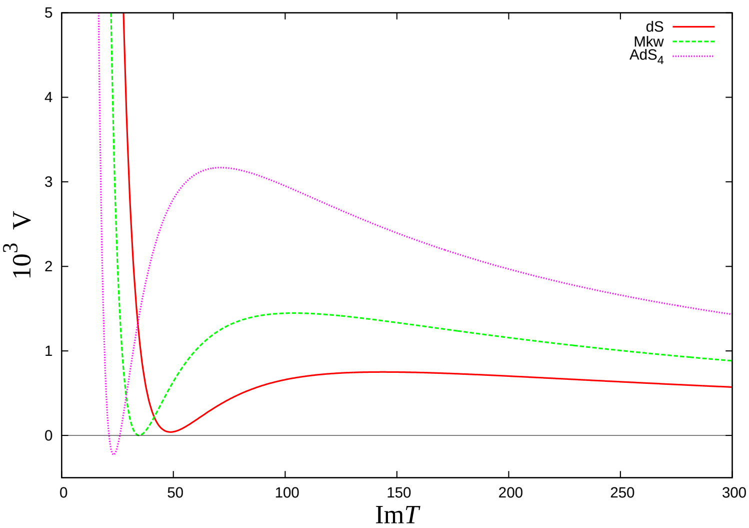

Figure 4.3: Plot of the potential energy, V, as a function of the modulus ImT. To obtain it, we have fixed all the moduli to their VEV but the lightest one, which mostly coincides with ImT. The magenta/dotted line (AdS_4) corresponds to $\epsilon_2 = 45$, the green/dashed one (Mkw) to $\epsilon_2 = 44.309$ and the red/solid line (dS) to $\epsilon_2 = 44$. Note that a tuning of the ϵ_2 parameter is required to obtain a Minkowski vacuum.

The supergravity algebras involving non-geometric flux backgrounds in any duality frame has been found to constitute the main set of algebra-based effective models where to perform a detailed search of dS/Mkw vacua, see table 4.8. One of the main points to be stressed here is that a plain minimisation of the scalar potential, which involves solving very high degree polynomials, is a very inefficient (and, probably, impossible) way of searching for vacua. On the other hand, an analytic calculation can be performed, using the decomposition of the scalar potential (4.41)-(4.43), to work out the stabilisation of the $\mathcal{S}$ and $\mathcal{T}$ moduli, that enter linearly the superpotential of eq. (4.27). After integrating out these fields, the resulting effective potential for the $\mathcal{Z}$ modulus can be tackled numerically.

4.5 Numerical analysis of type IIB effective models

Given the Kähler potential and the superpotential in (4.27), the dynamics of the moduli fields $\Phi \equiv (\mathcal{Z}, \mathcal{S}, \mathcal{T})$ is determined by the standard $\mathcal{N} = 1$ scalar potential in (3.54). Accordingly, the moduli fields are stabilised at the minimum of the potential energy, taking a VEV Φ_0 determined by the conditions

$$\left.\frac{\partial V}{\partial \Phi}\right|_{\Phi=\Phi_0} = 0. \tag{4.65}$$

From now on, our objective will be to solve the above system (4.65) of high degree polynomial equations together with the physical requirement of $V|_{\Phi=\Phi_0} \gtrsim 0$, namely, de Sitter (dS), almost Minkowski (Mkw) solutions. Our strategy will involve finding the exactly Mkw solutions to the minimisation conditions, and then looking for dS extrema continuously connected to them via a deformation of the parameter space.

4.5.1 Minimisation conditions

Since the moduli $\mathcal{S}$ and $\mathcal{T}$ enter the superpotential in (4.27) linearly, the scalar potential V computed from (3.54) can be written as

$$V = |\Gamma|^3 \, e^{\mathcal{K}} \left(m_0 + 2\, m_i\, x_i + M_{ij}\, x_i\, x_j \right) \quad \text{where} \quad i, j = 1, ..., 4 \; , \tag{4.66}$$

and

$$x = \Big(\text{Re}\mathcal{S} \, , \, \text{Re}\mathcal{T} \, , \, \text{Im}\mathcal{S} \, , \, \text{Im}\mathcal{T} \Big) \; . \tag{4.67}$$

Note that, because of the form of the superpotential in (4.27), m_0 and m_i depend on $(\, \mathcal{Z} \, , \, \epsilon_{1,2} \, , \, \xi_{3,7} \,)$, while the matrix M does not depend on the R-R flux parameters $\xi_{3,7}$.

The VEVs of the $\mathcal{S}_0$ and $\mathcal{T}_0$ moduli that extremise the potential at $V = 0$ can be computed analytically since they satisfy

$$\Big(\text{Re}\mathcal{S}_0 \, , \, \text{Re}\mathcal{T}_0 \, , \, \text{Im}\mathcal{S}_0 \, , \, \text{Im}\mathcal{T}_0 \Big) = -M^{-1} \, m\big|_{\mathcal{Z}=\mathcal{Z}_0} \; , \tag{4.68}$$

where we have assumed a non-degenerate M matrix. Otherwise there would be flat directions and the stabilisation of $\mathcal{S}$ and $\mathcal{T}$ would remain incomplete. It is worth mentioning that, when we plug a particular pair $\{\mathcal{P}_2(\mathcal{Z}), \mathcal{P}_3(\mathcal{Z})\}$ of polynomials from table 4.6, M becomes box diagonal and splits into two 2×2 matrices. In other words, axion and volume moduli do not mix[10] in the quadratic polynomial of (4.66).

Using eq. (4.68), the $V = 0$ condition reads

$$m_0 - M^{-1}_{ij} m_i m_j = 0 \; , \tag{4.69}$$

[10] The subtle cancellation of the cross terms is a consequence of the Jacobi identities of the supergravity algebra (2.46), in particular of the $\bar{H}_{x[bc}\, Q_{d]}^{ax} = 0$ constraints.

and provides us with the first constraint between the $\mathcal{Z}$ modulus and the $\epsilon_{1,2}$ and $\xi_{3,7}$ parameters at the Mkw vacua. The function appearing in (4.69),

$$\mathbb{V}(\mathcal{Z}) \equiv m_0 - M^{-1}_{ij} m_i m_j \ , \tag{4.70}$$

plays an important role in the calculation. The equations derived from $\partial_{\text{Re}\mathcal{Z}} V = \partial_{\text{Im}\mathcal{Z}} V = 0$ are just

$$\partial_{\text{Re}\mathcal{Z}}\, \mathbb{V} = 0 \qquad \text{and} \qquad \partial_{\text{Im}\mathcal{Z}} \mathbb{V} = 0 \ , \tag{4.71}$$

where again we have used $V = 0$ and eq. (4.68). The reduced potential $\mathbb{V}(\mathcal{Z})$ captures the Mkw extrema of V and some of their stability properties. In particular, tachyonic Mkw extrema in $\mathbb{V}(\mathcal{Z})$ have their origin in tachyonic Mkw extrema of the full potential V.

4.5.1.1 The $\mathfrak{nil}$ models

We now clarify the previous procedure by explaining the $\mathfrak{nil}$ case in detail. This algebra is defined by the superpotential

$$\mathcal{W} \ = \ |\Gamma|^{3/2} \left[3\,\mathcal{T} + \mathcal{S}\,(\epsilon_1 - 3\,\epsilon_2\,\mathcal{Z}) - \xi_3 \left(\epsilon_1\,\mathcal{Z}^3 + 3\,\epsilon_2\,\mathcal{Z}^2\right) + 3\,\xi_7\,\mathcal{Z}^3 \right] \ , \tag{4.72}$$

and the Kähler potential in (4.27).

The function m_0, derived from this superpotential, is given by [11]

$$m_0 \ = \ 4\,|\mathcal{Z}|^2 \left[\left(|\mathcal{Z}|^2\,(\epsilon_1\,\xi_3 - 3\,\xi_7) + 3\,\epsilon_2\,\xi_3\,\text{Re}\mathcal{Z} \right)^2 + 3\,\epsilon_2^2\,\xi_3^2\,\text{Im}\mathcal{Z}^2 \right] \ , \tag{4.73}$$

whereas the functions m_i are

$$\begin{aligned}
m_1 \ &= \ 4\,\text{Re}\mathcal{Z} \Big[\text{Re}\mathcal{Z} \left(3\,\epsilon_2\,\text{Re}\mathcal{Z} - \epsilon_1\right) \left(\text{Re}\mathcal{Z}\,(\epsilon_1\,\xi_3 - 3\,\xi_7) + 3\,\epsilon_2\,\xi_3 \right) \\
&+ \qquad 3\,\epsilon_2\,\text{Im}\mathcal{Z}^2 \left(\text{Re}\mathcal{Z}\,(\epsilon_1\,\xi_3 - 3\,\xi_7) + 2\,\epsilon_2\,\xi_3 \right) \Big] \ , \\
m_2 \ &= \ -12\,\text{Re}\mathcal{Z}^2 \left[\text{Re}\mathcal{Z}\,(\epsilon_1\,\xi_3 - 3\,\xi_7) + 3\,\epsilon_2\,\xi_3 \right] \ , \\
m_3 \ &= \ -4\,\text{Im}\mathcal{Z}^3 \left[\epsilon_1\ (\epsilon_1\,\xi_3 - 3\,\xi_7) + 3\,\epsilon_2^2\,\xi_3 \right] \ , \\
m_4 \ &= \ -12\,\text{Im}\mathcal{Z}^3\,(\epsilon_1\,\xi_3 - 3\,\xi_7) \ .
\end{aligned} \tag{4.74}$$

As mentioned above, the 4×4 symmetric matrix M splits into two 2×2 matrices, the first one acting on the axions $\text{Re}\mathcal{S}$ and $\text{Re}\mathcal{T}$ with

$$M_{11} = 4\ (3\,\epsilon_2\,\text{Re}\mathcal{Z} - \epsilon_1)^2 + 12\,\epsilon_2^2\,\text{Im}\mathcal{Z}^2 \quad , \quad M_{22} = 36 \quad , \quad M_{12} = -12\ (3\,\epsilon_2\,\text{Re}\mathcal{Z} - \epsilon_1)\,, \tag{4.75}$$

[11] To make the expressions lighter we replace $(\text{Im}\Phi)^q$ with $\text{Im}\Phi^q$, and similarly for any powers of $\text{Re}\Phi$.

and the second one on the volumes $\text{Im}\mathcal{S}$ and $\text{Im}\mathcal{T}$ with

$$M_{33} = 4\,(3\,\epsilon_2\,\text{Re}\mathcal{Z} - \epsilon_1)^2 + 12\,\epsilon_2^2\,\text{Im}\mathcal{Z}^2 \quad , \quad M_{44} = 12 \quad , \quad M_{34} = 0\,. \tag{4.76}$$

The absence of flat directions implies $\epsilon_2 \neq 0$. Otherwise, only the linear combination $3\,\mathcal{T} + \epsilon_1\,\mathcal{S}$ enters the superpotential in (4.72), and the axionic part of its orthogonal combination cannot be fixed.

At this stage, we do not know yet if there will be full, stable Mkw minima. If any, the axions of $\mathcal{S}$ and $\mathcal{T}$ will be fixed at the values

$$\begin{aligned} \epsilon_2\,\text{Re}\mathcal{S}_0 &= -\text{Re}\mathcal{Z}_0^2\,(\epsilon_1\,\xi_3 - 3\,\xi_7) - 2\,\text{Re}\mathcal{Z}_0\,\epsilon_2\,\xi_3\,, \\ 3\,\text{Re}\mathcal{T}_0 &= -\text{Re}\mathcal{S}_0\,\epsilon_1 + 3\,\text{Re}\mathcal{S}_0\,\text{Re}\mathcal{Z}_0\,\epsilon_2 + \text{Re}\mathcal{Z}_0^3\,(\epsilon_1\,\xi_3 - 3\,\xi_7) + 3\,\text{Re}\mathcal{Z}_0^2\,\epsilon_2\,\xi_3\,, \end{aligned} \tag{4.77}$$

while their volume partners will be given by

$$\text{Im}\mathcal{S}_0 = -\left.\frac{m_3}{M_{33}}\right|_{\mathcal{Z}=\mathcal{Z}_0} \quad , \quad \text{Im}\mathcal{T}_0 = -\left.\frac{m_4}{M_{44}}\right|_{\mathcal{Z}=\mathcal{Z}_0} \,. \tag{4.78}$$

Finally we analyse the $\mathcal{Z}$ modulus stabilisation at Mkw vacua, described by the reduced potential $\mathbb{V}(\mathcal{Z})$ in (4.70). The physical Mkw extrema conditions require both

$$\{\ \mathbb{V}\ ,\ \partial_{\text{Re}\mathcal{Z}}\,\mathbb{V}\ ,\ \partial_{\text{Im}\mathcal{Z}}\,\mathbb{V}\ \}_{\mathcal{Z}=\mathcal{Z}_0} = 0 \tag{4.79}$$

and

$$\{\ \det M\ ,\ M_{44}\ m_3\ ,\ M_{33}\ m_4\ \}_{\mathcal{Z}=\mathcal{Z}_0} \neq 0\,. \tag{4.80}$$

The last three conditions ensure a complete stabilisation of $\mathcal{S}$ and $\mathcal{T}$ at non-vanishing $\text{Im}\mathcal{S}_0$ and $\text{Im}\mathcal{T}_0$ values. Plugging the above expressions for $(m_0\,, m_i\,, M)$, it can be shown that these two condition sets are incompatible. Hence we can conclude that there are no Mkw extrema in the supergravity models based on the $\mathfrak{nil}$ B-field reduction.

4.5.2 Parameter space, discrete symmetries and strategy

We want to perform a detailed search of Minkowski extrema for the set of supergravity models based on the non-semisimple $\mathfrak{iso}(3)$ and $\mathfrak{su}(2) + \mathfrak{u}(1)^3$, as well as the semisimple $\mathfrak{so}(4)$ and $\mathfrak{so}(3,1)$ B-field reductions introduced in section 4.3.1. The task will be that of solving the set (4.79) of polynomial equations

$$\mathbb{V}|_{\mathcal{Z}=\mathcal{Z}_0} = 0 \quad , \quad \left.\frac{\partial\,\mathbb{V}}{\text{Re}\mathcal{Z}}\right|_{\mathcal{Z}=\mathcal{Z}_0} = 0 \quad , \quad \left.\frac{\partial\,\mathbb{V}}{\text{Im}\mathcal{Z}}\right|_{\mathcal{Z}=\mathcal{Z}_0} = 0\,. \tag{4.81}$$

The method we will use to find the solutions of (4.81) makes use of the symmetries and the scaling properties of the supergravity models which are now introduced.

- It is worth noticing that the form of the superpotential in (4.27), in particular that of the polynomials $\mathcal{P}_2(\mathcal{Z})$ in table 4.6, allows us to remove the factor $|\epsilon| \equiv \sqrt{\epsilon_1^2+\epsilon_2^2}$ from it (provided it is non-zero) by a rescaling of the $\mathcal{S}$ modulus and a redefinition of the ξ_3 parameter. Therefore, the angle $\tan\theta_\epsilon \equiv \dfrac{\epsilon_2}{\epsilon_1}$ is the only free parameter coming from the NS-NS $\bar{H}_3$ fluxes.

- Analogously, the (non-vanishing) combination $|\xi| \equiv \sqrt{|\epsilon|^2\,\xi_3^2+\xi_7^2}$ can be globally factorised in the superpotential by rescaling both the $\mathcal{S}$ and $\mathcal{T}$ moduli. This leaves the angle given by $\tan\theta_\xi \equiv \dfrac{\xi_7}{|\epsilon|\,\xi_3}$ as the free parameter coming from the R-R $\bar{F}_3$ fluxes.

These parameter redefinitions and moduli rescalings are given by

$$\epsilon_1 \rightarrow |\epsilon|\cos\theta_\epsilon \quad , \quad \epsilon_2 \rightarrow |\epsilon|\sin\theta_\epsilon \quad , \quad \xi_3 \rightarrow \frac{|\xi|}{|\epsilon|}\cos\theta_\xi \quad , \quad \xi_7 \rightarrow |\xi|\sin\theta_\xi \ , \tag{4.82}$$

together with

$$\mathcal{S} \rightarrow \frac{\mathcal{S}\,|\xi|}{|\epsilon|} \qquad \text{and} \qquad \mathcal{T} \rightarrow \mathcal{T}\,|\xi| \ , \tag{4.83}$$

generating a global factor in the superpotential of (4.27) and, therefore, also in the scalar potential built from (3.54),

$$\mathcal{W} \rightarrow |\Gamma|^{\frac{3}{2}}\,|\xi|\ \mathcal{W}(\Phi\,;\,\theta_\epsilon,\theta_\xi) \qquad \text{and} \qquad V \rightarrow \frac{|\Gamma|^3\,|\epsilon|}{|\xi|^2}\ V(\Phi\,;\,\theta_\epsilon,\theta_\xi) \ .$$

This also implies a rescaling of the F-term for all the moduli fields

$$F_{\mathcal{Z}} \rightarrow |\Gamma|^{\frac{3}{2}}\,|\xi|\,F_{\mathcal{Z}}(\Phi\,;\,\theta_\epsilon,\theta_\xi) \ , \ F_{\mathcal{S}} \rightarrow |\Gamma|^{\frac{3}{2}}\,|\epsilon|\,F_{\mathcal{S}}(\Phi\,;\,\theta_\epsilon,\theta_\xi) \ , \ F_{\mathcal{T}} \rightarrow |\Gamma|^{\frac{3}{2}}\,F_{\mathcal{T}}(\Phi\,;\,\theta_\epsilon,\theta_\xi) \, , \tag{4.84}$$

where $F_\Phi \equiv D_\Phi\mathcal{W}$ in eq. (3.54). Then, at any non-supersymmetric extremum with $F_{\Phi=\mathcal{Z},\mathcal{S},\mathcal{T}} \neq 0$, supersymmetry will be mostly broken by $F_{\mathcal{S}}$ ($F_{\mathcal{Z}}$) when the $|\epsilon|$ ($|\xi|$) parameter is large, and also by $F_{\mathcal{T}}$ when both $|\epsilon|$ and $|\xi|$ are small. Furthermore, the normalised moduli masses are also sensitive to these rescalings. From now on, we will always take $|\epsilon| = |\xi| = 1$ when presenting examples of moduli masses at an extremum of the potential.

After applying (4.82) and (4.83), the parameter space of the supergravity models can be understood as a 2-torus with coordinates $(\theta_\epsilon,\theta_\xi)$ shown in figure 4.4.

The set of discrete symmetries in (4.28), (4.29) and (4.30) now act on the moduli fields and the parameter space as follows:

i) $\mathcal{W}$ is invariant under

$$\mathcal{S} \;\rightarrow\; -\mathcal{S} \qquad , \qquad (\,\theta_\epsilon\,,\,\theta_\xi\,) \;\rightarrow\; (\,\theta_\epsilon+\pi\,,\,\pi-\theta_\xi\,) \ . \tag{4.85}$$

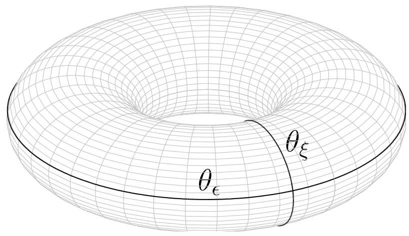

Figure 4.4: The parameter space $(\theta_\epsilon, \theta_\xi)$ as coordinates of a two-dimensional torus.

ii) $\mathcal{W}$ goes to $-\mathcal{W}$ under these two transformations:

$$\mathcal{T} \rightarrow -\mathcal{T} \quad , \quad (\theta_\epsilon\,,\theta_\xi) \rightarrow (\theta_\epsilon+\pi\,, 2\pi-\theta_\xi)\ . \tag{4.86}$$

$$\mathcal{Z} \rightarrow -\mathcal{Z} \quad , \quad (\theta_\epsilon\,,\theta_\xi) \rightarrow (2\pi-\theta_\epsilon\,, \theta_\xi+\pi)\ . \tag{4.87}$$

iii) Finally, since the parameters entering the superpotential are real, we can combine field conjugation with the above transformations to obtain an additional symmetry

$$(\mathcal{Z}\,,\mathcal{S}\,,\mathcal{T}) \rightarrow -(\mathcal{Z}\,,\mathcal{S}\,,\mathcal{T})^* \quad , \quad (\theta_\epsilon\,,\theta_\xi) \rightarrow (2\pi-\theta_\epsilon\,,\theta_\xi)\ , \tag{4.88}$$

which relates physical extrema at $\pm\theta_\epsilon$.

These symmetries of the supergravity models will be extensively used when scanning the parameter space looking for the physical solutions ($\text{Im}\Phi_0 > 0$) to the system (4.65).

The strategy to perform such a search will be the following: our scanning parameter is the angle θ_ϵ, which needs to be evaluated only in the interval $\theta_\epsilon \in [0,\pi]$ because of the symmetry (4.88). The value of θ_ξ can be obtained from the first equation in (4.81) since $\tan\theta_\xi$ enters it quadratically. Substituting $\theta_\xi(\theta_\epsilon,\mathcal{Z}_0)$ into the original system (4.81), it reduces to

$$\left.\frac{\partial\,\mathbb{V}}{\text{Re}\mathcal{Z}}\right|_{\mathcal{Z}=\mathcal{Z}_0} = h_1(\theta_\epsilon,\mathcal{Z}_0) = 0 \qquad \text{and} \qquad \left.\frac{\partial\,\mathbb{V}}{\text{Im}\mathcal{Z}}\right|_{\mathcal{Z}=\mathcal{Z}_0} = h_2(\theta_\epsilon,\mathcal{Z}_0) = 0\ , \tag{4.89}$$

where h_1 and h_2 are complicated functions depending on the supergravity model under consideration. Provided a value for the angle θ_ϵ, the VEV of $\mathcal{Z}_0$ can be numerically computed from (4.89). After that, and using the value obtained for $\theta_\xi(\theta_\epsilon,\mathcal{Z}_0)$, the VEVs for the moduli fields $\mathcal{S}$ and $\mathcal{T}$ can be obtained from (4.68).

In this sense, the modulus $\mathcal{Z}$ is the key field in the stabilisation process, whereas $\mathcal{S}$ and $\mathcal{T}$ simply get adjusted to generate the extremum of the potential. However, there are singular points given by $\text{Im}\mathcal{Z}_0 = 0$. We find that the value of the θ_ϵ parameter and the VEV of the $\text{Re}\mathcal{Z}$ modulus at such points can be obtained[12] from $\mathcal{P}_2(\mathcal{Z}_0) = \mathcal{P}_3(\mathcal{Z}_0) = 0$.

[12]Notice that these conditions correspond to the stabilisation of the $\mathcal{S}$ and $\mathcal{T}$ moduli at a globally supersymmetric extremum, namely $\partial_\mathcal{S}\mathcal{W} = \partial_\mathcal{T}\mathcal{W} = 0$.

4.5.3 Models based on non-semisimple B-field reductions

The first supergravity models we will deal with are those based on non-semisimple B-field reductions, namely, the $\mathfrak{iso}(3)$ and the $\mathfrak{su}(2)+\mathfrak{u}(1)^3$ reductions. These models exhibit a special feature: the functions h_1 and h_2 in (4.89) become homogeneous functions, so the set of Mkw extrema for these models has a scaling nature,

$$\mathcal{Z}_0(\theta_\epsilon) \propto |\tan\theta_\epsilon|^n \ . \tag{4.90}$$

4.5.3.1 The $\mathfrak{iso}(3)$ models

Let us start by exploring Minkowski solutions for the supergravity model based on the $\mathfrak{iso}(3)$ non-semisimple B-field reduction. This model is specified by the Kähler potential in (4.27) and the superpotential

$$\mathcal{W} \;=\; |\Gamma|^{3/2}\left[-3\,\mathcal{T}\,\mathcal{Z} \,+\, \mathcal{S}\,(3\,\epsilon_1\,\mathcal{Z}+\epsilon_2) - \xi_3\,\left(\epsilon_2\,\mathcal{Z}^3 - 3\,\epsilon_1\,\mathcal{Z}^2\right) + 3\,\xi_7\,\mathcal{Z}^2\right] \ . \tag{4.91}$$

Using the procedure introduced in the previous section, we find Mkw extrema in the $\epsilon_1 < 0$ range, as shown in figure 4.5. They are all rescaled solutions of the form

$$\mathcal{Z}_0(\theta_\epsilon) = |\tan\theta_\epsilon|\,(\pm 0.30920 + 0.11495\,i) \ , \tag{4.92}$$

and have a tachyonic direction, hence being unstable.

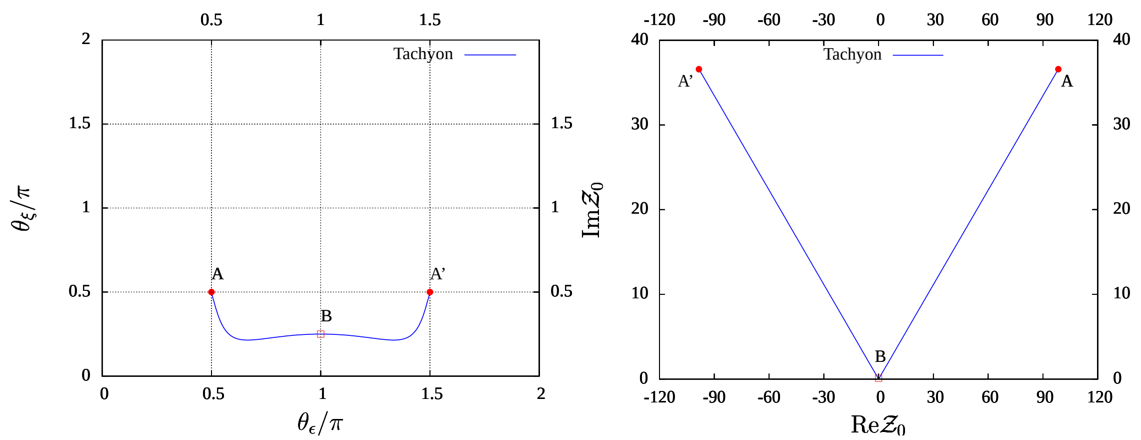

Figure 4.5: Left: location of the Mkw solutions within the parameter space for the supergravity models based on the $\mathfrak{iso}(3)$ B-field reduction, highlighting the singular points. Right: the set of VEVs of the modulus $\mathcal{Z}$, reflecting its scaling nature. The points A and A' correspond to a singular limit $|\mathcal{Z}_0| \to \infty$.

The set of singular points in the figure, as well as the supergravity algebras underlying the different regions in the plots, are summarised as follows:

i) Points A and A' have an underlying $\mathfrak{g} = \mathfrak{iso}(3) \oplus_{\mathbb{Z}_3} \mathfrak{u}(1)^6$ and are conjugate points with respect to the transformations in (4.85) and (4.88). As we flow towards them, the tachyon aligns with the Im$\mathcal{S}$ modulus direction and $|\Phi_0| \to \infty$ for all the moduli fields. Due to their underlying supergravity algebra, these points were excluded to have dS/Mkw extrema in table 4.8.

In the following, we will generically refer to such points as points of excluded supergravity algebras. They will show up as singularities in the moduli VEVs.

ii) All along the $\overline{\text{AA'}}$ line, including point B located at $(\theta_\epsilon, \theta_\xi) = (\pi, \frac{\pi}{4})$, there is a unique underlying supergravity algebra $\mathfrak{g} = \mathfrak{so}(4) \oplus_{\mathbb{Z}_3} \mathfrak{u}(1)^6$. As we flow towards this point B, the tachyon aligns with the Im$\mathcal{Z}$ modulus direction, and $|\Phi_0| \to 0$ for all the moduli fields, becoming again a singularity in the moduli VEVs.

Unlike the previous A and A' points, the supergravity algebra underlying the point B is not excluded to have dS/Mkw extrema in table 4.8. Therefore, with some abuse of the language, we will refer to these points as dynamical singularities in the moduli VEVs. Observe that the $\overline{\text{AA'}}$ line in the left plot of figure 4.5 is smooth at the singular point B.

4.5.3.2 The $\mathfrak{su}(2) + \mathfrak{u}(1)^3$ models

Let us continue with the second set of supergravity models based on a non-semisimple B-field reduction. Those models are based on the $\mathfrak{su}(2) + \mathfrak{u}(1)^3$ reduction. They are defined by (4.27) with the superpotential

$$\mathcal{W} \;=\; |\Gamma|^{3/2} \left[3\, \mathcal{T}\, \mathcal{Z} \,+\, \mathcal{S}\, \left(\epsilon_1\, \mathcal{Z}^3 + \epsilon_2\right) + \xi_3 \left(\epsilon_1 - \epsilon_2\, \mathcal{Z}^3\right) - 3\, \xi_7\, \mathcal{Z}^2 \right] \;. \tag{4.93}$$

The set of Minkowski solutions for this model is very similar to that previously analysed. This time, they correspond to solutions of the form

$$\mathcal{Z}_0(\theta_\epsilon) = |\tan\theta_\epsilon|^{\frac{1}{3}} \left(\pm 0.99368 + 0.55061\, i\right) \,, \tag{4.94}$$

and also have a tachyonic direction, being unstable.

The results are shown in the two plots of figure 4.6, where the singular points can be described as follows:

i) Points A and A' have an underlying $\mathfrak{g} = \mathfrak{iso}(3) + \mathfrak{nil}$ and, as in the previous case, are conjugate points with respect to the transformations in (4.85) and (4.88). As we flow towards these points, the tachyon aligns with the Im$\mathcal{S}$ modulus direction and also $|\Phi_0| \to \infty$ for all the moduli fields. Therefore they are again singularities associated to points of excluded supergravity algebras in table 4.8.

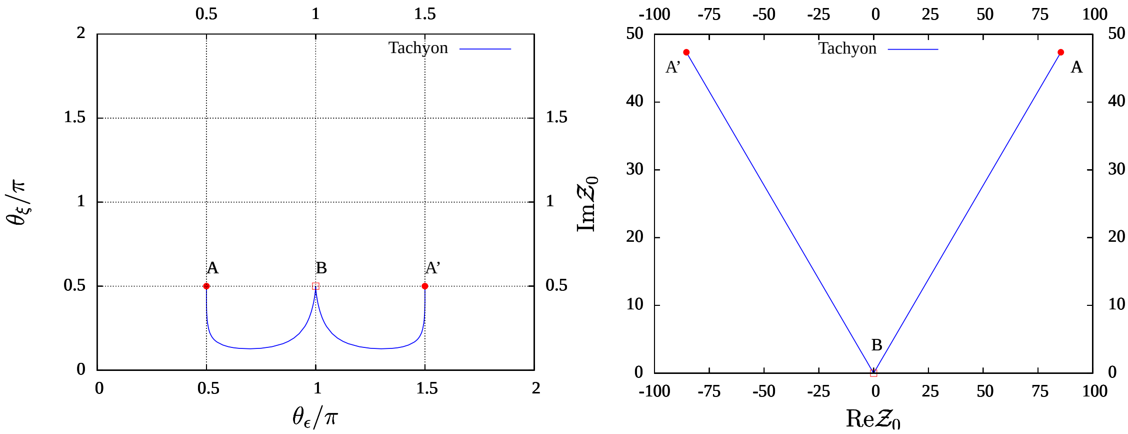

Figure 4.6: Left: location of the Mkw solutions within the parameter space for the supergravity models based on the $\mathfrak{su}(2)+\mathfrak{u}(1)^3$ B-field reduction, highlighting the singular points. Right: set of VEVs of the modulus $\mathcal{Z}$, reflecting its scaling nature. Again, the points A and A' are singular since $|\mathcal{Z}_0| \to \infty$.

ii) Along the $\overline{\text{AB}}$ and $\overline{\text{BA'}}$ lines, the supergravity algebra is $\mathfrak{g} = \mathfrak{so}(4) + \mathfrak{nil}$. However, this time point B corresponds to a different algebra, $\mathfrak{g} = \mathfrak{so}(4) + \mathfrak{u}(1)^6$, which cannot have Minkowski extrema as it is shown in table 4.8. As we flow towards this point B, $|\Phi_0| \to 0$ for all the moduli fields, resulting in a singularity in the moduli VEVs. Observe that the line of Mkw extrema is no longer smooth at this point, around which the tachyon aligns itself along the Im$\mathcal{S}$ modulus direction.

4.5.4 Models based on semisimple B-field reductions

In the final part of this section we concentrate on the supergravity models based on the semisimple B-field reductions of $\mathfrak{so}(4)$ and $\mathfrak{so}(3,1)$. Their distribution of Minkowski extrema is more involved than that of the previous models based on non-semisimple reductions. This is mainly because the scaling property in (4.90) no longer takes place.

As we will see, the distribution of Minkowski extrema draws closed curves in both the parameter space and the $\mathcal{Z}_0$ complex plane. Although the former has to be understood as a closed curve up to some of the discrete transformation in (4.85) and (4.86), the latter is a truly closed curve in the $\mathcal{Z}_0$ complex plane.

4.5.4.1 The $\mathfrak{so}(4)$ models

The first supergravity model based on a semisimple B-field reduction we are going to describe is that of the $\mathfrak{so}(4)$ reduction. This model is defined in eqs (4.27) with the superpotential given by

$$\begin{aligned} \mathcal{W} &= |\Gamma|^{3/2} \left[3\,\mathcal{T}\,\left(\mathcal{Z}^3 - \mathcal{Z}\right) + \mathcal{S}\,\left(\epsilon_1\,\mathcal{Z}^3 + 3\,\epsilon_2\,\mathcal{Z}^2 + 3\,\epsilon_1\,\mathcal{Z} + \epsilon_2\right) \right. \\ &+ \left. \xi_3\left(\epsilon_1 - 3\,\epsilon_2\,\mathcal{Z} + 3\,\epsilon_1\,\mathcal{Z}^2 - \epsilon_2\,\mathcal{Z}^3\right) - 3\,\xi_7\,\left(1 - \mathcal{Z}^2\right)\right] \;. \end{aligned} \tag{4.95}$$

As it happens for the supergravity models studied so far, there are only Minkowski solutions with a tachyonic direction. These unstable Mkw solutions are shown in figure 4.7, where the singular points highlighted in the plots are now explained:

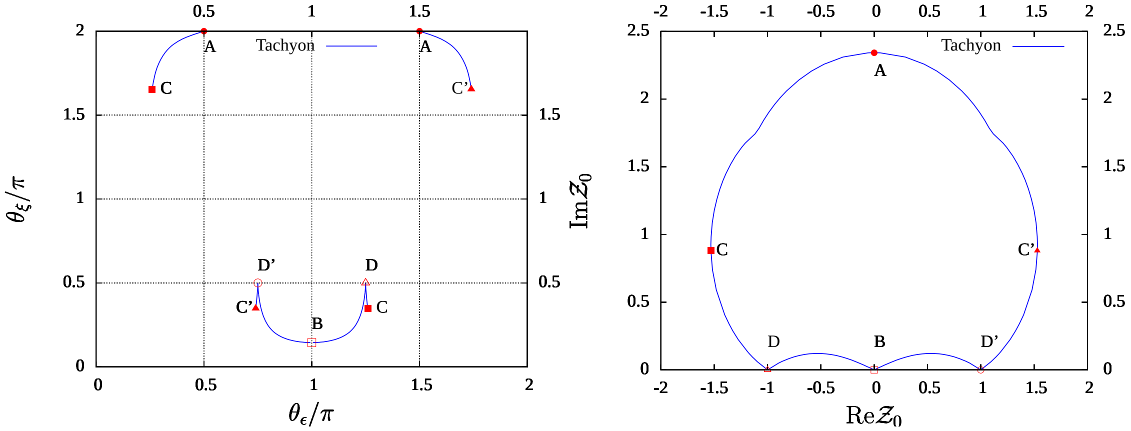

Figure 4.7: Left: location of the Mkw solutions within the parameter space for the supergravity models based on the $\mathfrak{so}(4)$ B-field reduction, highlighting the singular points. Right: set of VEVs of the modulus $\mathcal{Z}$. Note that, up to discrete transformations, the Mkw extrema describe closed curves in both plots.

i) Points D and D' have an underlying $\mathfrak{g} = \mathfrak{iso}(3) + \mathfrak{so}(4)$ and are conjugate points with respect to the transformation in (4.88). They are points of excluded supergravity algebras in table 4.8. As we flow towards these points, $\mathrm{Im}\mathcal{S}_0 \to \infty$ while $\mathrm{Im}\mathcal{T}_0$, $\mathrm{Im}\mathcal{Z}_0 \to 0$. The tachyonic direction in field space is aligned with the $\mathrm{Im}\mathcal{S}$ modulus direction.

ii) The $\overline{\text{DD'}}$ line, going through the singular point B, has an underlying $\mathfrak{g} = \mathfrak{so}(4)^2$ supergravity algebra. As we flow towards point B, the tachyon is still mostly aligned with $\mathrm{Im}\mathcal{S}$, and $\mathrm{Im}\Phi_0 \to 0$ for all the moduli fields, becoming once more a dynamical singularity in the moduli VEVs. However the axions behave differently when approaching the B point: $\mathrm{Re}\mathcal{Z}_0 \to 0$, $\mathrm{Re}\mathcal{S}_0 \to \pm\infty$ and $\mathrm{Re}\mathcal{T}_0 \to \mp\infty$, with the upper sign choice if approaching from the left, and the other way around when approaching from the right. Notice, again, that this $\overline{\text{DD'}}$ line in the left plot of figure 4.7 is smooth.

iii) The $\overline{\text{DD'}}$ line going through the singular points C, C' and A, has an underlying $\mathfrak{g} = \mathfrak{so}(3,1) + \mathfrak{so}(4)$ supergravity algebra. This path, shown in the left plot of figure 4.7, is discontinuous at points C, C' and A because of the vanishing of the Im$\mathcal{T}$ modulus. The pairs of points with identical labels are conjugate points with respect to the transformation in (4.86). As we flow towards points C and C', Im$\mathcal{T}_0 \to 0$, and the tachyonic direction aligns 50% in the Im$\mathcal{S}$ direction and 50% in the Re$\mathcal{S}$ one. Finally, moving towards point A, Im$\mathcal{T}_0 \to 0$ and the tachyon is aligned with the Im$\mathcal{Z}$ direction. These points C, C' and A are, then, dynamical singularities in the moduli VEVs.

4.5.4.2 The $\mathfrak{so}(3,1)$ models

The last, but not least, supergravity model based on a semisimple B-field reduction is $\mathfrak{so}(3,1)$. This model is defined in eqs (4.27) by the superpotential

$$\begin{aligned} \mathcal{W} \;=\; & |\Gamma|^{3/2}\left[-3\,\mathcal{T}\,\left(\mathcal{Z}^3+\mathcal{Z}\right)+\mathcal{S}\,\left(\epsilon_2+3\,\epsilon_1\,\mathcal{Z}-3\,\epsilon_2\,\mathcal{Z}^2-\epsilon_1\,\mathcal{Z}^3\right)\right. \\ - \; & \left.\xi_3\left(\epsilon_1-3\,\epsilon_1\,\mathcal{Z}^2-3\,\epsilon_2\,\mathcal{Z}+\epsilon_2\,\mathcal{Z}^3\right)+3\,\xi_7\,\left(1+\mathcal{Z}^2\right)\right]\,. \end{aligned} \tag{4.96}$$

The most interesting feature of this model is that it contains stable, Minkowski vacua within a certain region of the parameter space as well as unstable Mkw solutions, like those of the previously analysed models, in a different one.

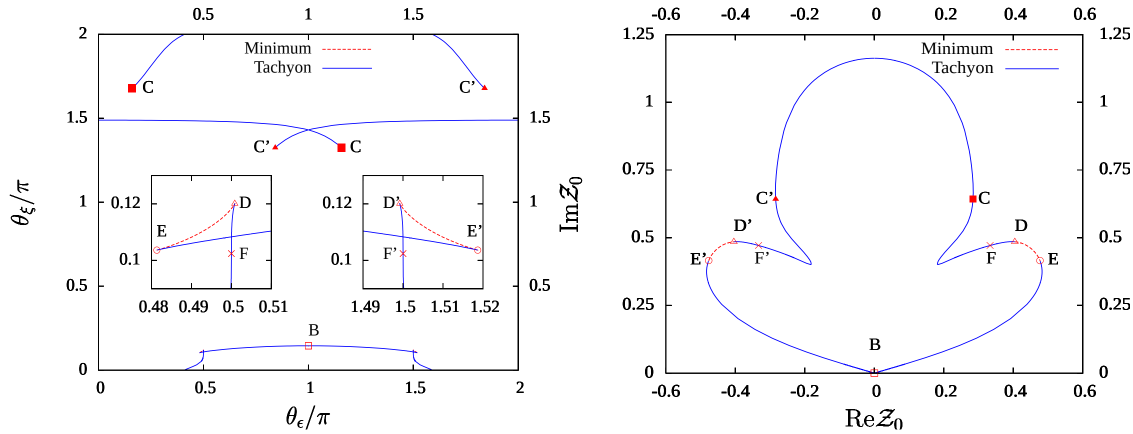

Figure 4.8: Left: location of the Mkw solutions within the parameter space for the supergravity models based on the $\mathfrak{so}(3,1)$ B-field reduction, highlighting the singular points. Right: set of VEVs of the modulus $\mathcal{Z}$. Notice that, up to discrete transformations, the Mkw extrema describe closed curves in both plots.

Another property of this model is that any point in the parameter space has a $\mathfrak{g} = \mathfrak{so}(3,1)^2$ supergravity algebra underlying it. Therefore, any singularity in the moduli VEVs is a dynamical singularity. The entire set of Minkowski solutions are shown in figure 4.8.

With respect to the highlighted points in the figure, let us divide the parameter space in three pieces: the $\overline{\text{DD'}}$ line going through the points C and C'; the $\overline{\text{EE'}}$ line going through the point B; and the $\overline{\text{DE}}$ & $\overline{\text{D'E'}}$ lines, containing the stable Mkw vacua:

i) At the points D, D', E and E', the Mkw extrema have a flat direction associated to volume directions [13]. This direction is, roughly, 58% Im$\mathcal{S}$ and 42% Im$\mathcal{T}$ at the D and D' points, whereas it becomes 72% Im$\mathcal{S}$, 25% Im$\mathcal{T}$ and 3% Im$\mathcal{Z}$ at the E and E' points.

ii) The $\overline{\text{EE'}}$ line contains the singular point B. When moving towards it, the tachyon mostly aligns with the Im$\mathcal{S}$ direction and Im$\Phi_0 \to 0$ for all the moduli fields. The axions behave differently when approaching this point: Re$\mathcal{Z}_0 \to 0$, Re$\mathcal{S}_0 \to \mp\infty$ and Re$\mathcal{T}_0 \to \pm\infty$, with the upper sign choice if flowing from the left and the other choice when flowing from the right. Again, the $\overline{\text{EE'}}$ line in the left plot of figure 4.8 is smooth.

iii) The $\overline{\text{DD'}}$ path goes through the singular points (F,F') and (C,C'). At (F,F') [14], it is discontinuous due to the double limits Im$\mathcal{S}_0 \to \frac{0}{0}$ and Im$\mathcal{T}_0 \to \frac{0}{0}$ in eq. (4.68). However, as we flow towards points C and C', a vanishing Im$\mathcal{S}_0 \to 0$ takes place, and the tachyonic direction mainly aligns with the Im$\mathcal{T}$ volume direction. These points are, again, dynamical singularities in the moduli VEVs. Observe that points equally labelled in figure 4.8 are conjugate points with respect to the transformation in (4.85).

iv) The $\overline{\text{DE}}$ & $\overline{\text{D'E'}}$ lines contain the stable Mkw vacua and will be explored separately.

There are two specially symmetric points which belong to part *iii*) of the parameter space. The first one comes from noticing that this piece exhibits the novel feature of having a crossing at the point $(\theta_\epsilon, \theta_\xi) = (\pi, 1.43082\pi)$. This crossing takes place in the parameter space, not in the moduli space, so two separate unstable Minkowski extrema

$$\mathcal{Z}_0 = \pm 0.27527 + 0.80635\, i\ ,\ \ |\epsilon||\xi|^{-1}\mathcal{S}_0 = \mp 0.87477 + 0.30709\, i\ ,\ \ |\xi|^{-1}\mathcal{T}_0 = \mp 0.44718 + 1.19429\, i, \tag{4.97}$$

with the tachyonic direction mostly along the Im$\mathcal{T}$ volume direction, coexist at this point. The second point, located at $(\theta_\epsilon, \theta_\xi) = (0, 1.48913\pi)$, gives rise to an axion-vanishing unstable Mkw solution

$$\mathcal{Z}_0 = 1.16280\, i \quad , \quad |\epsilon||\xi|^{-1}\mathcal{S}_0 = 0.30849\, i \quad , \quad |\xi|^{-1}\mathcal{T}_0 = 0.78019\, i\ , \tag{4.98}$$

invariant under the $\Phi \to -\Phi^*$ transformation of (4.88). The tachyonic direction is totally contained within the axion field space, with the relative contributions of 37% for Re$\mathcal{S}$, 40% for Re$\mathcal{T}$ and 23% for Re$\mathcal{Z}$.

[13] At these points, the 2×2 reduced Hessian built from $\mathbb{V}(\mathcal{Z})$ in eq. (4.70), becomes degenerate.

[14] These F and F' points can be analytically computed and correspond to $(\theta_\epsilon, \theta_\xi) = \left(\pm\frac{\pi}{2}\, ,\, \arctan\left(\frac{1}{3}\right)\right)$ together with the VEVs of $\mathcal{Z}_0 = \pm\frac{1}{3} + \frac{\sqrt{2}}{3} i$.

$\overline{\text{DE}}$ & $\overline{\text{D'E'}}$ lines of stable vacua.

Let us look into the region within the parameter space that contains totally stable Minkowski vacua, namely, the $\overline{\text{DE}}$ & $\overline{\text{D'E'}}$ lines shown in figure 4.8. Provided a value for θ_ϵ within

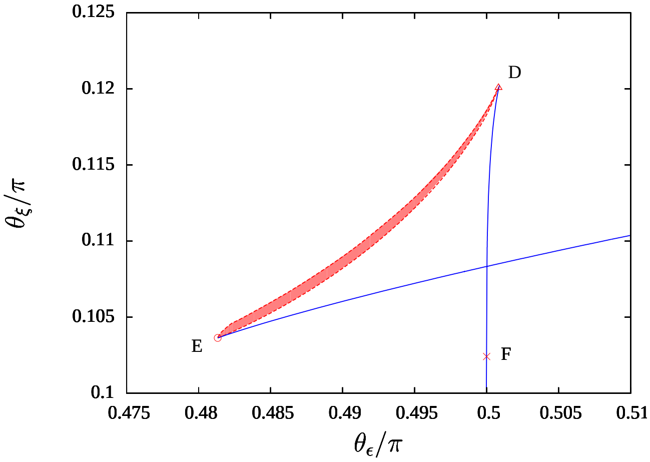

Figure 4.9: This figure shows the narrow band above the line of stable Mkw vacua containing stable, dS vacua.

the region $\overline{\text{DE}}$ (and equivalently for $\overline{\text{D'E'}}$), a stable dS vacuum emerges from varying the θ_ξ angle slightly with respect to its value at the Mkw vacuum,

$$\theta_\xi^{(\text{dS})} = \theta_\xi^{(\text{Mkw})} + \delta\theta_\xi \qquad \text{with} \qquad \delta\theta_\xi > 0 \ . \tag{4.99}$$

There is a critical value, $\delta\theta_\xi^*$, beyond which the dS vacuum no longer exists[15]. This behaviour is represented in figure 4.9. The dS vacua found in this way are deformations of the Mkw ones and are also stable along any direction in field space. Therefore, there is a narrow region above the line of Mkw vacua, shown in figure 4.9, which incorporates dS stable vacua. Moreover if we choose $\delta\theta_\xi < 0$, the original Mkw vacuum becomes stable AdS_4.

At these Mkw/dS vacua, supersymmetry is broken by a non-vanishing F-term for all the moduli fields[16], i.e. $F_{\Phi=\mathcal{Z},\mathcal{S},\mathcal{T}} \neq 0$. This agrees with the general results concerning the existence of non-supersymmetric, stable, Minkowski vacua stated in refs [183, 184]. Given that supersymmetry is broken by all directions considered here, seven complex ones in total, the constraint on the Kähler potential outlined in these works, formulated as the number of fields breaking supersymmetry being larger than three, is fulfilled.

[15] As an example, in the case of $\theta_\epsilon = \frac{49\,\pi}{100}$, the moduli VEVs at the Mkw vacuum are given by $\mathcal{Z}_0 = 0.45089 + 0.46042i\,,\ |\epsilon||\xi|^{-1}\mathcal{S}_0 = -1.07734 + 1.28783i$ and $|\xi|^{-1}\mathcal{T}_0 = 1.15629 + 0.60267i$. This Mkw vacuum is compatible with $\theta_\xi^{(\text{Mkw})} = 0.10821\,\pi\,,$ while the critical value for deforming it to dS (with $V_0 = 3.4\times 10^{-3}\,|\Gamma|^3\,|\epsilon|\,|\xi|^{-2}\,m_p^4$) is given by $\delta\theta_\xi^* = 0.00079\,\pi$.

[16] In the case of $\theta_\epsilon = \frac{49\,\pi}{100}$, the values of the F-terms at the Mkw vacuum are given by $|\Gamma|^{-\frac{3}{2}}\,|\xi|^{-1}\,F_{\mathcal{Z}} = 4.00933 + 3.48324i\,,\ |\Gamma|^{-\frac{3}{2}}\,|\epsilon|^{-1}\,F_{\mathcal{S}} = 0.46460 - 0.00623i$ and $|\Gamma|^{-\frac{3}{2}}\,F_{\mathcal{T}} = -4.67506 + 5.76899i$.

The (positive) smallest eigenvalue of the mass matrix is mostly associated to a combination of the Im$\mathcal{S}$ and Im$\mathcal{T}$ moduli fields. At the Mkw vacua, the rest of the moduli masses are about a couple of order of magnitudes above the lightest one, unlike in scenarios including gaugino condensation or other non-perturbative effects [12]. In the absence of large hierarchies we cannot split the stabilisation process into a $2+1$ fields problem, but the problem intrinsically becomes a 3 fields one. This property is related to the fact that all the moduli are stabilised due to fluxes, so one would not expect to have mysterious cancellations in the mass terms in order to generate a hierarchy.

At this stage, it would be interesting to know whether the potential has other minima with lower energy (i.e. AdS_4 vacua). The case of supersymmetric minima is tractable using the techniques developed in chapter 3. We find that they cannot coexist with the Mkw/dS vacua shown in figure 4.9. On the other hand, an exhaustive analysis of non-supersymmetric AdS_4 vacua is much more involved and we have not addressed it in this thesis.

4.6 Comparison with type IIA scenarios

The set of type IIB supergravity models we have explored in the previous sections are dual to type IIA generalised flux models through applying three T-duality transformations along internal space directions [7,8,95]. Several no-go theorems (see section 4.2) concerning the existence of Mkw/dS extrema in these type IIA generalised flux models have been stated as well as ways for circumventing them [13,14,16,17,137]. In this section we will use the mapping introduced in section 4.3.2 between the set of generalised flux models we derived in a type IIB with O3/O7-planes language, and their generalised type IIA dual flux models with O6-planes. Our purpose will be to investigate how the different sources of potential energy in (4.41), (4.42) and (4.43) do conspire to produce the Mkw extrema we have found.

The scalar potential in the type IIA dual supergravity models then splits as

$$V_{\text{IIA}} = V_{\text{NS-NS}} + V_{\text{loc}} + V_{\text{R-R}} \ , \tag{4.100}$$

with $V_{\text{NS-NS}} = V_{\bar{H}_3} + V_{\omega} + V_Q + V_R$ accounting for the generalised NS-NS fluxes, $V_{\text{loc}} = V_{\text{loc}}^{(1)} + V_{\text{loc}}^{(2)}$ accounting for the O6/D6 localised sources (types 1 and 2 shown in figures 4.1 and 4.2 respectively) and $V_{\text{R-R}} = V_{\bar{F}_0} + V_{\bar{F}_2} + V_{\bar{F}_4} + V_{\bar{F}_6}$ accounting for the R-R fluxes. It is worth recalling that the axions Re$\mathcal{S}$ and Re$\mathcal{T}$ enter the scalar potential only through the R-R piece $V_{\text{R-R}} \subset V_{\text{IIA}}$ in (4.43), which can be rewritten as

$$V_{\text{R-R}} = \sum_{p=0\,(even)}^{6} V_{\bar{F}_p} = e^{\mathcal{K}} \sum_{p=0\,(even)}^{6} \text{Im}\mathcal{Z}^{(6-p)} \Big(f_p(\text{Re}\Phi) \Big)^2 \ . \tag{4.101}$$

The functions f_p, with $p = 0, 2, 4$ and 6, depend on Re$\mathcal{S}$ and Re$\mathcal{T}$ linearly, as stated in section 4.3.2.

For dS/Mkw extrema to exist in these supergravity models, the terms in the scalar potential of (4.100) induced by the generalised NS-NS fluxes and the R-R fluxes have to satisfy the conditions in (4.18) and (4.21),

$$\begin{aligned} \left(V_\omega - V_{\bar{F}_2}\right) \;&+\; 2\left(V_Q - V_{\bar{F}_4}\right) \;+\; 3\left(V_R - V_{\bar{F}_6}\right) \;\geq\; 0\,, \\ \left(V_{\bar{F}_0} - V_{\bar{H}_3}\right) \;&+\; \left(V_Q - V_{\bar{F}_4}\right) \;+\; 2\left(V_R - V_{\bar{F}_6}\right) \;\geq\; 0\,, \end{aligned} \tag{4.102}$$

where all the R-R flux-induced terms, $V_{\bar{F}_p}$, are positive definite, as well as the $V_{\bar{H}_3}$ and V_R terms coming from the fluxes $\bar{H}_3$ and R, respectively (see section 4.3.2). The inequalities in (4.102) are saturated at the Mkw extrema. Therefore, if restricting ourselves to the set of geometric IIA flux models described above, there is a $V_{\bar{F}_0} \neq 0$ condition (non-vanishing Romans parameter) needed for having dS extrema.

It was shown in table 4.8 that the IIA duals of the IIB supergravity models based on the $\mathfrak{nil}$ and $\mathfrak{iso}(3)$ B-field reductions yield $V_Q = V_R = 0$, hence resulting in geometric IIA flux models [84,93,99,136,175,176,185]. This is also the case for the models based on the $\mathfrak{su}(2) + \mathfrak{u}(1)^3$ B-field reduction at the special circles $\theta_\epsilon = \pm\frac{\pi}{2}$ within the parameter space. Far from these circles as well as in those supergravity models based on the $\mathfrak{so}(4)$ and $\mathfrak{so}(3,1)$ B-field reductions, $V_Q \neq 0$ and/or $V_R \neq 0$, giving rise to non-geometric IIA flux models.

At this point, and before presenting our results in type IIA language, it is convenient to highlight the similarities and differences with related work published in the literature on the existence of de Sitter solutions and no-go theorems:

- As we already mentioned, our framework is also that of ref. [14], which we have extended to include the set of generalised fluxes needed to restore T-duality.
- Our initial setup does not contain KK5-branes [16, 17] or NS5-branes [13, 17].
- The minimisation procedure considers the dependence of the scalar potential on the axions which are treated as dynamical variables. In the above references, they are set to constant values and do not feature in the scalar potential.

There are also substantial differences between our work with that of ref. [137]. On the one hand, these authors consider Kähler and complex structure moduli in addition to the dilaton and volume moduli considered in the previous works reviewed here. However, the potential contains the effect of just geometric fluxes, in addition to the usual NS-NS 3-form flux, R-R fluxes and O6/D6 sources. Nevertheless they manage to find a couple of $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold models that, within their working numerical precision, are compatible with de Sitter vacua. These are both anisotropic models and cannot, therefore, be compared to ours. In any case it is worth mentioning that, throughout their analysis, these authors find plenty of solutions with one tachyonic direction, just as it happens in our analysis.

4.6.1 Minkowski extrema in geometric type IIA flux models

As we have stated above, there are three sets of type IIB supergravity models that become dual to geometric type IIA flux models with

$$V_Q = V_R = 0\ . \tag{4.103}$$

They are the models based on the $\mathfrak{nil}$ and $\mathfrak{iso}(3)$ B-field reductions together with those based on the $\mathfrak{su}(2) + \mathfrak{u}(1)^3$ reduction at the circles defined by $\theta_\epsilon = \pm\frac{\pi}{2}$ in the parameter space.

A common feature in all these IIA dual geometric models is that only the f_4 and f_6 functions appearing in (4.101) depend (linearly) on the Re$\mathcal{S}$ and Re$\mathcal{T}$ axions. Then, their stabilisation conditions, provided Im$\mathcal{Z}_0 \neq 0$, translate into

$$V_{\bar{F}_4} = V_{\bar{F}_6} = 0\ . \tag{4.104}$$

Substituting (4.103) and (4.104) into the inequalities of (4.102), we obtain, for any Minkowski extremum, that

$$V_{\bar{H}_3} = V_{\bar{F}_0} \qquad \text{and} \qquad V_\omega = V_{\bar{F}_2}\ , \tag{4.105}$$

so $V_{\text{NS-NS}} = V_{\text{R-R}} > 0$ at such extrema[17]. Then, the negative energy contribution needed to set $V_{\text{IIA}} = 0$ in (4.100) will come from the localised sources, i.e. $V_{\text{loc}} < 0$ (see figure 4.10).

The IIB supergravity models based on the $\mathfrak{nil}$ reduction were found in section 4.5.1.1 not to accommodate for Mkw extrema while those based on the $\mathfrak{su}(2) + \mathfrak{u}(1)^3$ reduction at the circles $\theta_\epsilon = \pm\frac{\pi}{2}$ were excluded to possess Mkw extrema in table 4.8. Therefore, the tachyonic Mkw extrema we found in the supergravity models based on the $\mathfrak{iso}(3)$ B-field reduction, constitute the entire set of geometric IIA dual Minkowski flux extrema for the isotropic $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. These extrema have an underlying $\mathfrak{g} = \mathfrak{so}(4) \oplus_{\mathbb{Z}_3} \mathfrak{u}(1)^6$ supergravity algebra.

The IIA dual contributions to the scalar potential at the geometric Mkw flux extrema are shown in figure 4.10 (where $m_p = 1/\sqrt{8\pi G_{\text{N}}} \approx 2 \times 10^{18}\,\text{GeV}$). Although they are plotted for a particular point within the parameter space, the profile of the contributions does not change when moving from one point to another, due to the scaling property in (4.90) explained in section 4.5.3. Observe that the negative energy contribution needed to obtain $V_{\text{IIA}} = 0$ comes from type 1 O6/D6 sources. Specifically, from O6-planes which carry negative charge. Moreover, additional positive energy coming from type 2 D6-branes

[17] Notice that due to the positiveness of $V_{\bar{F}_2}$, the V_ω contribution to the scalar potential coming from the (negative) curvature of the internal space (induced by the metric flux ω) results also positive as it was stated in refs [16, 17].

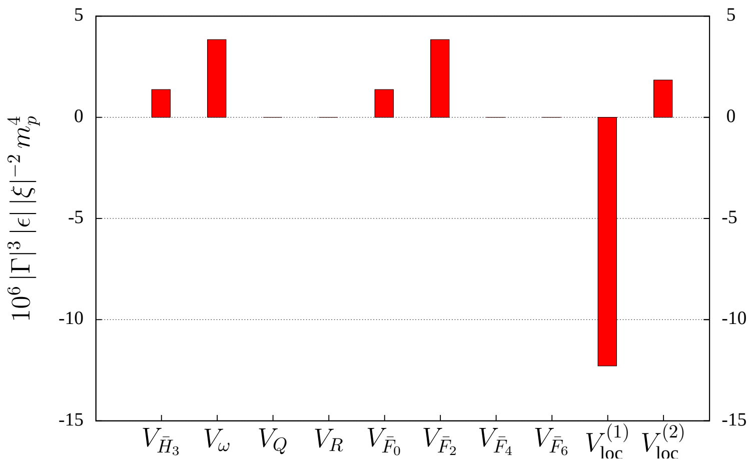

Figure 4.10: IIA dual contributions to the scalar potential at the Mkw extrema for the supergravity models based on the $\mathfrak{iso}(3)$ B-field reduction. They are computed at the circle $\theta_\epsilon = \frac{3\pi}{4}$ in the parameter space which implies $\theta_\xi = 0.22375\,\pi$ and the moduli VEVs of $\mathcal{Z}_0 = 0.30920 + 0.11495\,i$, $|\epsilon||\xi|^{-1}\mathcal{S}_0 = -0.00171 + 0.01276i$ and $|\xi|^{-1}\mathcal{T}_0 = 0.01579 + 0.00092i$. It can be seen that $V_Q = V_R = V_{\bar{F}_4} = V_{\bar{F}_6} = 0$ as well as $V_{\bar{H}_3} = V_{\bar{F}_0} > 0$ and $V_\omega = V_{\bar{F}_2} > 0$.

with positive charge is also required. These type 2 sources are forbidden in the $\mathbb{Z}_2$ orbifold compactifications of refs [7, 8, 95], so these geometric IIA dual Mkw extrema are not expected to exist there.

Finally, for these IIB supergravity models based on the $\mathfrak{iso}(3)$ B-field reduction, the IIA dual Romans parameter which generates the $V_{\bar{F}_0}$ contribution required for having dS extrema, reads

$$f_0^2 = 4\,|\Gamma|^3\,|\epsilon|^2\,|\xi|^2\,(\sin\theta_\epsilon)^2\,(\cos\theta_\xi)^2\ , \tag{4.106}$$

so it vanishes at the A, A' and B singular points shown in figure 4.5. Far from these points, an unstable dS extremum emerges from varying the θ_ξ angle slightly with respect to its value at the Mkw extremum, $\theta_\xi^{(\text{dS})} = \theta_\xi^{(\text{Mkw})} + \delta\theta_\xi$ with $\delta\theta_\xi > 0$, as it has been previously explained for the case of the stable dS vacua in the $\mathfrak{so}(3,1)$-based models. Also a critical value $\delta\theta_\xi^*$ appears beyond which dS solutions no longer exist.

4.6.2 Minkowski extrema in non-geometric type IIA flux models

Now we present the IIA dual energy contributions at the Mkw extrema for the supergravity models which are non-geometric type IIA generalised flux models. These models are those based on the $\mathfrak{su}(2) + \mathfrak{u}(1)^3$ (with $\theta_\epsilon \neq \pm\frac{\pi}{2}$), $\mathfrak{so}(4)$ and $\mathfrak{so}(3,1)$ B-field reductions.

4.6.2.1 The $\mathfrak{su}(2) + \mathfrak{u}(1)^3$ models

As it was stated in section 4.5.3, these models also have the scaling property in (4.90) of the geometric IIA dual models. Therefore, their profile of energy contributions, shown in figure 4.11, does not change from one point within the parameter space to another.

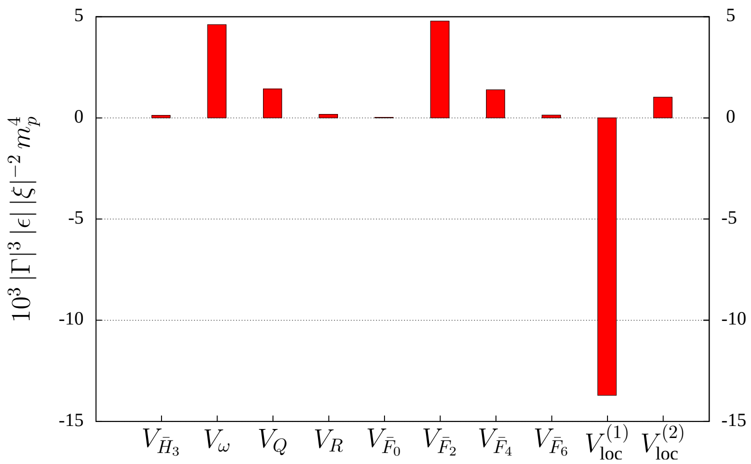

Figure 4.11: IIA dual contributions to the scalar potential at the Mkw extrema for the supergravity models based on the $\mathfrak{su}(2) + \mathfrak{u}(1)^3$ B-field reduction. They are computed, again, at the circle $\theta_\epsilon = \frac{3\pi}{4}$ in the parameter space implying this time $\theta_\xi = 0.13055\,\pi$ and the moduli VEVs of $\mathcal{Z}_0 = 0.99368 + 0.55061\,i\,,\ |\epsilon||\xi|^{-1}\mathcal{S}_0 = -1.01524 + 0.28041i$ and $|\xi|^{-1}\mathcal{T}_0 = 0.82169 + 0.01611i$.

These non-geometric type IIA dual flux models (note that $V_Q \neq 0$ and $V_R \neq 0$) need again of localised sources to achieve Minkowski (unstable) solutions. Analogously to the geometric case, type 1 O6-planes and type 2 D6-branes are required, as it can be seen in figure 4.11. Also each contribution in $V_{\text{NS-NS}}$ and $V_{\text{R-R}}$ is positive at the Mkw solutions. Finally, unstable dS solutions can again be obtained by deforming these Mkw extrema, as for the geometric IIA dual models.

4.6.2.2 The $\mathfrak{so}(4)$ models

The next supergravity models whose IIA duals become non-geometric flux models are those based on the semisimple $\mathfrak{so}(4)$ B-field reduction. The contributions to the potential energy at the Minkowski extrema do not fit a unique pattern, as it has been the case for the supergravity models analysed so far. Such contributions do depend on the point in the parameter space under consideration, since the scaling property in (4.90) is no longer present in these models.

In order to illustrate the above statement, let us recall the form of the contributions to the scalar potential coming from the localised sources. They were computed in sec-

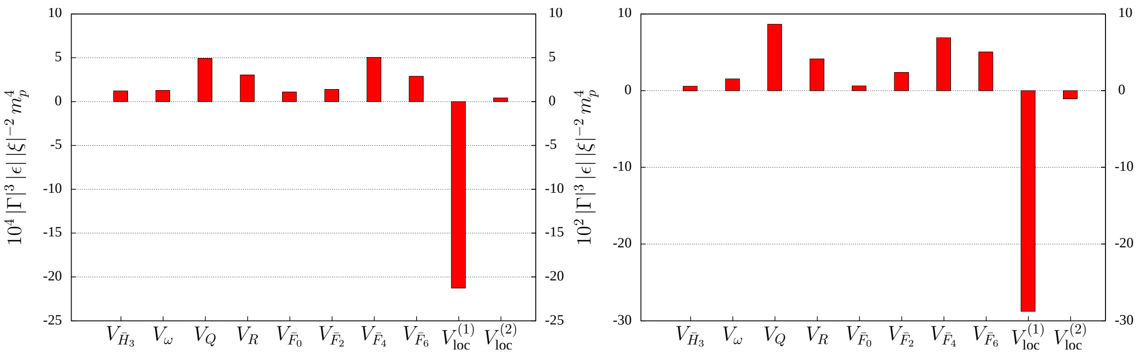

Figure 4.12: IIA dual contributions to the scalar potential at a Mkw extremum for the supergravity models based on the $\mathfrak{so}(4)$ B-field reduction. In the left plot, they are computed at the circle $\theta_\epsilon = \frac{1255}{1000}\pi$ which belongs to the $\overline{\rm DC}$ piece of the parameter space and implies $\theta_\xi = 0.40225\,\pi$ together with the moduli VEVs of $\mathcal{Z}_0 = -1.48023 + 0.59103i$, $|\epsilon||\xi|^{-1}\mathcal{S}_0 = -1.15549 + 3.44203i$ and $|\xi|^{-1}\mathcal{T}_0 = 0.13871 + 0.00708i$. In the right plot they are computed at the circle $\theta_\epsilon = \frac{3\pi}{8}$ which belongs to the $\overline{\rm CA}$ piece of the parameter space and implies $\theta_\xi = 1.93558\,\pi$ together with the moduli VEVs of $\mathcal{Z}_0 = -0.73422 + 2.12313i$, $|\epsilon||\xi|^{-1}\mathcal{S}_0 = -0.57723 + 0.74810i$ and $|\xi|^{-1}\mathcal{T}_0 = 0.33682 + 0.04399i$.

tion 4.3.2, and given by

$$V_{\rm loc}^{(1)} = -\frac{|\epsilon|\,|\Gamma|^3}{4\,|\xi|^2\,{\rm Im}\mathcal{T}^3}\cos\theta_\xi \qquad \text{and} \qquad V_{\rm loc}^{(2)} = \frac{3\,|\epsilon|\,|\Gamma|^3}{4\,|\xi|^2\,{\rm Im}\mathcal{T}^2\,{\rm Im}\mathcal{S}}\sin\theta_\xi\ , \tag{4.107}$$

for the supergravity models based on semisimple B-field reductions. Then $V_{\rm loc}^{(1)} = 0$ at the D and D' singular points shown in figure 4.7, while $V_{\rm loc}^{(1)} < 0$ in all the Mkw solutions. On the other side, $V_{\rm loc}^{(2)} = 0$ at the singular point A, whereas $V_{\rm loc}^{(2)} > 0$ for the Mkw solutions along the $\overline{\rm CC'}$ line that goes through point B, and $V_{\rm loc}^{(2)} < 0$ if doing so through point A. An example is shown in figure 4.12, where the sign of the energy contribution provided by type 2 localised sources is different for the two Mkw solutions. In the left plot type 2 D6-branes are required, while type 2 O6-planes are needed in the right one.

Finally, one observes that the flux-induced $\mathcal{P}_{2,3}(\mathcal{Z})$ polynomials for these models reduce to those of the geometric ($\mathfrak{iso}(3)$-based) models around $\mathcal{Z}=0$, as it can be seen from their form in table 4.6. As long as we take the limit $\theta_\epsilon \to \pi$, the profile (up to some scale factor) of the energy contributions to the Mkw extrema tend to that of the geometric models in figure 4.10. Once more, unstable dS extrema can be obtained by a continuous deformation of the Mkw solutions, namely, by taking $\theta_\xi \to \theta_\xi + \delta\theta_\xi$ for a given θ_ϵ circle.

4.6.2.3 The $\mathfrak{so}(3,1)$ models

Let us conclude by looking into the energy contributions to the Mkw extrema for the IIA duals of the supergravity models based on the $\mathfrak{so}(3,1)$ B-field reduction. As for the previous semisimple models, such contributions depend critically on the specific point within the parameter space under consideration.

The set of Minkowski solutions for this model is shown in figure 4.8, where a narrow region within the parameter space, that of the $\overline{\text{DE}}$ & $\overline{\text{D'E'}}$ lines, was found to contain stable vacua. At these stable vacua, $V_\omega < 0$ and $V_{\text{loc}}^{(1)} < 0$, while the rest of the contributions to the scalar potential are positive. Then, these stable vacua need type 1 O6-planes and type 2 D6-branes to exist. As long as we flow between the points D and E in figure 4.8, the main contributions to $\frac{|\xi|^2}{|\Gamma|^3\,|\epsilon|\,m_p^4} V_{\text{IIA}}$ change from being of order $\mathcal{O}(10^{-2})$ around the point D, to become of order $\mathcal{O}(1)$ around the point E. An intermediate point in the $\overline{\text{DE}}$ line is shown in figure 4.13.

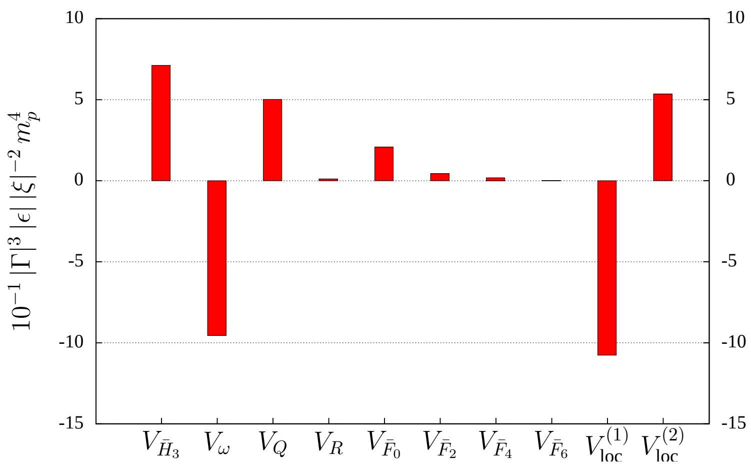

Figure 4.13: IIA dual contributions to the scalar potential at a stable Mkw vacuum for the supergravity models based on the $\mathfrak{so}(3,1)$ B-field reduction. They are computed at the circle $\theta_\epsilon = \frac{49\pi}{100}$ which belongs to the $\overline{\text{DE}}$ piece of the parameter space and implies $\theta_\xi = 0.10821\,\pi$, together with the moduli VEVs of $\mathcal{Z}_0 = 0.45089 + 0.46042i$, $|\epsilon||\xi|^{-1}\mathcal{S}_0 = -1.07734 + 1.28783i$ and $|\xi|^{-1}\mathcal{T}_0 = 1.15629 + 0.60267i$.

For these supergravity models, the contributions to the potential energy coming from localised sources are still given by (4.107). By inspection of figure 4.8, we conclude that there are unstable Mkw solutions having $V_{\text{loc}}^{(2)} \gtrless 0$. Even more, there is a particularly interesting solution with $V_{\text{loc}}^{(2)} = 0$. It is located at the point $(\theta_\epsilon, \theta_\xi) = (0.40904\pi, 0)$ within the parameter space, and its profile of the contributions to V_{IIA} is shown in figure 4.14.

Naturally, its image point under the transformation $\Phi \to -\Phi^*$ of (4.88) is also a solution with $V_{\text{loc}}^{(2)} = 0$. These unstable Mkw solutions are the only ones that would also exist in the $\mathbb{Z}_2$ orbifold compactification of refs [7, 8, 95], that does not allow type 2 O6/D6 sources. We will see in chapter 5 that, in the absence of such sources, these unstable solutions could presumably be lifted to solutions of a $\mathcal{N} = 4$ gauged supergravity [84, 155, 176, 178, 180, 186] built from an $\text{SL}(2,\mathbb{Z})_S$ electric-magnetic gauging.

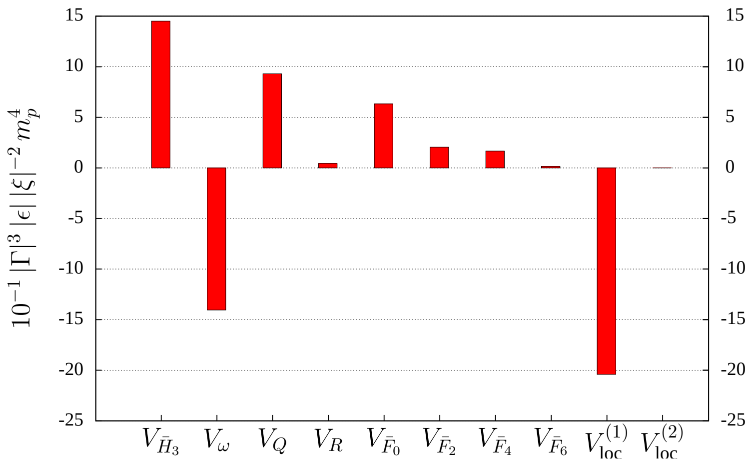

Figure 4.14: IIA dual contributions to the scalar potential at a unstable Mkw solution for the supergravity models based on the $\mathfrak{so}(3,1)$ B-field reduction. It is computed at the point $(\theta_\epsilon, \theta_\xi) = (0.40904\pi, 0)$ which belongs to the $\overline{\text{CD}}$ line in the parameter space. The VEVs for the moduli fields result in $\mathcal{Z}_0 = 0.18657 + 0.41905i\,,\ |\epsilon||\xi|^{-1}\mathcal{S}_0 = 0.12569 + 0.32326i$ and $|\xi|^{-1}\mathcal{T}_0 = 0.76855 + 0.49656i$.

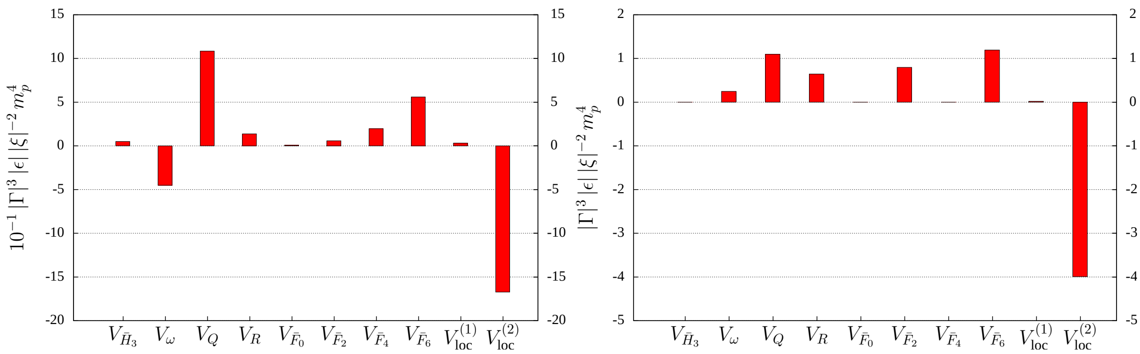

Figure 4.15: Left: IIA dual contributions to the scalar potential at the unstable Mkw solutions that coexist at the point $(\theta_\epsilon, \theta_\xi) = (\pi, 1.43082\pi)$ in the parameter space. Right: same for the axion-vanishing solution at the point of parameter space given by $(\theta_\epsilon, \theta_\xi) = (0, 1.48913\pi)$.

Furthermore, it can also be seen in figure 4.8 that, unlike in the previous supergravity models, unstable solutions with $V_{\text{loc}}^{(1)} > 0$ exist along the $\overline{\text{CC'}}$ line with $\pi < \theta_\xi < \frac{3\pi}{2}$. These solutions require type 1 D6-branes Sare compatible with $V_{\text{loc}}^{(2)} < 0$, so type 2 O6-planes have to be present. The point in the parameter space in which the two separate moduli solutions of (4.97) coexist, belongs to this set of solution and its sources of potential energy are shown in the left plot of figure 4.15. The point in the parameter space having the axion-vanishing moduli VEVs of (4.98), also belongs to this class. In this solution, $\text{Re}F_\Phi = 0$ and $V_{\bar{H}_3} = V_{\bar{F}_4} = 0$ together with $V_{\bar{F}_0} = 0$, as it is displayed in the right plot of figure 4.15.

Finally, the flux induced polynomials $\mathcal{P}_{2,3}(\mathcal{Z})$ in table 4.6 for this supergravity models, also reduce to those of the geometric IIA models in the limit case of $\mathcal{Z} \rightarrow 0$. Therefore, one would expect that, as we approach point B in figure 4.8, the profile of the potential energy contributions should match that of figure 4.10, again up to a scale factor. Indeed, $V_Q \rightarrow 0$ and $V_R \rightarrow 0$ when we approach this singular point, i.e. $|\Phi_0| \rightarrow 0$, of the moduli VEVs. As for all the previous supergravity models, dS extrema can again be obtained by continuously deforming the Mkw solutions.

Chapter 5

Supersymmetric Vacua in T- and S-duality Invariant Flux Models

The present chapter mainly follows the line started in chapter 3 when studying supersymmetric vacua in the T-duality invariant effective models and extends it to include the effect of a new non-geometric P-flux tensor induced by S-duality when both T- and S-duality are simultaneously considered. In other words, this chapter is devoted to look for supersymmetric moduli vacua of the $\mathcal{N} = 1$ four-dimensional effective supergravity theory defined by the Kähler potential and the superpotential in (2.67).

A large obstacle to investigating this effective theory is that the flux backgrounds are to be constrained by an enlarged set of non-trivial polynomial equations through Jacobi identities of the flux algebra. Additionally, non-zero flux-induced tadpoles relate the fluxes to the localised sources living in the space. This is further complicated when we consider S-duality transformations which introduce new flux objects, as well as superpositions of fluxes of the same tensor type. This means that they each contribute to both the Jacobi and the tadpole constraints because of the enhanced duality invariance of the effective theory.

Algebraic geometry is the mathematical discipline which involves the study of complex polynomial systems and their solution spaces [187]. Recent papers [174, 188, 189] demonstrate how previously unwieldy techniques, due to the size and complexity of the equations found in the supergravity descriptions, of algebraic geometry can be applied to finding solutions to the aforementioned flux constraints through such programs as *Singular* [173, 190], thanks to the continued increase in computer speeds and memories. Useful background material for the methods used in this chapter can be found in the appendices of ref. [188] or the first few chapters of ref. [187]. Ref. [174] provides a way of applying algebraic geometry to supergravity without having to learn the specifics of *Singular* or similar programs. Though the procedures outlined in refs [174, 188, 189] will be used in part, unfortunately they do not immediately lend themselves to the algebraic geometry methods we use in this chapter. However, there is another interface [191] between *Mathematica* and *Singular* which allows for direct access to many of *Singular*'s algorithms.

5.1 The non-geometric (Q, P) background fluxes.

First of all, we try to clarify the role played by the non-geometric P-flux in terms of deformations of Lie algebras. Before considering the algebraic problem of solving the constraints $PP = 0$ and $QP + PQ = 0$ in (2.49), we focus our attention on understanding the problem from a different point of view: that of the effect of the P-flux over the gauge subalgebra $\mathfrak{g}_{gauge}$ generated by the Q-flux in the T-duality invariant effective theory studied in the previous chapters.

5.1.1 A note on deformations of Lie algebras.

To start with, we present a brief introduction to the topic of deformations of Lie algebras, in which we take the notation and conventions from refs [192–194]. Let us start with a general Lie algebra $\mathcal{L}$ defined by its brackets[1]

$$[X^a, X^b] = C_c^{ab}\, X^c \ . \tag{5.1}$$

These relations define an algebra iff Jacobi identities are fulfilled, namely $C_e^{[ab}\, C_d^{c]e} = 0$. For our purposes, it will be interesting to define the second cohomology class of the algebra, $H^2(\mathcal{L}, \mathcal{L})$. It contains 2-cocycles $\varphi \in H^2(\mathcal{L}, \mathcal{L})$ that are closed under the action of an exterior derivation d without being coboundaries. More formally, a cocycle $\varphi \in H^2(\mathcal{L}, \mathcal{L})$ is a bilinear antisymmetric form that satisfies the constraint

$$\begin{aligned} d\varphi(X^a, X^b, X^c) \ :=\ & [X^a, \varphi(X^b, X^c)] + [X^c, \varphi(X^a, X^b)] + [X^b, \varphi(X^c, X^a)] \ + \\ & + \ \varphi(X^a, [X^b, X^c]) + \varphi(X^c, [X^a, X^b]) + \varphi(X^b, [X^c, X^a]) \ = \ 0 \ , \end{aligned} \tag{5.2}$$

for any X^a, X^b and X^c of $\mathcal{L}$.

Moreover, for φ to define a deformation of $\mathcal{L}$ that is also a Lie algebra, i.e. it also satisfies the new Jacobi identities, an additional integrability condition has to be imposed. The 2-cocycle φ is integrable if it satisfies

$$\varphi(\varphi(X^a, X^b), X^c) + \varphi(\varphi(X^c, X^a), X^b) + \varphi(\varphi(X^b, X^c), X^a) = 0 \ . \tag{5.3}$$

If both conditions, named the cohomology and the integrability conditions, are fulfilled then the linear deformation $\mathcal{L} + \varphi$ is also a Lie algebra [195], which we will denote as $\mathcal{L}_\varphi$

[1] We define generators with an upper index in analogy with the commutation relations $[X^a, X^b] = Q_c^{ab}\, X^c$ in (2.46).

with the deformed bracket

$$[X^a, X^b]_\varphi = C_c^{ab}\, X^c + \varphi(X^a, X^b) \ . \tag{5.4}$$

In particular, nullity of $H^2(\mathcal{L}, \mathcal{L})$ implies that any deformation $\mathcal{L}_\varphi$ is isomorphic to $\mathcal{L}$ and in that case $\mathcal{L}$ is called rigid or stable. However, in general, $\mathcal{L}_\varphi$ and $\mathcal{L}$ are not isomorphic.

To clarify the utility of deformed Lie algebras in the problem of S-duality and the non-geometric (Q, P) fluxes, let us consider a deformation $\varphi(X^a, X^b) := \alpha_c^{ab}\, X^c$ with $\alpha_c^{ab} = -\alpha_c^{ba}$, so the cohomology condition in (5.2) can be rewritten as

$$C_x^{[ab}\, \alpha_d^{c]x} + \alpha_x^{[ab}\, C_d^{c]x} = 0 \ , \tag{5.5}$$

while the integrability condition results in

$$\alpha_x^{[ab}\, \alpha_d^{c]x} = 0 \ . \tag{5.6}$$

At this point, the role of the non-geometric P-flux becomes clear by identifying

$$C_c^{ab} = Q_c^{ab} \qquad \text{and} \qquad \alpha_c^{ab} = P_c^{ab} \ . \tag{5.7}$$

The non-geometric Q-flux defines the gauge subalgebra of the T-duality invariant effective theory while the non-geometric P-flux can be implemented as deformations of this subalgebra by an element of its second cohomology class. The $P^2 = 0$ and $QP + PQ = 0$ constraints in (2.49) are simply the integrability (5.6) and cohomology (5.5) conditions for the non-geometric P-flux to define such deformations.

The gauge subalgebra of the T-duality invariant theory is trivially recovered when the deformation vanishes, i.e. $P = 0$, and just the original condition $Q^2 = 0$ remains unchanged. Another possibility to recover it is to fix $P = Q$, which is related to the $P = 0$ case by an $\mathrm{SL}(2, \mathbb{Z})_S$ transformation. This can be interpreted as a deformation of the original gauge subalgebra by itself.

5.1.2 Solving the integrability condition.

The integrability condition $P^2 = 0$ is straightforwardly solved by imposing that the P-flux becomes the structure constants of a Lie algebra $\mathfrak{g}_P$ belonging to the set of non-trivial six-dimensional Lie algebras compatibles with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ isotropic orbifold symmetries. As we already know from the previous chapters, there are only five isotropic non-trivial Lie algebras with such properties[2]: $\mathfrak{so}(3,1)$, $\mathfrak{so}(4)$, $\mathfrak{su}(2) + \mathfrak{u}(1)^3$, $\mathfrak{iso}(3)$ and $\mathfrak{nil}$. We do not consider the abelian $\mathfrak{u}(1)^6$ since it is equivalent to a trivial $P = 0$ background flux. This is totally analogous to what we made for solving the $Q^2 = 0$ condition in chapter 3.

[2] All these algebras are quasi-classical Lie algebras, i.e. they have an invariant non-degenerate metric built from their quadratic Casimir operator [192–194].

To solve both the $Q^2=0$ and $P^2=0$ conditions simultaneously, we pick two algebras, $\mathfrak{g}_Q$ and $\mathfrak{g}_P$ and specify their embedding into the the original Q and P fluxes through the modular parameters

$$\Gamma_Q = \begin{pmatrix} \alpha_q & \beta_q \\ \gamma_q & \delta_q \end{pmatrix} \qquad \text{and} \qquad \Gamma_P = \begin{pmatrix} \alpha_p & \beta_p \\ \gamma_p & \delta_p \end{pmatrix} . \tag{5.8}$$

We end up with a general parameterisation of the non-geometric fluxes $Q=Q(\alpha_q,\beta_q,\gamma_q,\delta_q)$ and $P=P(\alpha_p,\beta_p,\gamma_p,\delta_p)$, analogous to that derived in section 3.1 for the non-geometric Q-flux.

Recalling the Γ-matrices in (5.8), we can now define the two modular variables

$$\mathcal{Z}_Q = \frac{\alpha_q\, U + \beta_q}{\gamma_q\, U + \delta_q} \qquad \text{and} \qquad \mathcal{Z}_P = \frac{\alpha_p\, U + \beta_p}{\gamma_p\, U + \delta_p} . \tag{5.9}$$

Expressing the flux-induced polynomials in (2.70) and (2.71) due to the non-geometric Q and P fluxes in terms of these modular variables, we have

$$P_3(U) = (\gamma_q\, U + \delta_q)^3\, \mathcal{P}_3(\mathcal{Z}_Q) \qquad \text{and} \qquad P_4(U) = -(\gamma_p\, U + \delta_p)^3\, \mathcal{P}_4(\mathcal{Z}_P) . \tag{5.10}$$

The polynomials $\mathcal{P}_{3,4}(\mathcal{Z}_{Q,P})$ associated to $\mathfrak{g}_Q$ and $\mathfrak{g}_P$ can be simply read off from the table 5.1 after replacing $\mathcal{Z}$ by $\mathcal{Z}_Q$ and $\mathcal{Z}_P$ respectively.

algebra	$\mathcal{P}_{3,4}(\mathcal{Z})/3$ polynomials	modular roots
$\mathfrak{so}(3,1)$	$-\mathcal{Z}(\mathcal{Z}^2+1)$	$\mathcal{Z}=0\ ,\ +i\ ,\ -i$
$\mathfrak{so}(4)$	$\mathcal{Z}(\mathcal{Z}+1)$	$\mathcal{Z}=0\ ,\ \infty\ ,\ -1$
$\mathfrak{su}(2)+\mathfrak{u}(1)^3$	$\mathcal{Z}$	$\mathcal{Z}=0\ ,\ \infty$ (double)
$\mathfrak{iso}(3)$	$1-\mathcal{Z}$	$\mathcal{Z}=\infty$ (double) $,\ +1$
$\mathfrak{nil}$	1	$\mathcal{Z}=\infty$ (triple)

Table 5.1: Algebras and non-geometric flux-induced polynomials.

When using this parameterisation for the non-geometric fluxes, the roots structure of their flux-induced polynomials can be expressed in terms of the modular parameters, namely, the number and type of coincident roots becomes manifest. Moreover, it is not possible to do an $\mathrm{SL}(2,\mathbb{Z})_S$ transformation which alters this, so different algebras lead to different root structures (see table 5.1).

To analyse this we define the following two-dimensional generic vectors,

$$\begin{aligned} \mathcal{Z}_0 &= (\alpha,\beta) \quad &,\quad \mathcal{Z}_\infty &= (\gamma,\delta)\ , \\ \mathcal{Z}_{-1} &= (\alpha+\gamma,\beta+\delta) \quad &,\quad \mathcal{Z}_{+1} &= (\alpha-\gamma,\beta-\delta)\ , \\ \mathcal{Z}_{+i} &= i\left(\sqrt{\alpha^2+\gamma^2}, \frac{(\alpha\beta+\gamma\delta)+i|\Gamma|}{\sqrt{\alpha^2+\gamma^2}}\right) \quad &,\quad \mathcal{Z}_{-i} &= i\left(\sqrt{\alpha^2+\gamma^2}, \frac{(\alpha\beta+\gamma\delta)-i|\Gamma|}{\sqrt{\alpha^2+\gamma^2}}\right), \end{aligned} \tag{5.11}$$

in such a way that they carry the information about the roots values once they are contracted with the vector $\begin{pmatrix} U \\ 1 \end{pmatrix}$. Then the non-geometric flux-induced polynomials for each algebra can be easily reconstructed from its roots structure as

$$P_{3,4}(U) = 3 \prod_{\square = roots} \mathcal{Z}_{\square}^{Q,P} \begin{pmatrix} U \\ 1 \end{pmatrix} \ , \tag{5.12}$$

with $\square \equiv 0, \infty, -1, +1, +i, -i$ according with the modular roots, as it is shown in table 5.1. As an example, we reconstruct the cubic $P_3(U)$ induced by the non-geometric Q-flux in the case of an algebra $\mathfrak{so}(4)$. In this case, it reads

$$\begin{aligned} P_3(U) &= 3\, \mathcal{Z}_0^Q \begin{pmatrix} U \\ 1 \end{pmatrix} \cdot \mathcal{Z}_\infty^Q \begin{pmatrix} U \\ 1 \end{pmatrix} \cdot \mathcal{Z}_{-1}^Q \begin{pmatrix} U \\ 1 \end{pmatrix} \\ &= 3\, (\alpha_q\, U + \beta_q)\, (\gamma_q\, U + \delta_q)\ \left[(\alpha_q + \gamma_q)\, U + (\beta_q + \delta_q) \right] \\ &= 3\, (\gamma_q\, U + \delta_q)^3\, \mathcal{Z}_Q\, (\mathcal{Z}_Q + 1)\ . \end{aligned} \tag{5.13}$$

Note that $\mathfrak{so}(3,1)$ is unique in the above results, in that it generates non-geometric flux-induced polynomial whose roots are certain to be complex, given the real and non-degenerate nature of the Γ's matrices.

5.1.3 Solving the cohomology condition.

In terms of the flux entries, the cohomology $QP + PQ = 0$ constraints are those in (2.75). Since the expressions for the entries of Q and P are in terms of the modular parameters the cohomology condition puts constraints on their possible values. Finding the space of valid flux entries is difficult because the constraints are polynomials in terms of the eight modular parameters. However, these polynomials form the generators of the ideal $\langle QP + PQ \rangle$ in the ring of polynomials $\mathbb{C}[\alpha_q, \ldots, \delta_p]$ and so we can use an algebraic geometry method of prime decomposition to split $\langle QP + PQ \rangle$ into its prime ideals, J_i. One such method is the Gianni-Trager-Zacharias (GTZ) algorithm, which is implemented within *Singular*. Each prime ideal has a solution space, the variety $\mathbb{V}_i$, which is a subset of $\mathbb{V}$, the variety of $\langle QP + PQ \rangle$ and because we are working with prime ideals, their varieties do not intersect other than at a finite number of disjoint points. Therefore, given the decomposition

$$\langle\, QP + PQ\, \rangle \;=\; J_1 \cap \ldots \cap J_n \ , \tag{5.14}$$

in order to satisfy $QP + PQ = 0$, we need only to solve the set of equations $f_{i,j} = 0$, where $J_i = \langle\, f_{i,1}, f_{i,2}, \ldots, f_{i,m}\, \rangle$, though to completely account for all possible solutions each prime ideal must be analysed. An ideal automatically has at least one prime ideal but in the case of some of the $(\mathfrak{g}_Q, \mathfrak{g}_P)$ pairings, we find as many as three prime ideals of varying complexity. These relate the Γ_Q and Γ_P modular matrices in (5.8) and so restrict the transformations which are needed to bring the Q and P fluxes (understood as structure constants) to their standard form giving rise to the flux-induced polynomials of table 5.1.

For purpose of illustration we consider the example of $\mathfrak{g}_Q = \mathfrak{su}(2) + \mathfrak{u}(1)^3$ and $\mathfrak{g}_P = \mathfrak{so}(4)$. We have individual parameterisation of the following format,

- Q-flux fixing the gauge subalgebra in the T-duality invariant supergravity to be $\mathfrak{g}_Q = \mathfrak{su}(2) + \mathfrak{u}(1)^3$,

$$\begin{array}{lcl} c_0 = \beta_q\, \delta_q^2 & , & c_3 = -\alpha_q\, \gamma_q^2 \quad , \\ c_1 = \beta_q\, \delta_q\, \gamma_q & , & c_2 = -\alpha_q\, \gamma_q\, \delta_q \quad , \\ \tilde{c}_2 = \gamma_q^2\, \beta_q & , & \tilde{c}_1 = -\alpha_q\, \delta_q^2 \quad . \end{array} \tag{5.15}$$

- P-flux fixing the original gauge subalgebra in the T-duality invariant supergravity to be deformed by $\mathfrak{g}_P = \mathfrak{so}(4)$,

$$\begin{array}{lcl} d_0 = \beta_p\, \delta_p\, (\beta_p + \delta_p) & , & d_3 = -\, \alpha_p\, \gamma_p\, (\alpha_p + \gamma_p) \quad , \\ d_1 = \beta_p\, \delta_p\, (\alpha_p + \gamma_p) & , & d_2 = -\, \alpha_p\, \gamma_p\, (\beta_p + \delta_p) \quad , \\ \tilde{d}_2 = \gamma_p^2\, \beta_p + \alpha_p^2\, \delta_p & , & \tilde{d}_1 = -\, (\gamma_p\, \beta_p^2 + \alpha_p\, \delta_p^2) \quad . \end{array} \tag{5.16}$$

This leads to a $\langle QP + PQ \rangle$ cohomology condition ideal which has three prime ideals in its decomposition,

$$\begin{array}{lcl} J_1 & = & \langle\, \alpha_q \beta_p - \beta_q \alpha_p \ ,\ \gamma_q \delta_p - \delta_q \gamma_p \,\rangle \quad , \\ J_2 & = & \langle\, \alpha_q \delta_p - \beta_q \gamma_p \ ,\ \gamma_q \beta_p - \delta_q \alpha_p \,\rangle \quad , \\ J_3 & = & \langle\, \gamma_q(\beta_p + \delta_p) - \delta_q(\alpha_p + \gamma_p) \,\rangle \quad . \end{array} \tag{5.17}$$

These constraints can be rewritten in terms of entries in two-dimensional vectors

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad , \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \Rightarrow \quad u_1 v_2 - u_2 v_1 = 0 \quad \Leftrightarrow \quad \mathbf{u} \times \mathbf{v} = 0\ . \tag{5.18}$$

If two vectors satisfy $\mathbf{u} \times \mathbf{v} = 0$ then they are parallel, which we denote by $\mathbf{u} \parallel \mathbf{v}$. With this notation and using the vectors given in (5.11), the cohomology condition becomes

$$\begin{array}{lcl} J_1 & = & \langle\, \mathcal{Z}_0^Q \times \mathcal{Z}_0^P \ ,\ \mathcal{Z}_\infty^Q \times \mathcal{Z}_\infty^P \,\rangle \ \Leftrightarrow \ \mathcal{Z}_0^Q \parallel \mathcal{Z}_0^P \ ,\ \mathcal{Z}_\infty^Q \parallel \mathcal{Z}_\infty^P \quad . \\ J_2 & = & \langle\, \mathcal{Z}_0^Q \times \mathcal{Z}_\infty^P \ ,\ \mathcal{Z}_\infty^Q \times \mathcal{Z}_0^P \,\rangle \ \Leftrightarrow \ \mathcal{Z}_0^Q \parallel \mathcal{Z}_\infty^P \ ,\ \mathcal{Z}_\infty^Q \parallel \mathcal{Z}_0^P \quad . \\ J_3 & = & \langle\, \mathcal{Z}_\infty^Q \times \mathcal{Z}_{-1}^P \,\rangle \ \Leftrightarrow \ \mathcal{Z}_\infty^Q \parallel \mathcal{Z}_{-1}^P \quad . \end{array} \tag{5.19}$$

In each case the prime ideal's generating functions can be rewritten as a vanishing cross product. In fact, this happens for all prime ideals of all possible pairings $(\mathfrak{g}_Q, \mathfrak{g}_P)$. Therefore, the prime ideals of $\langle QP+PQ \rangle$ can be viewed as geometric constraints on the position of the vectors representing the roots of the cubic polynomials $P_3(U)$ and $P_4(U)$. Specifically, when the polynomials themselves are computed, this is equivalent to $P_3(U)$ and

$P_4(U)$ sharing some roots. It is worth noticing here that the $J_1 = 0$ and $J_2 = 0$ solutions also imply the piecewise vanishing $QP = PQ = 0$, unlike $J_3 = 0$. Moreover, $J_1 = 0$ can be translated into $\mathcal{Z}_P \propto \mathcal{Z}_Q$ while $J_2 = 0$ implies $\mathcal{Z}_P \propto S\mathcal{Z}_Q$, where S is the inversion generator in (1.22).

The full list of the vector alignments arising from the different prime ideals of the cohomology condition are given in table 5.2 for each algebra pairing.

	$\mathfrak{g}_P$ deformation				
$\mathfrak{g}_Q$ original	$\mathfrak{so}(4)$	$\mathfrak{so}(3,1)$	$\mathfrak{su}(2)+\mathfrak{u}(1)^3$	$\mathfrak{iso}(3)$	$\mathfrak{nil}$
$\mathfrak{so}(4)$	$\mathcal{Z}_0^Q \parallel \mathcal{Z}_0^P$, $\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_\infty^P$ $\mathcal{Z}_0^Q \parallel \mathcal{Z}_\infty^P$, $\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_0^P$ $\mathcal{Z}_{-1}^Q \parallel \mathcal{Z}_{-1}^P\,(*)$	$\mathcal{Z}_{-1}^Q \parallel \mathcal{Z}_0^P\,(*)$	$\mathcal{Z}_0^Q \parallel \mathcal{Z}_0^P$, $\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_\infty^P$ $\mathcal{Z}_0^Q \parallel \mathcal{Z}_\infty^P$, $\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_0^P$ $\mathcal{Z}_{-1}^Q \parallel \mathcal{Z}_\infty^P\,(*)$	$\mathcal{Z}_{-1}^Q \parallel \mathcal{Z}_{+1}^P\,(*)$	$\mathcal{Z}_{-1}^Q \parallel \mathcal{Z}_\infty^P\,(*)$
$\mathfrak{so}(3,1)$	$\mathcal{Z}_0^Q \parallel \mathcal{Z}_{-1}^P\,(*)$	$\mathcal{Z}_{+i}^Q \parallel \mathcal{Z}_{+i}^P$, $\mathcal{Z}_{-i}^Q \parallel \mathcal{Z}_{-i}^P$ $\mathcal{Z}_{+i}^Q \parallel \mathcal{Z}_{-i}^P$, $\mathcal{Z}_{-i}^Q \parallel \mathcal{Z}_{+i}^P$ $\mathcal{Z}_0^Q \parallel \mathcal{Z}_0^P\,(*)$	$\mathcal{Z}_0^Q \parallel \mathcal{Z}_\infty^P\,(*)$	$\mathcal{Z}_0^Q \parallel \mathcal{Z}_{+1}^P\,(*)$	$\mathcal{Z}_0^Q \parallel \mathcal{Z}_\infty^P\,(*)$
$\mathfrak{su}(2)+\mathfrak{u}(1)^3$	$\mathcal{Z}_0^Q \parallel \mathcal{Z}_0^P$, $\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_\infty^P$ $\mathcal{Z}_0^Q \parallel \mathcal{Z}_\infty^P$, $\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_0^P$ $\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_{-1}^P\,(*)$	$\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_0^P\,(*)$	$\mathcal{Z}_0^Q \parallel \mathcal{Z}_\infty^P$, $\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_0^P$ $\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_\infty^P$	$\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_{+1}^P\,(*)$	$\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_\infty^P$
$\mathfrak{iso}(3)$	$\mathcal{Z}_{+1}^Q \parallel \mathcal{Z}_{-1}^P\,(*)$	$\mathcal{Z}_{+1}^Q \parallel \mathcal{Z}_0^P\,(*)$	$\mathcal{Z}_{+1}^Q \parallel \mathcal{Z}_\infty^P\,(*)$	$\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_\infty^P$ $\mathcal{Z}_{+1}^Q \parallel \mathcal{Z}_{+1}^P\,(*)$	$\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_\infty^P$ $\mathcal{Z}_{+1}^Q \parallel \mathcal{Z}_\infty^P\,(*)$
$\mathfrak{nil}$	$\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_{-1}^P\,(*)$	$\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_0^P\,(*)$	$\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_\infty^P$	$\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_\infty^P$ $\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_{+1}^P\,(*)$	$\mathcal{Z}_\infty^Q \parallel \mathcal{Z}_\infty^P$

Table 5.2: Cohomology condition in terms of the root alignments. The branches labelled by $(*)$ disappear under the more restrictive condition $QP = PQ = 0$. Under the inversion $S \to -1/S$ transformation, the algebras $\mathfrak{g}_Q$ and $\mathfrak{g}_P$ are exchanged resulting in the symmetry of this table (and all the forthcoming ones).

At this point, two comments must be done concerning the $\mathrm{SL}(2,\mathbb{Z})^7$-duality invariance of the solutions found, as discussed in section 2.4.3.

1. Most of these solutions (those labelled by $(*)$) disappear under the more restrictive condition $QP = PQ = 0$ that occurs in an $\mathrm{SL}(2,\mathbb{Z})^7$-duality invariant supergravity. In other words, not all the pairings are allowed in the $\mathrm{SL}(2,\mathbb{Z})^7$-duality invariant theory [155].

2. Apart from each algebra being deformed by itself, there are the following possibilities in an $\mathrm{SL}(2,\mathbb{Z})^7$-duality invariant supergravity: $\mathfrak{so}(4)$ can be deformed by $\mathfrak{su}(2)+\mathfrak{u}(1)^3$; $\mathfrak{su}(2)+\mathfrak{u}(1)^3$ can be deformed by $\mathfrak{so}(4)$ and by $\mathfrak{nil}$; $\mathfrak{iso}(3)$ can be deformed by $\mathfrak{nil}$ and $\mathfrak{nil}$ can be deformed by $\mathfrak{su}(2)+\mathfrak{u}(1)^3$ and by $\mathfrak{iso}(3)$.

We note how in several $(\mathfrak{g}_Q, \mathfrak{g}_P)$ pairings there are two, even three, different ways (branches) to solve the cohomology condition. In section 5.3 we will provide an example based on $\mathfrak{g}_Q = \mathfrak{su}(2) + \mathfrak{u}(1)^3$ and $\mathfrak{g}_P = \mathfrak{so}(4)$ for which supersymmetric Minkowski vacua only exist in one of these branches, i.e. $\mathcal{Z}^Q_\infty \parallel \mathcal{Z}^P_{-1}$. According to table 5.2, this solution disappears when imposing $\mathrm{SL}(2,\mathbb{Z})^7$-duality invariance on the flux backgrounds. However, supersymmetric AdS$_4$ solutions can be found in the other branches, i.e. $\mathcal{Z}^Q_0 \parallel \mathcal{Z}^P_0$ together with $\mathcal{Z}^Q_\infty \parallel \mathcal{Z}^P_\infty$.

5.2 The gauge $(\bar{F}_3, \bar{H}_3)$ background fluxes.

Now, we shall proceed to solve the set of constraints in (2.76) coming from the $\mathrm{SL}(2,\mathbb{Z})$-singlet Jacobi identity in (2.50). Schematically, these constraints can be written as a linear system

$$(\Phi_Q)_i^{\ j}\, b_j \quad = \quad (\Phi_P)_i^{\ j}\, a_j \ , \tag{5.20}$$

where Φ_Q and Φ_P are 4×4 rank two matrices depending on the non-geometric Q and P fluxes respectively.

Since modular variables are more transparent to work with, we decide to use a universal parameterisation for NS-NS $\bar{H}_3$ and R-R $\bar{F}_3$ fluxes based on the complete decomposition

$$P_2(U) = (\gamma_q\, U + \delta_q)^3\, \mathcal{P}_2(\mathcal{Z}_Q) \quad , \quad P_1(U) = -(\gamma_p\, U + \delta_p)^3\, \mathcal{P}_1(\mathcal{Z}_P)\ , \tag{5.21}$$

with

$$\mathcal{P}_2(\mathcal{Z}_Q) = \sum_{i=0}^{3} \epsilon_i\, \mathcal{Z}_Q^i \qquad \text{and} \qquad \mathcal{P}_1(\mathcal{Z}_P) = \sum_{i=0}^{3} \rho_i\, \mathcal{Z}_P^i\ . \tag{5.22}$$

Under this decomposition, the NS-NS $\bar{H}_3$ flux entries are parameterised as

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} -\beta_q^3 & -\beta_q\delta_q^2 & -\beta_q^2\delta_q & -\delta_q^3 \\ \alpha_q\beta_q^2 & \frac{1}{3}\delta_q\,(2\beta_q\gamma_q + \alpha_q\delta_q) & \frac{1}{3}\beta_q\,(\beta_q\gamma_q + 2\alpha_q\delta_q) & \gamma_q\delta_q^2 \\ -\alpha_q^2\beta_q & -\frac{1}{3}\gamma_q\,(\beta_q\gamma_q + 2\alpha_q\delta_q) & -\frac{1}{3}\alpha_q\,(2\beta_q\gamma_q + \alpha_q\delta_q) & -\gamma_q^2\delta_q \\ \alpha_q^3 & \alpha_q\gamma_q^2 & \alpha_q^2\gamma_q & \gamma_q^3 \end{pmatrix} \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} \tag{5.23}$$

and those for R-R $\bar{F}_3$ flux, a_i, have the same form[3] upon replacing the subscript $q \to p$ and $\epsilon_i \to \rho_i$.

Fixing a pairing $(\mathfrak{g}_Q, \mathfrak{g}_P)$ and using the parameterisation $Q = Q(\alpha_q, \beta_q, \gamma_q, \delta_q)$ and $P = P(\alpha_p, \beta_p, \gamma_p, \delta_p)$ of the non-geometric fluxes as well as that in (5.23) for the gauge fluxes $(\bar{F}_3, \bar{H}_3)$, the above system in (5.20) can be rewritten as

$$(\tilde{\Phi}_Q)_i^{\ j}\, \epsilon_j \quad = \quad (\tilde{\Phi}_P)_i^{\ j}\, \rho_j \ , \tag{5.24}$$

[3] These universal parameterisations are well defined because their Jacobian have determinants $-|\Gamma_Q|^6/9$ and $-|\Gamma_P|^6/9$ so they never vanish, provided the isomorphisms used for bringing non-geometric fluxes to their standard form are not singular.

where $\tilde{\Phi}_Q$ and $\tilde{\Phi}_P$ depend on the modular matrices Γ_Q and Γ_P defined in (5.8). Both $\tilde{\Phi}_Q$ and $\tilde{\Phi}_P$ are linear transformations and therefore the solutions space of (5.24) can be obtained from the intersection of their images

$$\mathfrak{I}_{QP} \equiv \mathfrak{Im}(\tilde{\Phi}_Q) \cap \mathfrak{Im}(\tilde{\Phi}_P) \ . \tag{5.25}$$

The parameters ϵ_i and ρ_i belong to the $\tilde{\Phi}_Q$ and $\tilde{\Phi}_P$ anti-images of $\mathfrak{I}_{QP}$ respectively,

$$\begin{array}{rcl} \vec{\epsilon} & \in & \tilde{\Phi}_Q^{-1}(\mathfrak{I}_{QP}) \ , \\ \vec{\rho} & \in & \tilde{\Phi}_P^{-1}(\mathfrak{I}_{QP}) \ . \end{array} \tag{5.26}$$

Therefore we denote a background for the $\bar{H}_3$ and $\bar{F}_3$ fluxes solving (5.24) by a pair of vectors $(\vec{\epsilon}, \vec{\rho})$ satisfying (5.26). The main features of this background, such as its dimension or its flux-induced C_8' tadpole, are severely restricted by the non-geometric background we have previously imposed. Furthermore, we are able to distinguish between two non-geometric flux configurations by seeing whether or not $\mathcal{I}_{QP}$ becomes trivial.

- Non-geometric type A configuration: A non-geometric background satisfying

$$\mathfrak{I}_{QP} = \{\mathbf{0}\} \ , \tag{5.27}$$

fixes the NS-NS and R-R background fluxes to be $\vec{\epsilon} \in \ker(\tilde{\Phi}_Q)$ $(\bar{H}_3 Q = 0)$ and $\vec{\rho} \in \ker(\tilde{\Phi}_P)$ $(\bar{F}_3 P = 0)$. This has dimension 4 and, according to (2.61), does not generate a flux-induced C_8' tadpole,

$$N_7' = 0 \qquad (\text{type A}). \tag{5.28}$$

- Non-geometric type B configuration: A non-geometric background satisfying

$$\mathfrak{I}_{QP} \neq \{\mathbf{0}\} \ , \tag{5.29}$$

results in a less restricted one for the NS-NS and R-R fluxes. It is a six-dimensional background for which a flux-induced C_8' tadpole can be generated. This can always be written as

$$N_7' = \Delta_Q \ |\Gamma_Q|^3 + \Delta_P \ |\Gamma_P|^3 \qquad (\text{type B}) \ , \tag{5.30}$$

with Δ_Q and Δ_P depending on ϵ_i and ρ_i respectively[4] and vanishing in the special case of $\vec{\epsilon} \in \ker(\tilde{\Phi}_Q)$ and $\vec{\rho} \in \ker(\tilde{\Phi}_P)$.

Let us explain a little bit more about the preceding classification. Starting with a non-geometric background for the Q and P fluxes, that satisfies both the integrability and the cohomology conditions, it will be either a type A or a type B configuration. For

[4] $\ker(\tilde{\Phi}_Q)$, $\ker(\tilde{\Phi}_P)$, Δ_Q and Δ_P differ for each pairing $(\mathfrak{g}_Q, \mathfrak{g}_P)$, being easily computed in each case.

this to be a type B it has to fulfil the condition in (5.29), which can be rephrased as a single roots alignment as shown in table 5.3 for all the possible pairings $(\mathfrak{g}_Q, \mathfrak{g}_P)$. If the non-geometric background we are working with generates that alignment, we are dealing with a type B configuration. Otherwise it is type A configuration. This is determined by the way (branch) we followed for solving the cohomology condition (see table 5.2).

$\mathfrak{g}_Q$ original	$\mathfrak{g}_P$ deformation				
	$\mathfrak{so}(4)$	$\mathfrak{so}(3,1)$	$\mathfrak{su}(2)+\mathfrak{u}(1)^3$	$\mathfrak{iso}(3)$	$\mathfrak{nil}$
$\mathfrak{so}(4)$	$\mathcal{Z}_{-1}^Q \parallel \mathcal{Z}_{-1}^P$	$\mathcal{Z}_{-1}^Q \parallel \mathcal{Z}_{0}^P$	$\mathcal{Z}_{-1}^Q \parallel \mathcal{Z}_{\infty}^P$	$\mathcal{Z}_{-1}^Q \parallel \mathcal{Z}_{+1}^P$	$\mathcal{Z}_{-1}^Q \parallel \mathcal{Z}_{\infty}^P$
$\mathfrak{so}(3,1)$	$\mathcal{Z}_{0}^Q \parallel \mathcal{Z}_{-1}^P$	$\mathcal{Z}_{0}^Q \parallel \mathcal{Z}_{0}^P$	$\mathcal{Z}_{0}^Q \parallel \mathcal{Z}_{\infty}^P$	$\mathcal{Z}_{0}^Q \parallel \mathcal{Z}_{+1}^P$	$\mathcal{Z}_{0}^Q \parallel \mathcal{Z}_{\infty}^P$
$\mathfrak{su}(2)+\mathfrak{u}(1)^3$	$\mathcal{Z}_{\infty}^Q \parallel \mathcal{Z}_{-1}^P$	$\mathcal{Z}_{\infty}^Q \parallel \mathcal{Z}_{0}^P$	$\mathcal{Z}_{\infty}^Q \parallel \mathcal{Z}_{\infty}^P$	$\mathcal{Z}_{\infty}^Q \parallel \mathcal{Z}_{+1}^P$	$\mathcal{Z}_{\infty}^Q \parallel \mathcal{Z}_{\infty}^P$
$\mathfrak{iso}(3)$	$\mathcal{Z}_{+1}^Q \parallel \mathcal{Z}_{-1}^P$	$\mathcal{Z}_{+1}^Q \parallel \mathcal{Z}_{0}^P$	$\mathcal{Z}_{+1}^Q \parallel \mathcal{Z}_{\infty}^P$	$\mathcal{Z}_{+1}^Q \parallel \mathcal{Z}_{+1}^P$	$\mathcal{Z}_{+1}^Q \parallel \mathcal{Z}_{\infty}^P$
$\mathfrak{nil}$	$\mathcal{Z}_{\infty}^Q \parallel \mathcal{Z}_{-1}^P$	$\mathcal{Z}_{\infty}^Q \parallel \mathcal{Z}_{0}^P$	$\mathcal{Z}_{\infty}^Q \parallel \mathcal{Z}_{\infty}^P$	$\mathcal{Z}_{\infty}^Q \parallel \mathcal{Z}_{+1}^P$	$\mathcal{Z}_{\infty}^Q \parallel \mathcal{Z}_{\infty}^P$

Table 5.3: Roots alignment in non-geometric type B configurations.

To illustrate this, we consider an example where $\mathfrak{g}_Q = \mathfrak{su}(2) + \mathfrak{u}(1)^3$ and $\mathfrak{g}_P = \mathfrak{so}(4)$. Solving the cohomology condition through the $\mathcal{Z}_{\infty}^Q || \mathcal{Z}_{-1}^P$ branch (see table 5.2), leaves us with a non-geometric type B configuration (see table 5.3). The $\ker(\tilde{\Phi}_Q)$ is expanded by (ϵ_0, ϵ_3) while that of $\tilde{\Phi}(\mathfrak{g}_P)$ is expanded by (ρ_0, ρ_3) for this pairing. In this case, the NS-NS and R-R fluxes account for six degrees of freedom and generate a flux-induced C_8' tadpole given in (5.30) with $\Delta_Q = \epsilon_2/3$ and $\Delta_P = (\rho_2 - \rho_1)/3$.

$\mathfrak{g}_Q$	$\mathcal{P}_2(\mathcal{Z}_Q)$
$\mathfrak{so}(3,1)$	$\epsilon_3\, \mathcal{Z}_Q^3 - 3\, \epsilon_0\, \mathcal{Z}_Q^2 - 3\, \epsilon_3\, \mathcal{Z}_Q + \epsilon_0$
$\mathfrak{so}(4)$	$\epsilon_3\, \mathcal{Z}_Q^3 + \epsilon_0$
$\mathfrak{su}(2)+\mathfrak{u}(1)^3$	$\epsilon_3\, \mathcal{Z}_Q^3 + \epsilon_0$
$\mathfrak{iso}(3)$	$\epsilon_1\, \mathcal{Z}_Q + \epsilon_0$
$\mathfrak{nil}$	$\epsilon_1\, \mathcal{Z}_Q + \epsilon_0$

Table 5.4: $\bar{H}_3$ flux-induced polynomials in the non-geometric type A configurations.

The $\bar{H}_3$ and $\bar{F}_3$ background fluxes determine the flux-induced $\mathcal{P}_2(\mathcal{Z}_Q)$ and $\mathcal{P}_1(\mathcal{Z}_P)$ polynomials in the superpotential. Fixing a non-geometric type A configuration, $\mathcal{P}_2(\mathcal{Z}_Q)$ is shown in table 5.4 for each $\mathfrak{g}_Q$ algebra. The equivalent expression for the polynomial $\mathcal{P}_1(\mathcal{Z}_P)$, resulting from the $\mathfrak{g}_P$ algebra, is obtained upon replacing $\epsilon_i \leftrightarrow \rho_i$ and $\mathcal{Z}_Q \leftrightarrow \mathcal{Z}_P$.

5.3 Supersymmetric solutions.

In this section, we provide some examples of supersymmetric vacua of the T and S-duality invariant effective supergravity given by the standard Kähler potential and the moduli potential induced by the $(\bar{F}_3, Q)$ and $(\bar{H}_3, P)$ fluxes using the methods we have developed in this work. We will focus on solutions with the axiodilaton S and Kähler T moduli being completely stabilised.

The starting point is the $\mathcal{N} = 1$ four-dimensional effective theory defined by the Kähler potential and the superpotential in (2.67). The latter can be rewritten as

$$\begin{aligned} W &= -(\gamma_p\, U + \delta_p)^3 \left[\left(\sum_{i=0}^{3} \rho_i\, \mathcal{Z}_P^i \right) + 3\, T\, S\, \mathcal{P}_4(\mathcal{Z}_P) \right] \\ &+ (\gamma_q\, U + \delta_q)^3 \left[S \left(\sum_{i=0}^{3} \epsilon_i\, \mathcal{Z}_Q^i \right) + 3\, T\, \mathcal{P}_3(\mathcal{Z}_Q) \right] , \end{aligned} \tag{5.31}$$

with $\mathcal{P}_3(\mathcal{Z}_Q)$, $\mathcal{P}_4(\mathcal{Z}_P)$ taken from table 5.1 according with a fixed pairing $(\mathfrak{g}_Q, \mathfrak{g}_P)$ and $\mathcal{Z}_Q$ and $\mathcal{Z}_P$ the modular variables in (5.9). In general, $\mathcal{Z}_Q \neq \mathcal{Z}_P$, and we will have to deal with two modular variables instead of just one, $\mathcal{Z}$. Each pairing $(\mathfrak{g}_Q, \mathfrak{g}_P)$ gives rise to a specific superpotential due to the relationship between the roots structure of a polynomial and its associated algebra.

A supersymmetric vacuum implies the vanishing of the F-terms

$$F_T = \partial_T W + \frac{3iW}{2\,\mathrm{Im}T} = 0 \quad , \quad F_S = \partial_S W + \frac{iW}{2\,\mathrm{Im}S} = 0 \quad , \quad F_U = \partial_U W + \frac{3iW}{2\,\mathrm{Im}U} = 0 \,, \tag{5.32}$$

bringing about either Minkowski or AdS$_4$ solutions because the potential in (2.62) at the minimum is given by $V_0 = -3\, e^{K_0} |W_0|^2 \leq 0$. Restricting our search to Minkowski solutions, i.e. $V_0 = 0$, simplifies the supersymmetric equations of motion in (5.32) to

$$\partial_S W = \partial_T W = \partial_U W = W = 0 \,. \tag{5.33}$$

Working with the generic expression in (2.67) for the superpotential, the Kähler moduli and axiodilaton equations of motion fix both moduli to

$$\begin{aligned} S_0 &= -\frac{P_3(U_0)}{P_4(U_0)} = \left(\frac{\gamma_q\, U + \delta_q}{\gamma_p\, U + \delta_p} \right)^3 \left. \frac{\mathcal{P}_3(\mathcal{Z}_Q)}{\mathcal{P}_4(\mathcal{Z}_P)} \right|_{U_0} , \\ T_0 &= -\frac{P_2(U_0)}{P_4(U_0)} = \left(\frac{\gamma_q\, U + \delta_q}{\gamma_p\, U + \delta_p} \right)^3 \left. \frac{\sum_{i=0}^{3} \epsilon_i\, \mathcal{Z}_Q^i}{\mathcal{P}_4(\mathcal{Z}_P)} \right|_{U_0} , \end{aligned} \tag{5.34}$$

where S_0, T_0 and U_0 are moduli values at the vacuum. As we have already discussed in the previous chapters, these VEVs are subject to physical considerations. ImS_0 must be positive because it is the inverse of the string coupling constant g_s. Im$T_0 = e^{-\phi} A$ where A is the area of a 4-dimensional subtorus, so it also has to be positive. Also, for the modular variables $\mathcal{Z}_Q$ and $\mathcal{Z}_P$ at the minimum, it happens that Im$\mathcal{Z}_Q$ =Im$U_0 |\Gamma_Q| / |\gamma_q U_0 + \delta_q|^2$ and Im$\mathcal{Z}_P$ =Im$U_0 |\Gamma_P| / |\gamma_p U_0 + \delta_p|^2$. Therefore, necessarily Im$\mathcal{Z}_Q \neq 0$ and Im$\mathcal{Z}_P \neq 0$ because for Im$U_0 = 0$ the internal space is degenerate. Without loss of generality, we choose again Im$U_0 > 0$.

The remaining $W = 0$ and $\partial_U W = 0$ conditions can be rewritten, using the stabilisation in (5.34), as [8]

$$\begin{aligned} E(U_0) &= P_1(U_0)\, P_4(U_0) - P_2(U_0)\, P_3(U_0) = 0 \, , \\ E'(U_0) &= 0 \, , \end{aligned} \tag{5.35}$$

provided[5] $P_4(U_0) \neq 0$. The prime denotes differentiation with respect to U and, therefore, $E(U)$ has a double root. The root must, given our definition for the Kähler potential, be complex and therefore $E(U)$ contains a double copy of complex conjugate pairs, accounting for 4 of its 6 roots. Therefore, we have the following factorisation property of $E(U)$,

$$E(U) = (f_2\, U^2 + f_1\, U + f_0)\, \tilde{E}(U) \, , \tag{5.36}$$

with $\tilde{E}(U) \equiv (g_2\, U^2 + g_1\, U + g_0)^2$ accounting for the double root that becomes complex iff $g_1^2 - 4\, g_2\, g_0 < 0$.

Information about the nature of the six roots of $E(U)$ can be immediately obtained from the generic superpotential polynomials once a $(\mathfrak{g}_Q, \mathfrak{g}_P)$ pairing is chosen and the full set of Jacobi identities, i.e. integrability, cohomology and singlet Jacobi constraints, are applied. Four cases are automatically discarded because their $E(U)$ possesses at least four real roots, so they can never have a double complex root for the Minkowski vacua to be physically viable, i.e. Im$U_0 \neq 0$. The number of real roots for each $(\mathfrak{g}_Q, \mathfrak{g}_P)$ pairing is summarised[6] in table 5.5. A priori, all branches with $E(U)$ having a number of real roots less than three could accommodate supersymmetric Minkowski solutions. This is a necessary but not sufficient condition for the existence of Minkowski vacua because for $E(U)$ to split into the form of eq. (5.36), additional constraints on $\bar{H}_3$ and $\bar{F}_3$ fluxes are needed. Therefore, several branches in table 5.5 will exclude Minkowski vacua, even though they have a sufficient number of complex roots and we will provide an example of this.

[5]This has to be the case for Im$U_0 \neq 0$ in all $\mathfrak{g}_P$ but $\mathfrak{g}_P = \mathfrak{so}(3,1)$ that has complex roots $\mathcal{Z}_P = \pm i$. For this singular case, $P_4(U_0) = 0$ implies $P_i(U_0) = 0$ for $i = 1, 2, 3, 4$ as can be seen from (5.33). Then S and T can not be simultaneously stabilised in a supersymmetric Minkowski vacuum.

[6]Entries in table 5.5 are in one to one correspondence with entries in table 5.2.

$\mathfrak{g}_Q$ original	$\mathfrak{g}_P$ deformation				
	$\mathfrak{so}(4)$	$\mathfrak{so}(3,1)$	$\mathfrak{su}(2)+\mathfrak{u}(1)^3$	$\mathfrak{iso}(3)$	$\mathfrak{nil}$
$\mathfrak{so}(4)$	2 2 1	1	2 2 1	1	1
$\mathfrak{so}(3,1)$	1	2 2 1	1	1	1
$\mathfrak{su}(2)+\mathfrak{u}(1)^3$	2 2 1	1	2 2	1	2
$\mathfrak{iso}(3)$	1	1	1	**4** 1	**4** 1
$\mathfrak{nil}$	1	1	2	**4** 1	**6**

Table 5.5: Number of real roots of $E(U)$ defined in (7.8) after imposing the full set of Jacobi constraints.

Despite this, several results can be read from table 5.5 :

i) There are no supersymmetric Minkowski solutions in the $(\mathfrak{nil},\mathfrak{nil})$ case because all $E(U)$ roots become real for this pairing.

ii) For supersymmetric Minkowski solutions to exist in $(\mathfrak{iso}(3),\mathfrak{iso}(3))$, $(\mathfrak{iso}(3),\mathfrak{nil})$ and $(\mathfrak{nil},\mathfrak{iso}(3))$ pairings, it is necessary to have non-geometric type B configurations (see table 5.3), generating an eventually non vanishing flux-induced C_8' tadpole.

iii) The rest of the pairings are richer and supersymmetric Minkowski solutions could, in principle, exist in all branches that solve the cohomology condition (see table 5.2).

5.3.1 Some simple solutions

For our first example, we shall continue to investigate the case $\mathfrak{g}_Q = \mathfrak{su}(2)+\mathfrak{u}(1)^3$ deformed by $\mathfrak{g}_P = \mathfrak{so}(4)$, in order to show how simple supersymmetric solutions can be easily obtained using these methods. For the sake of simplicity, we will look for $\bar{H}_3$ and $\bar{F}_3$ background fluxes with $\vec{\epsilon} \in \ker(\tilde{\Phi}_Q)$ and $\vec{\rho} \in \ker(\tilde{\Phi}_P)$, so $N_7' = 0$ but the net charges N_7 and $\tilde{N}_7$ are considered as free variables. In these solutions, $\mathcal{P}_2(\mathcal{Z}_Q)$ and $\mathcal{P}_1(\mathcal{Z}_P)$ can be obtained from table 5.4 leaving us with a set $(\epsilon_0, \epsilon_3\,;\, \rho_0, \rho_3)$ of free parameters in the superpotential determining the $\bar{H}_3$ and $\bar{F}_3$ background fluxes.

Taking the relevant polynomials from the table 5.1 and the table 5.4, the superpotential in (5.31) becomes

$$\begin{aligned} W &= -(\gamma_p\, U + \delta_p)^3 \left[(\rho_3\, \mathcal{Z}_P^3 + \rho_0) + 3\, T\, S\, \mathcal{Z}_P (\mathcal{Z}_P + 1) \right] \\ &+ \quad (\gamma_q\, U + \delta_q)^3 \left[S\, (\epsilon_3\, \mathcal{Z}_Q^3 + \epsilon_0) + 3\, T\, \mathcal{Z}_Q \right] \;, \end{aligned} \tag{5.37}$$

and the tadpole cancellation conditions can be expressed in terms of the roots as

$$\begin{aligned} N_3 &= A_{33}\, (\mathcal{Z}_0^Q \times \mathcal{Z}_0^P)^3 + A_{30}\, (\mathcal{Z}_\infty^Q \times \mathcal{Z}_0^P)^3 + A_{03}\, (\mathcal{Z}_0^Q \times \mathcal{Z}_\infty^P)^3 + A_{00}\, (\mathcal{Z}_\infty^Q \times \mathcal{Z}_\infty^P)^3 \;, \\ N_7 &= \rho_3\, (\mathcal{Z}_0^Q \times \mathcal{Z}_0^P)\, (\mathcal{Z}_\infty^Q \times \mathcal{Z}_0^P)^2 + \rho_0\, (\mathcal{Z}_0^Q \times \mathcal{Z}_\infty^P)\, (\mathcal{Z}_\infty^Q \times \mathcal{Z}_\infty^P)^2 \;, \\ \tilde{N}_7 &= -\epsilon_3\, (\mathcal{Z}_0^Q \times \mathcal{Z}_0^P)(\mathcal{Z}_0^Q \times \mathcal{Z}_\infty^P)(\mathcal{Z}_0^Q \times \mathcal{Z}_{-1}^P) - \epsilon_0\, (\mathcal{Z}_\infty^Q \times \mathcal{Z}_0^P)\, (\mathcal{Z}_\infty^Q \times \mathcal{Z}_\infty^P)\, (\mathcal{Z}_\infty^Q \times \mathcal{Z}_{-1}^P), \end{aligned} \tag{5.38}$$

with $A_{ij} = -\rho_i\, \epsilon_j$.

We now impose the constraints coming from one of the prime ideals of the cohomology condition, of which there are three to choose for this pairing, as shown in table 5.2 and explicitly stated in (5.19). The case $J_1 = 0$ is automatically fulfilled with an embedding $\Gamma_P = \Gamma_Q \equiv \Gamma$, or equivalently $\mathcal{Z}_P = \mathcal{Z}_Q \equiv \mathcal{Z}$, while the $J_2 = 0$ results are equivalent to this after applying a T-duality transformation $\mathcal{Z} \to -1/\mathcal{Z}$. The case $J_3 = 0$ is a little bit different from the previous ones. It can not be transformed into $J_{1,2} = 0$ and the resultant solutions are distinct from those of the first two branches. We will solve for each of the three branches and clarify their relation to the existence of both AdS_4 and Minkowski vacua.

Simple type A AdS$_4$ solutions.

Imposing[7] $J_1 = 0$ and just fixing the modular embeddings to be

$$\Gamma_P = \Gamma_Q \equiv \Gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \;, \tag{5.39}$$

provides us with a much simplified superpotential given by

$$\frac{W}{(\gamma\, U + \delta)^3} = -(\rho_3\, \mathcal{Z}^3 + \rho_0) + S\, (\epsilon_3\, \mathcal{Z}^3 + \epsilon_0) + 3\, T\, \mathcal{Z} - 3\, T\, S\, \mathcal{Z}\, (\mathcal{Z} + 1) \;, \tag{5.40}$$

where

$$\mathcal{Z} \equiv \Gamma\, U = \frac{\alpha\, U + \beta}{\gamma\, U + \delta} \;. \tag{5.41}$$

[7] Imposing $J_2 = 0$ is T-dual to $J_1 = 0$ just with $\mathcal{Z} \to -1/\mathcal{Z}$.

Under the transformation $U \to \mathcal{Z}$ in (5.41), the effective theory is known to suffer a Kähler transformation such that $e^{K} W \to e^{\mathcal{K}} \mathcal{W}$ with

$$\begin{aligned} \mathcal{K} &= -3\ln\Big(-i(T-\bar{T})\Big) - \ln\Big(-i(S-\bar{S})\Big) - 3\ln\Big(-i(\mathcal{Z}-\bar{\mathcal{Z}})\Big) \quad , \\ \mathcal{W} &= |\Gamma|^{3/2}\left[-(\rho_3\,\mathcal{Z}^3+\rho_0) + S\,(\epsilon_3\,\mathcal{Z}^3+\epsilon_0) + 3\,T\,\mathcal{Z} - 3\,T\,S\,\mathcal{Z}\,(\mathcal{Z}+1)\right] \quad , \end{aligned} \tag{5.42}$$

and the tadpole cancellation conditions in (5.38) simplify to

$$\begin{aligned} N_3 &= |\Gamma|^3(A_{03}-A_{30}) = |\Gamma|^3(\epsilon_0\,\rho_3 - \epsilon_3\,\rho_0) \ , \\ N_7 &= \tilde{N}_7 = 0 \ . \end{aligned} \tag{5.43}$$

It is worth noting that, by simply imposing the embedding in (5.39), it becomes impossible to have non-geometric type B configurations, as we can see from table 5.3. Indeed, the alignment $\mathcal{Z}_\infty || \mathcal{Z}_{-1}$ results in $|\Gamma| = 0$ and the isomorphism is no longer valid. So whenever we impose the embedding of (5.39), automatically $\epsilon_1 = \epsilon_2 = \rho_1 = \rho_2 = 0$ and then $N_7' = N_7 = \tilde{N}_7 = 0$.

It can also be proven that this system does not possess Minkowski vacua. To do this, let us compute restrictions on the NS-NS $\bar{H}_3$ and R-R $\bar{F}_3$ background fluxes needed for the polynomial $E(U)$ to be factorised as in (5.36). From table 5.5 we know that $E(U)$ has at least two real roots. Factorising out and dropping these real roots, $E(U) \to \tilde{E}(U)$, it can be shown that for $\tilde{E}(U)$ to possess a double complex root, the $\bar{H}_3$ and $\bar{F}_3$ background fluxes must satisfy

$$\begin{aligned} 8\,\epsilon_0\,\rho_3 + (\epsilon_3 - 9\,\rho_3)\,\rho_0 &= 0 \ , \\ (\epsilon_3-\rho_3)^3 - 8\,\rho_3^2\,\rho_0 &= 0 \ , \end{aligned} \tag{5.44}$$

so that $g_1^2 - 4\,g_2\,g_0 = 12\left(\rho_3^{1/3}\,\rho_0^{1/3}\right)^2 \geq 0$, fixing all six roots of $E(U)$ to be real and producing non physical vacua, i.e. $\mathrm{Im}U_0 = 0$.

However, we find that supersymmetric AdS$_4$ vacua with all the moduli stabilised can exist without introducing localised sources. This result is new compared to the T-duality invariant effective theory which was deeply studied in chapter 3. Let us fix $\epsilon_3 = \rho_3 = 0$ so as to have $N_3 = 0$ and, for instance, $\rho_0 = 2\,\epsilon_0$. Solving the F-flatness conditions in (5.32) we obtain

$$\mathcal{Z}_0 = -1.0434 + 0.4758\,i \quad , \quad S_0 = -2.3802 + 4.1685\,i \quad , \quad \epsilon_0^{-1}\,T_0 = -0.4022 + 1.1483\,i \ , \tag{5.45}$$

with a vacuum energy $V_0\,\epsilon_0/|\Gamma|^3 = -2.3958$ and with $N_3 = N_7 = \tilde{N}_7 = N_7' = 0$. In terms of the original complex structure modulus, $U_0 = \Gamma^{-1}\,\mathcal{Z}_0$ with Γ the modular matrix given in (5.39). Fixing for example $\beta = \gamma = 0$, this solution corresponds to $a_0 = -2\,\epsilon_0\,\delta^3$, $b_0 = -\epsilon_0\,\delta^3$, $\tilde{c}_1 = \tilde{d}_1 = -\alpha\,\delta^2$ and $\tilde{d}_2 = \alpha^2\,\delta$. Large positive values of the ϵ_0 parameter translate into large absolute values of the NS-NS and R-R fluxes and also a large internal volume.

Simple type B Minkowski solutions.

Now we explore the case $J_3 = 0$, or equivalently $\mathcal{Z}^Q_\infty || \mathcal{Z}^P_{-1}$. As an example of this alignment involving just two modular parameters let us take

$$\Gamma_Q = \begin{pmatrix} \alpha & -\delta \\ \alpha & \delta \end{pmatrix} \quad , \quad \Gamma_P = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} . \tag{5.46}$$

This results in a two-dimensional family of non-geometric type B configurations. Substituting directly into (5.34) we obtain

$$S_0 = \frac{\alpha\, U_0}{\delta} - \frac{\delta}{\alpha U_0} \qquad \text{and} \qquad T_0 = \frac{1}{3\,\alpha\,\delta}\, \frac{\epsilon_3\,(\alpha U_0 - \delta)^3 + \epsilon_0\,(\delta + \alpha U_0)^3}{U_0\,(\delta + \alpha U_0)} . \tag{5.47}$$

Let us compute again restrictions on the NS-NS $\bar{H}_3$ and R-R $\bar{F}_3$ background fluxes needed for the polynomial $E(U)$ to be factorised as in (5.36). From table 5.5, this time $E(U)$ has at least one real root. Factorising out this real root, $E(U) \to (f_1\, U + f_0)\, \tilde{E}(U)$, this imposes

$$\rho_0 = 0 \quad , \quad \epsilon_0 = -\epsilon_3 = \frac{\rho_3}{8} \quad , \quad f_1 = g_1 = 0 \quad , \quad \frac{g_0}{g_2} = \left(\frac{\delta}{\alpha}\right)^2 \tag{5.48}$$

and therefore $g_1^2 - 4\, g_2\, g_0 < 0$, producing physical vacua $U_0 = i\left(\frac{\delta}{\alpha}\right)$. Substituting directly in (5.47), the moduli get stabilised to

$$U_0 = \left(\frac{\delta}{\alpha}\right) i \quad , \quad S_0 = 2\, i \quad , \quad T_0 = \frac{\rho_3}{12}\,(1 + i) . \tag{5.49}$$

This family is physical for $\rho_3 > 0$ and $|\Gamma_P| > 0$. The tadpole conditions for these supersymmetric Minkowski vacua are

$$N_3 = \frac{\rho_3}{4} \quad , \quad N_7 = \rho_3 \quad , \quad \tilde{N}_7 = |\Gamma_P|^3\, \frac{\rho_3^2}{4} , \tag{5.50}$$

with $|\Gamma_P| = \alpha\,\delta$, so $N_3 > 0$, $N_7 > 0$ and $\tilde{N}_7 > 0$ is required.

In terms of the original fluxes, this solution corresponds to $c_3 = -\alpha^3$, $c_2 = \tilde{c}_2 = -\tilde{d}_2 = -\alpha^2\,\delta$, $c_1 = \tilde{c}_1 = \tilde{d}_1 = -\alpha\,\delta^2$ and $c_0 = -\delta^3$ for non-geometric fluxes; $b_0 = -\delta^3\,\frac{\rho_3}{4}$ and $b_2 = -\alpha^2\,\delta\,\frac{\rho_3}{4}$ for the NS-NS flux; and $a_3 = \alpha^3\,\rho_3$ for the R-R flux. Again, large values of the ρ_3 parameter translate into large absolute values of the NS-NS and R-R fluxes and a large internal volume. However, this also increases the number of localised sources and therefore their backreaction, which we are not taking into account.

5.3.2 More Minkowski vacua examples.

In our previous example, we gave simple Minkowski solutions with all moduli stabilised in a physical vacuum with a vanishing flux-induced C_8' tadpole, i.e. $N_7' = 0$. Now, we provide Minkowski solutions with $N_7' \neq 0$ (examples 3 and 4).

Our main goal in this work has been to develop a systematic method to compute supersymmetric Minkowski vacua based on different $(\mathfrak{g}_Q, \mathfrak{g}_P)$ pairings which fulfil all algebraic constraints. To show how these methods work, we conclude by presenting several simple non-geometric type B configurations involving all the six-dimensional Lie algebras compatible with the orbifold symmetries. Besides finding analytic VEVs for the moduli, we also relate them to the net charge of localised sources which can exist, as well as some features of such vacua.

Example 1: vacua with unstabilised complex structure modulus.

Let us work out a simple family of Minkowski solutions with a vanishing flux-induced C_8' tadpole for which all the moduli but the complex structure modulus are fixed by the fluxes. These solutions were previously found in ref. [110] and we now clarify their flux structure.

Let us fix the non-geometric Q and P fluxes to be isomorphic to $\mathfrak{g}_Q = \mathfrak{so}(4)$ and $\mathfrak{g}_P = \mathfrak{iso}(3)$ respectively[8] under the modular embeddings

$$\Gamma_Q = \begin{pmatrix} \alpha_q & 0 \\ 0 & \delta_q \end{pmatrix} \quad , \quad \Gamma_P = \begin{pmatrix} \alpha_p & \beta_p \\ 0 & \delta_p \end{pmatrix} . \tag{5.51}$$

The cohomology condition for this pairing has an unique branch (see table 5.2) implying $\mathcal{Z}^{Q}_{-1} \parallel \mathcal{Z}^{P}_{+1}$ and it is a type B configuration, as is shown in table 5.3. This relates the modular matrices in (5.51) so that, $\alpha_q = \lambda\, \alpha_p$ and $\delta_q = \lambda\,(\beta_p - \delta_p)$.

Taking for simplicity $\vec{\epsilon} \in \ker(\tilde{\Phi}_Q)$ and $\vec{\rho} \in \ker(\tilde{\Phi}_P)$ results in $\epsilon_1 = \epsilon_2 = \rho_2 = \rho_3 = 0$. Moreover, we will also fix $\epsilon_3 = 0$ and therefore, substituting into (5.34), we obtain

$$S_0 = -\lambda^3 \left(\frac{\alpha_p}{\delta_p^2}\right) (\beta_p - \delta_p)\, U_0 \qquad \text{and} \qquad T_0 = -\frac{\lambda^3\, \epsilon_0\, (\beta_p - \delta_p)^3}{3\, \delta_p^2\, (\alpha_p\, U_0 + (\beta_p - \delta_p))} . \tag{5.52}$$

When plugging the above stabilisation of the axiodilaton and Kähler moduli into the superpotential we obtain

$$W(U_0) = -\left(\frac{\alpha_p}{\delta_p^2}\right) \left(\lambda^6\, (\beta_p - \delta_p)^4\, \epsilon_0 + \delta_p^4\, \rho_1\right) U_0 - \delta_p^2 \left(\delta_p\, \rho_0 + \beta_p\, \rho_1\right) , \tag{5.53}$$

so that, for Minkowski solutions to exist, $\partial_U W = W = 0$. Moreover, because of $\alpha_p\, \delta_p \neq 0$ (otherwise $|\Gamma_P| = 0$), Minkowski vacua with the complex structure modulus unstabilised do exist provided

$$\lambda^6\, (\beta_p - \delta_p)^4\, \epsilon_0 + \delta_p^4\, \rho_1 = 0 , \tag{5.54}$$

$$\delta_p\, \rho_0 + \beta_p\, \rho_1 = 0 . \tag{5.55}$$

[8] In this case, $\Delta_Q = (\epsilon_2 - \epsilon_1)/3$ and $\Delta_P = -\rho_3 - \rho_2/3$.

Under these restrictions for ρ_0 and ρ_1, the tadpole cancellation conditions simplify to

$$N_3 = \tilde{N}_7 = N_7' = 0 \qquad \text{and} \qquad N_7 = \frac{\lambda^9\, \epsilon_0}{3} \left(\frac{\alpha_p^3}{\delta_p^2} \right) (\beta_p - \delta_p)^5 \ . \tag{5.56}$$

From the axiodilaton and Kähler stabilisation in (5.52), taking a physical vacuum with $\text{Im}U_0 > 0$ implies that

$$\lambda\, \alpha_p\, (\beta_p - \delta_p) < 0 \qquad \text{and} \qquad \lambda\, \alpha_p\, (\beta_p - \delta_p)\, \epsilon_0 > 0 \ , \tag{5.57}$$

for $\text{Im}S_0 > 0$ and $\text{Im}T_0 > 0$ and therefore $\epsilon_0 < 0$. Otherwise the vacuum is not physical. Therefore $N_7 > 0$ and so D7-branes are needed. Several configurations of these necessary D7-branes were presented in ref. [110]. Large values of $|\lambda|$ and $|\epsilon_0|$ favour a small string coupling constant and increase the internal volume, i.e. $g_s \propto 1/|\lambda|^3$ and $V_{int} \propto |\epsilon_0|^{3/2}$, for a fixed Γ_P modular matrix and a given VEV for the complex structure modulus, U_0.

Example 2: vacua with a hierarchy of fluxes.

In this example we work out a family of solutions with a richer structure of localised sources. This time we fix the non-geometric Q and P fluxes to be isomorphic to $\mathfrak{g}_Q = \mathfrak{so}(4)$ and $\mathfrak{g}_P = \mathfrak{so}(4)$ respectively[9]. Just to illustrate some vacua with this algebraic structure, we fix the modular embeddings to be

$$\Gamma_Q = \begin{pmatrix} \alpha & \delta \\ -\lambda\,\alpha & \lambda\,\delta \end{pmatrix} \quad , \quad \Gamma_P = \begin{pmatrix} (1-\lambda)\,\alpha & 0 \\ 0 & (1+\lambda)\,\delta \end{pmatrix} \ , \tag{5.58}$$

with $\alpha\,\delta \neq 0$ and $\lambda\,(1-\lambda^2) \neq 0$ for the isomorphism to be well defined.

The cohomology condition has three branches, as seen in table 5.2, and the embeddings in (5.58) satisfy $\mathcal{Z}_{-1}^{Q} \parallel \mathcal{Z}_{-1}^{P}$, giving a type B configuration (see table 5.3). For simplicity we will fix again $\vec{\epsilon} \in \ker(\tilde{\Phi}_Q)$ and $\vec{\rho} \in \ker(\tilde{\Phi}_P)$, and so this time $\epsilon_1 = \epsilon_2 = \rho_1 = \rho_2 = 0$ which results in $N_7' = 0$. Under this fluxes choice, $E(U)$ has 1 real root, as table 5.5 shows. We find that $E(U)$ can be factorised as in (5.36) with $g_1 = 0$ and

$$\epsilon_3 = \frac{1-\lambda^2}{8\lambda}\Big((\lambda-1)^3\rho_3 + (\lambda+1)^3\rho_0 \Big) \quad , \quad \epsilon_0 = \frac{1-\lambda^2}{8\lambda^4}\Big((\lambda-1)^3\rho_3 - (\lambda+1)^3\rho_0 \Big) \ , \tag{5.59}$$

so that

$$\frac{g_0}{g_2} = \left(\frac{\delta}{\alpha}\right)^2 \qquad \text{and} \qquad \frac{f_0}{f_1} = -\left(\frac{\delta}{\alpha}\right) \frac{(\lambda-1)^3\rho_3}{(\lambda+1)^3\rho_0} \ . \tag{5.60}$$

Since $g_1^2 - 4\, g_2\, g_0 < 0$, these are physical vacua with $U_0 = i\left(\frac{\delta}{\alpha}\right)$. From eqs (5.34), the axiodilaton and Kähler moduli get stabilised to

$$\begin{aligned} S_0 &= \left(\frac{2\lambda}{\lambda^2-1}\right) i \ , \\ T_0 &= \frac{\lambda^2-1}{12\lambda(\lambda^2+1)} \left(\frac{(\lambda+1)^4}{\lambda^2-1}\,\rho_0 - \frac{(\lambda-1)^4}{\lambda^2-1}\,\rho_3 + i\left((\lambda-1)^2\,\rho_3 + (\lambda+1)^2\rho_0 \right) \right) . \end{aligned} \tag{5.61}$$

[9] In this case $\Delta_Q = (\epsilon_2 - \epsilon_1)/3$ and $\Delta_P = (\rho_2 - \rho_1)/3$.

The resultant tadpole conditions for these vacua are

$$N_3 = \frac{|\Gamma_Q|^3}{2\lambda^2}(\lambda^2-1)\left((\lambda-1)^6\,\tilde{\rho}_3^2 + (\lambda+1)^6\,\tilde{\rho}_0^2\right) \ , \tag{5.62}$$

together with

$$\begin{aligned} N_7 &= \frac{|\Gamma_Q|^3}{2\lambda}(\lambda^2-1)\left((\lambda-1)^2\,\tilde{\rho}_3 + (\lambda+1)^2\,\tilde{\rho}_0\right) \quad , \\ \tilde{N}_7 &= \frac{|\Gamma_Q|^3}{8\lambda^3}\left(\lambda^2-1\right)^3\left((\lambda-1)^2\,\tilde{\rho}_3 + (\lambda+1)^2\,\tilde{\rho}_0\right) \quad , \end{aligned} \tag{5.63}$$

where $\rho_3 = 4\lambda\tilde{\rho}_3$ and $\rho_0 = 4\lambda\tilde{\rho}_0$. Then $N_3 > 0$, $N_7 > 0$ and $\tilde{N}_7 > 0$ is necessary for vacua to be physical[10].

In terms of the original fluxes, this solution corresponds to $c_3 = -\alpha^3\,\lambda\,(\lambda-1)$, $c_2 = \tilde{c}_2 = \alpha^2\,\delta\,\lambda\,(\lambda+1)$, $c_1 = \tilde{c}_1 = -\alpha\,\delta^2\,\lambda\,(\lambda-1)$, $c_0 = \delta^3\,\lambda\,(\lambda+1)$ and $\tilde{d}_1 = \alpha\,\delta^2\,(\lambda^2-1)\,(\lambda+1)$, $\tilde{d}_2 = \alpha^2\,\delta\,(\lambda^2-1)\,(\lambda-1)$ for non-geometric fluxes; $b_0 = \delta^3\,(\lambda^2-1)\,(\lambda-1)^3\,\tilde{\rho}_3$, $b_1 = -\alpha\,\delta^2\,(\lambda^2-1)\,(\lambda+1)^3\,\tilde{\rho}_0$, $b_2 = \alpha^2\,\delta\,(\lambda^2-1)\,(\lambda-1)^3\,\tilde{\rho}_3$ and $b_3 = -\alpha^3\,(\lambda^2-1)\,(\lambda+1)^3\,\tilde{\rho}_0$ for NS-NS flux and $a_0 = -4\,\delta^3\,\lambda\,(\lambda+1)^3\,\tilde{\rho}_0$ and $a_3 = -4\,\alpha^3\,\lambda\,(\lambda-1)^3\,\tilde{\rho}_3$ for R-R flux.

By considering the fluxes' dependency on the parameter λ, we note that generically a hierarchy between $\bar{F}_3, \bar{H}_3$ and non-geometric Q, P fluxes occurs, in which the NS-NS and R-R fluxes, i.e. $a_i \propto \lambda^4$, $b_j \propto \lambda^5$ are large compared to the non-geometric fluxes, i.e. $c_i \propto \lambda^2$, $d_j \propto \lambda^3$, given $\lambda > 1$ for $\text{Im}S_0 > 0$. However, there is a critical value $\lambda_0 = 1+\sqrt{2}$ for which $g_s \geq 1$ if $\lambda \geq \lambda_0$. Hence, there is a narrow range, $1 < \lambda < \lambda_0$, for which non-perturbative string effects can be neglected, i.e. $\lambda = 2$ implies $g_s = 3/4$. Finally, large values of the $\tilde{\rho}_0$ and $\tilde{\rho}_3$ parameters favour a large internal volume.

Example 3: vacua with a non vanishing flux-induced C_8' tadpole.

We now consider a simple family of solutions with a non vanishing flux-induced C_8' tadpole for which all moduli get stabilised. Let us fix the non-geometric Q and P fluxes to be isomorphic to $\mathfrak{g}_Q = \mathfrak{so}(3,1)$ and $\mathfrak{g}_P = \mathfrak{so}(4)$ respectively[11]. Examples belonging to this pairing were also found in ref. [8].

For simplicity, we fix the modular embeddings to be

$$\Gamma_Q = \begin{pmatrix} \alpha & \delta \\ \alpha & -\delta \end{pmatrix} \quad , \quad \Gamma_P = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \ , \tag{5.64}$$

so that the cohomology condition for this pairing has an unique branch $\mathcal{Z}_0^Q \parallel \mathcal{Z}_{-1}^P$ as is shown in table 5.2. It is a non-geometric type B configuration (see table 5.3) and therefore, has a potentially non vanishing flux-induced C_8' tadpole. The modular embeddings

[10] Fixing $|\Gamma_Q| > 0$ implies $\lambda > 0$ for $\text{Im}U_0 > 0$, $(\lambda^2-1) > 0$ for $\text{Im}S_0 > 0$ and $(\lambda-1)^2\,\tilde{\rho}_3 + (\lambda+1)^2\,\tilde{\rho}_0 > 0$ for $\text{Im}T_0 > 0$. This fixes the net charge of the tadpoles.

[11] In this case $\Delta_Q = -\epsilon_2/3 - \epsilon_0 = -\epsilon_0'/24$ and $\Delta_P = (\rho_2 - \rho_1)/3$.

in (5.64) belong to this branch.

First of all, we will redefine our NS-NS flux parameters as

$$\begin{pmatrix} \epsilon_3' \\ \epsilon_1' \\ \epsilon_2' \\ \epsilon_0' \end{pmatrix} = 8 \begin{pmatrix} 3 & 1 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} \epsilon_3 \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_0 \end{pmatrix} . \tag{5.65}$$

Solutions with NS-NS and R-R fluxes for which $\vec{\epsilon} \notin \ker(\tilde{\Phi}_Q)$ and $\vec{\rho} \notin \ker(\tilde{\Phi}_P)$ can be given parametrically in terms of the (κ_1, κ_2) parameters as

$$\epsilon_3' = \kappa_1 + \kappa_2 \quad , \quad \epsilon_0' = \kappa_1 - \kappa_2 \quad , \quad \rho_1 = \kappa_2 \quad , \quad \rho_2 = \kappa_1 \; , \tag{5.66}$$

with $(\epsilon_1', \epsilon_2')$ expanding the $\ker(\tilde{\Phi}_Q)$ and (ρ_0, ρ_3) expanding the $\ker(\tilde{\Phi}_P)$ being completely free. For simplicity, we will deal just with a non vanishing κ_2 parameter plus the R-R fluxes ρ_0 and ρ_3. All the Jacobi identities are by construction satisfied. In general, $E(U)$ has 1 real root (see table 5.5) for this algebra pairing, but under this specific fluxes configuration, it has two real roots. Factorising out these real roots, $E(U) \rightarrow \tilde{E}(U)$, and requiring it to factorise as in (5.36) we find

$$f_1 = g_1 = \rho_0 = 0 \quad , \quad \rho_3 = \frac{4}{3}\kappa_2 \quad , \quad \frac{g_0}{g_2} = \left(\frac{\delta}{\sqrt{2}\alpha}\right)^2 \quad , \quad f_0\, g_2^2 = -16\, \alpha^4\, \delta\, \kappa_2 \; . \tag{5.67}$$

These values give $g_1^2 - 4\, g_2\, g_0 < 0$, producing physical vacua with $U_0 = i\left(\frac{\delta}{\sqrt{2}\alpha}\right)$.

Using the VEVs in (5.34), the moduli get stabilised to

$$U_0 = \left(\frac{\delta}{\sqrt{2}\alpha}\right) i \quad , \quad S_0 = \sqrt{2}\, i \quad , \quad T_0 = -\frac{\kappa_2}{27}\,(1 + \sqrt{2}\, i) \; , \tag{5.68}$$

which is physical for $\kappa_2 < 0$ and $|\Gamma_P| > 0$. The tadpole conditions for these vacua are

$$\tilde{N}_7 \;=\; 0 \; , \tag{5.69}$$

$$N_3 \;=\; \frac{\kappa_2}{15}\, N_7 = -\frac{\kappa_2}{3}\, N_7' = \frac{2}{9}\, |\Gamma_P|^3\, \kappa_2^2 \; , \tag{5.70}$$

with $|\Gamma_P| = \alpha\, \delta$, so $N_3 > 0$, $N_7 < 0$ and $N_7' > 0$ is required.

In terms of the original fluxes, this solution corresponds to $c_3 = 2\,\alpha^3$, $c_2 = \tilde{c}_2 = 2\,\tilde{d}_2 = 2\,\alpha^2\,\delta$, $c_1 = \tilde{c}_1 = 2\,\tilde{d}_1 = -2\,\alpha\,\delta^2$ and $c_0 = -2\,\delta^3$ for non-geometric fluxes; $b_0 = -\frac{\kappa_2}{6}\,\delta^3$ for NS-NS flux and $a_3 = \frac{4}{3}\,\kappa_2\,\alpha^3$, $a_1 = \frac{1}{3}\,\kappa_2\,\alpha\,\delta^2$ for R-R flux. The string coupling constant turns out to be $g_s = 1/\sqrt{2}$ and the internal volume increases for large values of $|\kappa_2|$. This also increases the number of localised sources cancelling the flux-induced tadpoles.

Example 4: vacua with a non defined flux-induced C_8 tadpole sign.

Finally, and for the sake of completeness, we fix the non-geometric Q and P fluxes to be isomorphic to $\mathfrak{g}_Q = \mathfrak{so}(4)$ and $\mathfrak{g}_P = \mathfrak{nil}$ respectively[12].

Now we fix the modular embeddings to be

$$\Gamma_Q = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \quad , \quad \Gamma_P = \begin{pmatrix} \alpha & -\delta \\ \alpha & \delta \end{pmatrix} \, , \tag{5.71}$$

with $\alpha\,\delta \neq 0$ for the isomorphism to be well defined. In this case, we obtained a single cohomology condition, see table 5.2, $\mathcal{Z}^{Q}_{-1} \parallel \mathcal{Z}^{P}_{\infty}$ which is satisfied by (5.71) and is again a type B configuration (see table 5.3). Once more, solutions with NS-NS and R-R fluxes for which $\vec{\epsilon} \notin \ker(\tilde{\Phi}_Q)$ and $\vec{\rho} \notin \ker(\tilde{\Phi}_P)$ can be given parametrically,

$$\epsilon_1 = -4\,(\kappa_1 - 3\,\kappa_2) \quad , \quad \epsilon_2 = -4\,(\kappa_1 + 3\,\kappa_2) \quad , \quad \rho_3 = \kappa_2 \quad , \quad \rho_2 = \kappa_1 \, , \tag{5.72}$$

depending on the (κ_1, κ_2) parameters and with (ϵ_0, ϵ_3) expanding the $\ker(\tilde{\Phi}_Q)$ and (ρ_0, ρ_1) expanding the $\ker(\tilde{\Phi}_P)$, being completely free.

For this pairing, $E(U)$ has 1 real root as was shown in table 5.5. Again, we find that $E(U)$ can be factorised as in (5.36) with $g_1 = f_1 = 0$ and

$$\epsilon_3 = \frac{2B^2}{A} - 2B + 4A \quad , \quad \epsilon_0 = -4A \quad , \quad \kappa_1 = \frac{1}{4}(B - 5A) \quad , \quad \kappa_2 = \frac{B-A}{4} \, , \tag{5.73}$$

together with

$$\frac{g_0}{g_2} = \left(\frac{\delta}{\alpha}\right)^2 \frac{A}{B} \qquad , \qquad f_0\, g_0^2 = -2\,A\,\delta^5 \, , \tag{5.74}$$

where $A = \rho_1 - \rho_0$ and $B = \rho_1 - 5\rho_0$. Then $g_1^2 - 4\,g_2\,g_0 < 0$ provided $AB > 0$ and there are physical vacua with $U_0 = i\left(\frac{\delta}{\alpha}\right)\left(\frac{\sqrt{A}}{\sqrt{B}}\right)$. In addition, from eqs (5.34), the axiodilaton and Kähler moduli get stabilised to

$$S_0 = \frac{\sqrt{A}\sqrt{B}}{(A+B)^2}\Big(2\sqrt{A}\sqrt{B} + i\,(B-A)\Big) \qquad \text{and} \qquad T_0 = \frac{4A}{3(A+B)}\Big(A + i\,\sqrt{A}\sqrt{B}\Big) \, . \tag{5.75}$$

The resultant tadpole conditions for these vacua are

$$N_3 = \frac{16}{3}|\Gamma_Q|^3\,\big(\,(B-A)^2 + AB\,\big) \, , \tag{5.76}$$

as well as

$$N_7 = -\frac{2}{3}|\Gamma_Q|^3(B-2A) \quad , \quad \tilde{N}_7 = -|\Gamma_Q|^3\frac{2(A+B)^2}{A} \quad , \quad N_7' = -4|\Gamma_Q|^3(B-A) \, , \tag{5.77}$$

[12] In this case $\Delta_Q = (\epsilon_2 - \epsilon_1)/3$ and $\Delta_P = -\rho_3$.

from which it follows that $N_3 > 0$, N_7 has no defined sign, $\tilde{N}_7 < 0$ and $N_7' < 0$ is required for physical vacua[13].

In terms of the original fluxes, this solution corresponds to $d_3 = -\alpha^3$, $-d_2 = \tilde{d}_2 = \tilde{c}_2 = \alpha^2\,\delta$, $d_1 = -\tilde{d}_1 = -\tilde{c}_1 = \alpha\,\delta^2$ and $d_0 = \delta^3$ for non-geometric fluxes; $b_0 = 4\,\delta^3\,A$, $b_1 = \frac{2}{3}\,\alpha\,\delta^2\,(A+B)$, $b_2 = \frac{4}{3}\,\alpha^2\,\delta\,(B-2A)$ and $b_3 = 2\,\alpha^3\left(\frac{(B-A)^2}{A} + (A+B)\right)$ for NS-NS flux and $a_0 = 2\,\delta^3\,A$, $a_2 = \frac{2}{3}\,\alpha^2\,\delta\,(B-2A)$ for R-R flux. This family of solutions gives rise to $g_s > 1$ for $A, B > 0$ and then, non-perturbative string effects can not be neglected.

5.4 Lifting to $\mathcal{N}=4$ gauged supergravities

The flux algebra in (2.24) was originally obtained by performing dimensional reduction of the $\mathcal{N}=1$ 10d heterotic theory into a 6d internal space which incorporates heterotic metric fluxes as well as a non-trivial background $\bar{H}_3$ for the NS-NS field strength $H_3 = dB_2$ [157]. Notice that the heterotic string does not contain R-R gauge potentials C_p in its spectrum. After reducing without breaking supersymmetries, one obtains 4d gauged supergravities with an extended $\mathcal{N}=4$ supersymmetry.

It is well known that the heterotic and the type I theories are related by S-duality and that the type I theory in turn represents a $\mathcal{N}=2 \rightarrow \mathcal{N}=1$ orientifold of the type IIB theory including O9-planes (see section 1.2.1). Therefore, in the type IIB with O9-planes orientifold theory, the correct starting point for an electric flux algebra would then be given by [161]

$$\begin{aligned} [Z_a\,,\,Z_b] &= \omega_{ab}^{c}\,Z_c + \bar{F}_{abc}\,X^c\,, \\ \left[Z_a\,,\,X^b\right] &= -\,\omega_{ac}^{b}\,X^c\,, \\ \left[X^a\,,\,X^b\right] &= 0\,, \end{aligned} \tag{5.78}$$

and no longer by that in (2.24). The main difference is that the R-R gauge flux $\bar{F}_3$ is now entering the isometry-isometry commutators, unlike in (2.24) where the NS-NS gauge flux $\bar{H}_3$ does. Consequently, the ω_{ab}^{c} fluxes in (5.78) are now type II metric fluxes.

According to (4.32) and (4.33), and after applying six T-dualities along the compact internal space coordinates in order to go from a type IIB with O9-panes to a type IIB with O3-planes orientifold theory, the electric flux algebra in (5.78) transforms into [155, 161]

$$\begin{aligned} \left[X^a\,,\,X^b\right] &= Q_c^{ab}\,X^c + \tilde{F}^{abc}\,Z_c\,, \\ \left[Z_a\,,\,X^b\right] &= Q_a^{bc}\,Z_c\,, \\ [Z_a\,,\,Z_b] &= 0\,, \end{aligned} \tag{5.79}$$

[13] Fixing $|\Gamma_Q| > 0$, then $A, B > 0$ for $\mathrm{Im}T_0 > 0$ and $(B-A) > 0$ for $\mathrm{Im}S_0 > 0$. This fixes the net charge of tadpoles but N_7 depends on the sign of $(B-2A)$, with $N_7 > 0$ for $(B-2A) < 0$ and $N_7 < 0$ for $(B-2A) > 0$.

where $\tilde{F}^{abc} = \frac{1}{3!}\,\epsilon^{abcdef}\,\bar{F}_{def}$. The algebra in (5.79) clearly differs from that of (2.46) in a crucial point, as originally noticed in ref. [161]: it always corresponds to a non-semisimple Lie algebra unlike the one in (2.46). This makes sense since the algebra in (5.79) has been obtained by performing duality transformations upon that in (2.24), and the semisimple/non-semisimple character of the algebra holds under these transformations.

As introduced in section 2.4, the half-maximal $\mathcal{N}=4$ four-dimensional gauged supergravities are specified by two constant embedding tensors, $\xi_{\alpha A}$ and $f_{\alpha ABC}$, under the global symmetry [9, 160]

$$\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SO}(6,6,\mathbb{Z}) \ , \tag{5.80}$$

where the index $\alpha = \pm$ denotes electric-magnetic components of the embedding tensors, and the index $A = (1, \ldots, 6\,,\,\bar{1}, \ldots, \bar{6})$ refers to the $\mathrm{SO}(6,6,\mathbb{Z})$ components when using light-cone coordinates[14] as in ref. [161]. By setting $\xi_{\alpha A} = 0$ (as it happens in the case of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ toroidal orbifold), the gauge group fulfils the general commutation relations

$$\Big[T^{A\alpha}\,,\,T^{B\beta}\Big] = f^{\alpha AB}{}_{C}\,T^{C\beta} \ , \tag{5.81}$$

with the generators splitting of $T^{A\alpha} = (Z_a^{\alpha}\,,\,X^{a\alpha})$. In addition, the structure constants $f^{\alpha AB}{}_{C}$ are forced to obey the Jacobi identities arising from this algebra.

At this point it becomes clear that the flux algebra in (2.46) corresponds to the identification [161]

$$\begin{array}{lclcl} f_{+abc} = \bar{H}_{abc} & , & f_{+a\bar{b}\bar{c}} = f_{+a}{}^{bc} = Q_a^{bc} & , \\ f_{-abc} = \bar{F}_{abc} & , & f_{-a\bar{b}\bar{c}} = f_{-a}{}^{bc} = P_a^{bc} & , \end{array} \tag{5.82}$$

between flux parameters and structure constants of the algebra in (5.81). Notice that we have also included in (5.82) what would be the magnetic $\bar{F}_3$ and P partners of the $\bar{H}_3$ and Q fluxes under an $\mathrm{SL}(2,\mathbb{Z})$ type IIB self-duality transformation.

In contrast, the flux algebra in (5.79) corresponds to identify the flux parameters and the structure constants in a different manner [161],

$$\begin{array}{lclcl} f_{+\bar{a}\bar{b}\bar{c}} = f_{+}{}^{abc} = \tilde{F}^{abc} & , & f_{+a\bar{b}\bar{c}} = f_{+a}{}^{bc} = Q_a^{bc} & , \\ f_{-\bar{a}\bar{b}\bar{c}} = f_{-}{}^{abc} = \tilde{H}^{abc} & , & f_{-a\bar{b}\bar{c}} = f_{-a}{}^{bc} = P_a^{bc} & . \end{array} \tag{5.83}$$

Under the identification of (5.83), the flux pair $(\bar{F}_3, Q)$ corresponds to $\mathrm{SL}(2,\mathbb{Z})$-electric fluxes whereas $(\bar{H}_3, P)$ to $\mathrm{SL}(2,\mathbb{Z})$-magnetic ones, hence in agreement with the results derived from the spinorial embedding of the flux parameters introduced in section 2.4.1.

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[14] Working with these coordinates, the SO(6,6) metric is taken to be off-diagonal, i.e. $\eta_{AB} = \begin{pmatrix} 0 & \mathbb{I}_6 \\ \mathbb{I}_6 & 0 \end{pmatrix}$, and the SO(6,6) vector index A has the (down/up) splitting of ${}^{A} = ({}_{a}\,,\,{}^{a})$.

Taking into account both SL$(2,\mathbb{Z})$-electric (Z_a, X^a) and SL$(2,\mathbb{Z})$-magnetic generators $(\bar{Z}_a, \bar{X}^a)$, it is easy to see that the set of isometry generators $\{Z_a\,,\,\bar{Z}_a\}$ spans a 12-dimensional abelian ideal. Therefore the 24-dimensional flux algebra specified by the structure constants in (5.83) turns out with a general semidirect sum structure

$$\mathfrak{g} = \mathfrak{g}_{(Q,P)} \oplus \mathfrak{u}(1)^{12} \;, \tag{5.84}$$

where the piece $\mathfrak{g}_{(Q,P)}$ represents a 12-dimensional algebra spanned by the gauge generators $\{X^a\,,\,\bar{X}^a\}$ and totally determined by the non-geometric (Q,P)-fluxes.

In the absence of fluxes, the algebra in (5.84) reduces to the abelian case $\mathfrak{u}(1)^{24}$. However, when gauge fluxes $(\bar{F}_3, \bar{H}_3)$ are switched on, the possibilities for the 24-dimensional flux algebra in (5.84) are summarised as follows:

i) If $Q = P = 0$, the algebra induced by the gauge fluxes $(\bar{F}_3, \bar{H}_3)$ becomes the non-trivial

$$\mathfrak{g} = \mathfrak{u}(1)^{12} \oplus \mathfrak{u}(1)^{12} \sim \mathfrak{nil}_{24}(2) \;. \tag{5.85}$$

By non-trivial we mean that it is not abelian. Specifically, it becomes nilpotent of order two because the lower central series becomes zero after two steps (abelian would be nilpotent of order 1).

ii) If $Q \neq 0$ and $P = 0$, the generators $\{Z_a\,,\,\bar{Z}_a\,,\,\bar{X}^a\}$ span an enlarged 18-dimensional solvable ideal. In particular, the solvable ideal turns out to be again nilpotent of order 2. The algebra in (5.84) is in this case given by

$$\mathfrak{g} = \mathfrak{g}_Q \oplus \mathfrak{nil}_{18}(2) \;, \tag{5.86}$$

where $\mathfrak{g}_Q$ is the 6-dimensional algebra specified by the non-geometric Q-flux.

iii) If $Q = 0$ and $P \neq 0$, a nilpotent 18-dimensional ideal is now spanned by the $\{Z_a\,,\,\bar{Z}_a\,,\,X^a\}$ generators. In this case, the algebra in (5.84) takes the form of

$$\mathfrak{g} = \mathfrak{g}_P \oplus \mathfrak{nil}_{18}(2) \;, \tag{5.87}$$

with $\mathfrak{g}_P$ now being the 6-dimensional algebra induced by the non-geometric P-flux.

iv) If $Q \neq 0$ and $P \neq 0$ then the algebra results no longer simplified and corresponds to the generic form given in (5.84).

The fluxes in (5.83), and consequently the flux-induced algebras listed above, are restricted by a set of quadratic constraints [160] coming from the closure of the algebra as well as from the orthogonality of charges [161]. These constraints are needed in order to have a consistent 12-dimensional gauging. They turn out to imply the absence of 7-branes (notice that their presence would break the $\mathcal{N} = 4$ supersymmetry) and then the vanishing of the SL$(2,\mathbb{Z})$-triplet of flux-induced tadpoles in (2.57) for the $(C_8, C_8', \tilde{C}_8)$ R-R gauge potentials [155, 161].

Irrespective of the identification between flux parameters and structure constants of the $\mathcal{N}=4$ gauged supergravity algebra of (5.81), the $\mathcal{N}=1$ effective theory invariant under T- and S-duality transformations arising from the type IIB $\mathbb{Z}_2 \times \mathbb{Z}_2$ isotropic orientifold with O3/O7-planes is specified by the Kähler potential and the superpotential in (2.67). However, from the moduli flux vacua found in the previous chapters, only those compatible with the lack of 7-branes could in principle correspond to an underlying $\mathcal{N}=4$ gauged supergravity.

Unfortunately all the supersymmetric Minkowski solutions we found in the T- and S-duality invariant effective models required 7-branes in order to cancel flux-induced tadpoles. Nevertheless, in the case of the T-duality invariant supergravity models, the single moduli flux vacuum shown in figure 4.14 did not need of such 7-branes. This non-supersymmetric Minkowski solution was compatible with switching off the non-geometric P-flux and turned out to be unstable (tachyonic).

Chapter 6

Modular Inflation in Supergravity Flux Models

The topic of inflation coming from string models is experiencing a comeback driven by both progress in our theoretical understanding of the models and the increasing precision of observations. Whereas the former has been commonly identified, in its phenomenological aspects, with the publication of moduli stabilisation mechanisms in a dS vacuum (KKLT, generalised fluxes, etc.), the experimental data released by the WMAP collaboration [27,28] give support to the inflationary picture, as well as provide us with stringent limits on some of the cosmological parameters, something that can immediately be implemented within inflationary model building.

This revival comprises different string models where the inflaton is normally a modulus, either parameterising the distance between branes (as proposed originally by Dvali and Tye in ref. [111]), or the geometry/structure of the compactified space. This second option, also denoted as *modular inflation*, has been studied ever since the first models of moduli stabilisation were put forward [128]. Inflation was not working at all because either the moduli were flat at all orders in perturbation theory or, when including non perturbative effects, their potential would be too steep to inflate [129]. Moreover we have to add the fact that, at that stage, all models of moduli stabilisation predicted a negative vacuum energy. Despite of all these problems, a few partially successful examples were built [29,130,131].

As we have seen in the chapter 4, progress on finding dS vacua has been driven by our increasing understanding of (generalised) flux compactifications and their crucial role in stabilising moduli. However, before the inclusion of non-geometric fluxes, several mechanisms to solve the long standing problem of the appearance of AdS_4 vacua had been proposed. In particular, in the context of the type IIB theory, see also ref. [105], the KKLT construction in ref. [12] considered the presence of anti-D3-branes as the source of an additional term in the scalar potential, which breaks supersymmetry explicitly and may uplift the minima from AdS_4 to dS. In order to improve the aesthetics and the control over the effective theory, an alternative, fully supersymmetric, approach based on the presence of a non-vanishing (Fayet-Iliopoulos) D-term as the uplifting contribution to the potential, was proposed in ref. [101]. In its original formulation, this mechanism had important inconsistencies, that were fixed in ref. [109]. In that paper, a consistent formulation of these so-called "uplifting" D-terms in the context of $\mathcal{N}=1$ supergravity was established, implying the presence of chiral matter living on D-branes. Explicit viable examples were given, where a Kähler modulus T and the chiral matter get stabilised at phenomenologically relevant values and positive vacuum energy. The scenario was mainly determined by the requirements of supersymmetry and gauge invariance and, therefore, left little room for fine-tuning the parameters entering the superpotential, which increases the predictive power[1].

In this final chapter we carry on along the lines of modular inflation in this new context of uplifted supergravities[2] and propose and study a scenario based on the setup of ref. [109] plus an extra singlet, where suitable inflation takes place. In order to do that we take into account all possible observational constraints which, together with the symmetries of the model, will determine the structure and size of the different couplings in the superpotential.

6.1 Gauge fluxes, de Sitter vacua and inflation

In this section we briefly review the scenario considered in ref. [109]. It describes an $\mathcal{N}=1$ effective supergravity coming from type IIB superstring theory, although it can also be realised in the context of heterotic strings.

Along the lines of the moduli stabilisation mechanism proposed by KKLT in ref. [12], we assume that all moduli (axiodilaton and complex structure) but one, an overall T modulus, have been stabilised at a high scale due to the presence of gauge fluxes $(\bar{F}_3, \bar{H}_3)$ which induce the GVW superpotential shown in (2.21). In addition, non-perturbative effects involving the Kähler modulus T are also taken into account such that the resulting effective model is then given by the Kähler potential and the superpotential

$$\begin{aligned} K &= -3\log(T+\bar{T}) \;, \\ W &= W_{flux} + W_{n.p} = W_0 + A\, e^{-aT} \;. \end{aligned} \tag{6.1}$$

Notice the change of convention between the volume component of the Kähler modulus in (6.1) and that in (2.67). The W_0 parameter appears as an effective quantity resulting

[1] Alternative approaches to achieve the desired uplifting can be found in refs [103, 106, 108].

[2] Other, related work, in the topic of string/brane inflation can be found in refs [30, 107, 112–114, 116–124, 132–134, 196–198].

from having stabilised the axiodilaton and the complex structure moduli at a higher scale, whereas a and A are constant parameters associated to the non-perturbative term. This model stabilises the Kähler modulus T in a supersymmetric AdS_4 minimum of the scalar potential $V = V_F$ built from the standard form in (2.62). For this vacuum to be *uplifted* to dS, additional anti-D3-branes were also considered generating a contribution to the scalar potential given by

$$V = V_F + \frac{k}{(\mathrm{Re}T)^2} \,. \tag{6.2}$$

For a suitable value of the parameter k, the original AdS_4 vacuum is lifted to a dS one. Nevertheless, the term $\frac{k}{(\mathrm{Re}T)^2}$ breaks supersymmetry in an explicit manner and renders corrections difficult to be computed reliably.

6.1.1 De Sitter vacua via D-terms uplifting

As it is common in type IIB orientifold models with O3/O7-planes, we also assume that gaugino condensation happens, with gauge group $SU(N)$, due to stacks of N D7-branes wrapped on some 4-cycle of the Calabi-Yau space. For each $SU(N)$ there typically appears a $U(1)$ factor. Some of these $U(1)$'s, or combinations of them, can be anomalous.

As proposed by BKQ in ref. [101], the corresponding D part of the scalar potential,

$$V_D \sim \frac{\pi\, E^2}{(\mathrm{Re}T)^3} \,, \tag{6.3}$$

can provide the required uplifting of the potential in order to have de Sitter vacua through a tuning of the parameter E. This has the advantage that supersymmetry is not explicitly broken (as it was by the anti-D3-brane contributions originally considered in KKLT), so corrections can be more reliably computed and kept under control. However, the setup of ref. [101] had some serious inconsistencies [91, 104], arising from the lack of gauge invariance of the formulation.

Nevertheless, as shown in ref. [109], this scheme can be made gauge invariant, which imposes the presence of chiral matter transforming typically as $(N, \bar{N})$ with abelian charges $(q, \bar{q})$, for the whole setup to be consistent. This mechanism was confirmed in explicit string constructions in ref. [199]. As stressed in ref. [109], the constraints coming from enforcing gauge invariance are sufficiently strong to determine the form of the superpotential. Moreover, the presence of the anomalous $U(1)$ group generates a Fayet-Iliopoulos term which enters the D part of the scalar potential and is responsible for the uplifting.

Based on this information one can construct a simple model assuming that the symmetries in the hidden sector are dictated by a unique $SU(N)$ times the anomalous $U(1)$. In its simplest possible form, the model contains:

- A Kähler modulus T.
- Two chiral multiplets, Q, $\bar{Q}$, transforming as $(N, \bar{N})$ with abelian charges $(q, \bar{q})$. This is equivalent to have only one flavour, i.e., $N_f = 1$.

The modulus field transforms non trivially under the U(1) group,

$$T \to T + i\,\frac{\delta_{GS}}{2}\epsilon \quad , \tag{6.4}$$

and the corresponding transformation of the Lagrangian compensates the $\mathrm{SU}(N)^2 \times \mathrm{U}(1)_X$ and $\mathrm{U}(1)_X^3$ anomalies, according to the Green-Schwarz mechanism. The resulting condition reads

$$\delta_{GS} = -\frac{(q+\bar{q})}{2\pi k_N} = -\frac{N(q^3+\bar{q}^3)}{3\pi k_X} \quad , \tag{6.5}$$

where k_N, k_X are $\mathcal{O}(1)$ constant factors that enter in the definition of the corresponding gauge couplings [86, 200, 201].

It is also well known that the theory gets strongly coupled in the infrared. As a consequence, gaugino condensation takes place at some scale, Λ, and also squark meson condensates

$$M^2 = 2\,Q\bar{Q} \quad , \tag{6.6}$$

are formed. A non perturbative superpotential term is generated [202–204],

$$W_{n.p} = (N-1)\left(\frac{2\,\Lambda^{3N-1}}{M^2}\right)^{\frac{1}{N-1}} = (N-1)\left(\frac{2}{M^2}\right)^{\frac{1}{N-1}}\; e^{\frac{-4\,\pi\,k_N\,T}{N-1}} \quad , \tag{6.7}$$

which is, as expected, invariant under U(1) transformations.

The $\mathcal{N}=1$ supergravity model that we will be dealing with is then defined in terms of the the Kähler potential[3] and the superpotential

$$\begin{aligned} K &= -3\,\log(T+\bar{T}) + |Q|^2 + |\bar{Q}|^2 = -3\,\log(T+\bar{T}) + |M|^2 \;, \\ W &= W_0 + W_{n.p} \;, \end{aligned} \tag{6.8}$$

where W_0 is the (real) effective flux parameter and $W_{n.p}$ is the non-perturbative superpotential given in (6.7).

As usual in $\mathcal{N}=1$ supergravity, the scalar potential is the sum of an F-part and a D-part,

$$V = V_{\mathrm{F}} + V_{\mathrm{D}} \;. \tag{6.9}$$

Using Planck units $m_p^{-2} = 8\,\pi\,G_{\mathrm{N}} = 1$, the V_{F} piece is given by

$$V_{\mathrm{F}} = e^K\left[K_{ij}^{-1} D_i W D_j \bar{W} - 3|W|^2\right] \;, \tag{6.10}$$

[3] We are assuming here a minimal Kähler potential for the matter fields. This is a simplification, but it does not affect the main results and conclusions. We will come back to this point in section 6.1.3.

where subindices denote Kähler derivatives with respect to all scalar fields as in (2.62). In our case, $V_{\rm F}$ can be computed using the complete superpotential in (6.8). In addition, we have used $|Q|^2 = |\bar{Q}|^2$, which is dynamically dictated by the cancellation of the SU(N) D-term (in the $N_f = 1$ case). The D part of the potential associated with the U(1) is then given by

$$V_{\rm D} = \frac{\pi}{2k_X(T+\bar{T})}\left((q+\bar{q})|M|^2 - \frac{3\delta_{GS}}{T+\bar{T}}\right)^2 , \tag{6.11}$$

which is positive definite.

For reasonable values of N, q, $\bar{q}$ and k_N it is possible to choose W_0 in such a way that:

i) There are minima of the F-potential with broken supersymmetry and negative vacuum energy.

ii) $V_{\rm D}$ is non-zero and sizeable enough to uplift these vacua from AdS_4 to dS.

As explained in ref. [109], the absence of physical vacua with unbroken supersymmetry comes from the fact that the conditions $D_T W = D_M W = 0$ can only be fulfilled simultaneously in the decompactification limit, $\mathrm{Re}\, T \to \infty$. This can be easily checked by computing

$$\frac{1}{K_T} D_T W - \frac{1}{K_M} D_M W = W_{n.p}\left(\frac{T+\bar{T}}{3}a + \frac{b}{|M|^2}\right) , \tag{6.12}$$

with (a,b) defined by expressing eq. (6.7) as $W_{n.p} \propto M^{-b}\, e^{-a\,T}$. The positiveness of the (a,b) parameters implies that this equation has no solution consistent with $D_T W = D_M W = 0$ for physical values $0 < \mathrm{Re}\, T < \infty$ of the modulus field.

Let us now review the main features of the vacuum structure of these models. For the remainder of this analysis it is particularly convenient to split the complex fields T and M in the following way,

$$T = T_{\rm R} + i\, T_{\rm I} \qquad \text{and} \qquad M = \rho_{\rm M}\, e^{i\alpha_{\rm M}} . \tag{6.13}$$

As we have mentioned, there is always a supersymmetric vacuum at $T_{\rm R} \to \infty$. Besides this minimum, for given values of $\rho_{\rm M}$ and $T_{\rm R}$, the potential gets always minimal when $\alpha_{\rm M}$ and $T_{\rm I}$ are such that W_0 and $W_{n.p}$ are aligned in the complex plane. For real W_0 this translates into the condition

$$\varphi_{n.p} \equiv -\frac{2}{N-1}(\alpha_{\rm M} + 2\,\pi\, k_N\, T_{\rm I}) = n\,\pi , \tag{6.14}$$

where $\varphi_{n.p}$ is the phase of $W_{n.p}$ and n is even or odd depending on the details of the model [109]. Actually, $\alpha_{\rm M}$ and $T_{\rm I}$ appear in the potential only through the $\varphi_{n.p}$ combination, which can thus be set to its minimising value. Hence, the minimisation can be

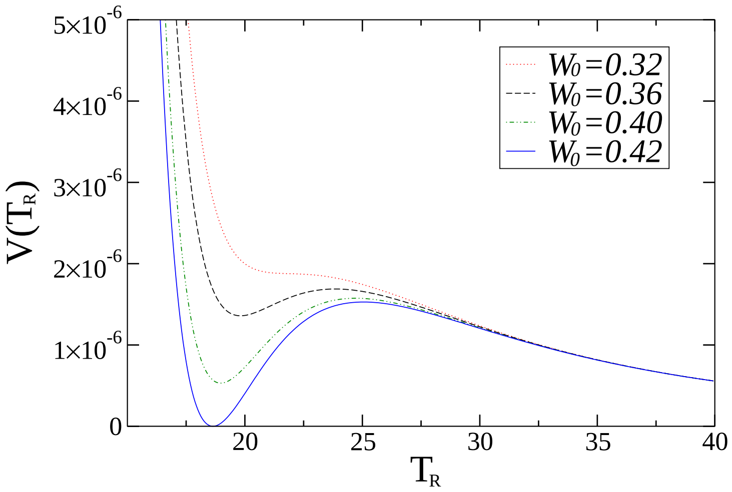

Figure 6.1: Plot of the scalar potential, as a function of $T_{\rm R}$, for the example shown in the text and $\rho_{\rm M}$ chosen as the value minimising the potential at each $T_{\rm R}$.

reduced to a two variable problem, namely to find the values of $(T_{\rm R}\,,\,\rho_{\rm M})$ at the minimum.

It is worth noticing that the energy of the vacuum is controlled by the size of the flux parameter W_0. In the figure 6.1 we plot the dependence of the scalar potential with W_0 as a function of $T_{\rm R}$, with the non-perturbative phase, $\varphi_{n.p}$, fixed to its minimising value. The matter condensate, $\rho_{\rm M}^2$, changes along the path and is determined by the extremal condition $\delta_{\rho_{\rm M}} V = 0$. The particular model is defined by $N = 20$, $q = 1$, $\bar{q} = 1/10$ and $k_N = 1/2$. The remaining parameters k_X and $\delta_{\rm GS}$ are fixed by the anomaly cancellation condition in (6.5).

As discussed in ref. [109], the value of W_0 sets the overall scale of the F-potential. On the other hand, from (6.11), the size of $V_{\rm D}$ is always $\mathcal{O}(N^{-1}T_{\rm R}^{-3})$ in Planck units. Therefore, too large values of W_0 result in a too large (and negative) $V_{\rm F}$, then the up-lifting by $V_{\rm D}$ is not efficient enough to promote the minimum from negative to positive. Conversely, too small values of W_0 would imply that $V_{\rm D}$ dominates too much and we lose the minimum. This explains the allowed range of W_0 values shown in figure 6.1 for the example at hand. Notice, also from that figure, that the natural scale for the potential is about five or six orders of magnitude below the Planck scale. Consequently, to obtain a Minkowski vacuum, or de Sitter with a small cosmological constant consistent with observation, requires the tuning of W_0, as usual.

Concerning the dependence of the potential on the other fields involved in the problem, V increases monotonically with $\rho_{\rm M}$ as this departs from its value at the minimum, and has a periodic behaviour along the axionic direction defined by the relative phase $\varphi_{n.p}$

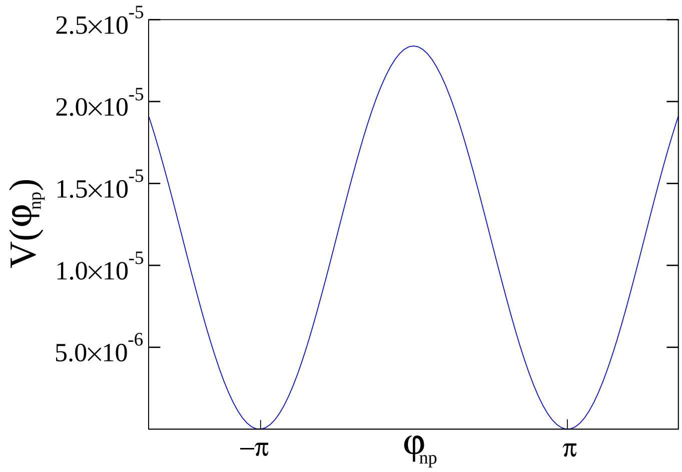

Figure 6.2: Periodic structure of the potential along the relative phase $\varphi_{n.p}$. The model is the same as in the previous figure and $(T_{\rm R}\,,\,\rho^2_{\rm M})$ have been fixed to their values at the minimum.

given in (6.14). The other independent phase does not appear in the potential, as already mentioned. The latter behaviour is represented in figure 6.2, which shows (for the same example as in figure 6.1) the scalar potential as a function of $\varphi_{n.p}$, with $(T_{\rm R}\,,\,\rho^2_{\rm M})$ fixed to their values at the minimum.

6.1.2 Chances and problems to implement inflation

Now we consider the problem of finding a plausible candidate to inflaton in these scenarios. For that matter it is convenient to recall the most ubiquitous obstacles that one finds to implement inflation in concrete models.

6.1.2.1 The η and initial condition problems

In supersymmetric theories the inflaton, ϕ, is generically one (real) component of some complex scalar field, Φ. Successful inflation requires that the slow-roll parameters defined (in Planck units) as

$$\epsilon \equiv \frac{1}{2}\left[\frac{V'}{V}\right]^2 \quad , \quad \eta \equiv \frac{V''}{V} \ , \tag{6.15}$$

(the primes denote differentiation with respect to ϕ) must be $\ll 1$. On the other hand, it is well known that supergravity theories generically present the so-called "η problem" [29], which refers to the fact that η is naturally of order one due to the presence of the e^K factor in (6.10).

Take, for example, the simplest form of the Kähler potential, i.e. $K = |\Phi|^2$, which gives rise to canonical kinetic terms for Φ. If we write eq. (6.10) as

$$V = e^K \tilde{V} \ , \tag{6.16}$$

and expand the exponential to first order, we see that

$$V \sim (1 + |\Phi|^2 + \ldots) \tilde{V} \ . \tag{6.17}$$

In other words, Φ, and thus ϕ, acquires a mass term of order $\tilde{V}$ (in Planck units) and the corresponding contribution to η in (6.15) is of order one, which totally disfavours slow-roll.

Some mechanisms have been proposed to avoid or alleviate this η problem. In particular it is obvious that if ϕ does not enter the Kähler potential, K, it will not pick up a mass term coming from the expansion of the e^K term. However, this is neither sufficient nor necessary to avoid the η problem, since one also has to take into account the possible dependence on Φ of $\tilde{V}$ in (6.16). Schematically,

$$V \sim (1 + |\Phi|^2 + \ldots) \left[\tilde{V}\right]_{\Phi=0} + e^K \left[\frac{\partial^2 \tilde{V}}{\partial\Phi\partial\bar{\Phi}}\right]_{\Phi=0} |\Phi|^2 \ + \ \cdots \ . \tag{6.18}$$

Consequently, what should be required is that the various $|\Phi|^2$ terms involved in (6.18) cancel among themselves.

Beside the η problem, we must worry about the initial condition for the inflaton. In other words, it is desirable that we do not need to invoke tuning or unreasonable assumptions of any kind about the initial value of ϕ required to have a successful subsequent period of inflation. Note that this is typically a problem of naturalness. To that respect, an attractive possibility is that of eternal topological inflation [205–207], which is realized in models where the inflaton potential has a saddle point between different, degenerate vacua. This gives rise to domain walls forming and, in the presence of an expanding universe, the false vacuum inside the walls serves as a site of inflation. This is both topological and eternal because, even though the field will be driven towards one of the minima, the core of the wall grows exponentially with time, providing us with a very generic initial state. Whether topological inflation actually takes place or not for a given potential is a delicate question which has been studied for only a few examples [208]. Generally speaking, a sufficiently flat potential satisfying the slow-roll conditions with a large VEV looks to be necessary, according to these previous results.

6.1.2.2 Candidates to inflaton

Figures 6.1 and 6.2 suggest that both $T_{\rm R}$ and $\varphi_{n.p}$ could be inflaton candidates. The structure of the potential along both directions shows a maximum (storing large potential energy), which might be the starting point of topological inflation. However things are not that smooth. The $T_{\rm R}$ field has an obvious η problem since it appears explicitly in the Kähler potential given in (6.8). The same holds for the matter condensate $\rho_{\rm M}$.

The chances of $\varphi_{n.p}$ could be better in principle since it does not appear in the Kähler potential. Actually, using $T_{\rm I}$ (which is one of the components of $\varphi_{n.p}$ defined in (6.14)) as the inflaton was the basic strategy followed in refs [133, 134], where they considered a setup with no matter fields. However, as discussed above, this is not a guarantee to avoid the η problem due to the presence of other dangerous contributions to η, see eqs (6.16) and (6.18).

Let us analyse this possibility in more depth. For the sake of the simplicity, let us ignore for the moment the matter fields, writing $W_{n.p} = Ae^{-aT}$. Then the $T_{\rm I}$-dependent terms in V are proportional to $\cos(-aT_{\rm I})$, having a similar size to other terms and, thus, to the whole V. Noting that the canonically normalised field is $\hat{T}_{\rm I} = (\sqrt{6}/2T_{\rm R})T_{\rm I}$, it is straightforward, from eq. (6.18), to see that $\eta = \mathcal{O}(1) \times (aT_{\rm R})^2$. Since, usually, $aT_{\rm R} > \mathcal{O}(1)$ to provide the required suppression for the potential, then η is naturally larger than $\mathcal{O}(1)$, which prevents $\hat{T}_{\rm I}$ from inflating. Actually, the η problem was also found in refs [133,134], and was solved by tuning the parameters of the model appropriately [4].

In our case, we can be more precise since the size of $W_{n.p}$ is greatly constrained from the above-mentioned fact that the size of $V_{\rm F}$ must be of the same order as $V_{\rm D}$, and the latter is basically fixed [5]. More precisely, from eqs (6.11) and (6.5) (see also eq. (6.32)) the size of $V_{\rm D}$ is

$$V_{\rm D} \sim \frac{27}{128\pi N k_N^3 T_R^3} \frac{(q+\bar{q})^3}{(q^3+\bar{q}^3)} \simeq \frac{\mathcal{O}(1)}{15 N k_N^3 T_R^3} \; . \tag{6.19}$$

Now, to estimate $W_{n.p}$ we can use the condition $V_{\rm F} \sim V_{\rm D}$ (necessary for a successful uplifting). Actually, since $V_{\rm F}$ is a sum of terms, none of these should exceed $V_{\rm D}$ unless there are unlikely delicate cancellations between them. So we can concentrate, e.g., on the term $e^K K^{T\bar{T}} |W_T|^2$, which depends on $W_{n.p}$ in a clear way. From eq. (6.7), the value of $W_{n.p}$ is proportional to $\rho_{\rm M}^{-2/(N-1)}$. Usually $\rho_{\rm M}^2$ is small, but for $N = \mathcal{O}(10)$ this factor is $\mathcal{O}(1)$ (or maybe larger). Then the condition $e^K K^{T\bar{T}} |W_T|^2 \lesssim V_{\rm D}$ translates into

$$e^{-8\pi k_N T_{\rm R}/(N-1)} \lesssim \mathcal{O}(10^{-3}) \times \frac{1}{N(k_N T_{\rm R})^2} \; . \tag{6.20}$$

For example for $N, T_{\rm R} = \mathcal{O}(10)$, which is a reasonable choice, the condition becomes $k_N\, T_{\rm R}/(N-1) \sim 1/2$ which, in turn, implies $\eta = \mathcal{O}(20)$. This result is quite robust since,

[4] The setup in these references has two different (racetrack) exponentials in W, which makes the structure of V more involved, but the previous schematic argument still applies.

[5] In refs [133,134] the uplifting of the potential was provided by an explicitly non-supersymmetric term $\delta V = E/T_{\rm R}^2$ which, in the spirit of KKLT, could arise from anti-D3-branes. Unlike our uplifting $V_{\rm D}$ potential, the size of E is not constrained (or it is uncertain), so it represents an extra degree of freedom that can be tuned. Besides, the exponents of $W_{n.p}$ were taken as continuously varying quantities (contrary to our case, where they go as $(N-1)^{-1}$ in (6.7)), which allowed their tuning.

for other ranges of values of N and $T_{\rm R}$, the exponent in (6.20) cannot change much if the balance between F and D terms is to be maintained. These results are confirmed by our numerical analysis.

An alternative argument to exclude topological inflation driven by the non-perturbative phase $\varphi_{n.p}$ in figure 6.2 is that, as long as it changes from $\varphi_{n.p} = \pm\pi$ to $\varphi_{n.p} = 0$, the minimum and the saddle point merge together and all the structure of the scalar potential disappears giving rise to a runaway behaviour. This is shown in figure 6.3.

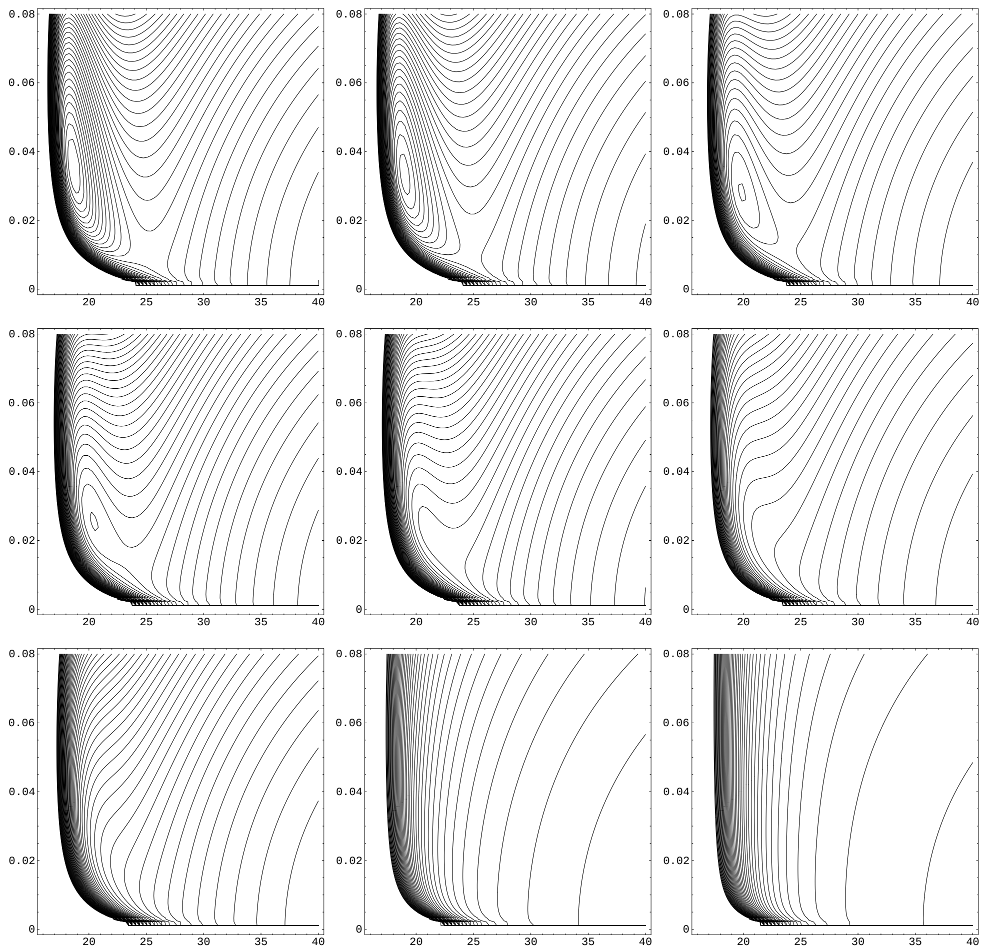

Figure 6.3: Sections of the scalar potential $V(\,\mathrm{Re}\,T, |M|, \varphi_{n.p}\,)$ as a function of $(\mathrm{Re}\,T, |M|)$ when the non-perturbative phase varies from $\varphi_{n.p} = \pm\pi$ (upper-left plot) to $\varphi_{n.p} = 0$ (lower-right plot).

Therefore, we can conclude that, in its simplest form, the modulus-condensate scenario does not contain a suitable inflaton. This suggests to explore natural modifications to the simplest setup, which we do in the next section.

6.1.3 A simple inflationary model

The most obvious extensions of the model are to allow either more matter flavours (until now we have fixed $N_f = 1$) or a second gaugino condensate, but in both cases the problems persist.

Another natural modification is to add matter charged under the anomalous U(1) group and coupled to the squark condensate. Then the superpotential $W = W_0 + W_{n.p}$, with $W_{n.p}$ given by eq. (6.7), can get extra terms such as

$$W \to W + \lambda\, M^a\, X^b \ , \tag{6.21}$$

where λ is a coupling constant, and X is a new singlet. This potential exhibits several degenerate vacua for $a > 0$, $b \geq 2$ and non zero values of the singlet X, and a saddle point at $\langle X \rangle = 0$, which could be useful for the implementation of topological inflation. However, as already pointed out in ref. [109], this kind of coupling between singlet and matter condensate forces the anomalous charges of X and M to be such that the Fayet–Iliopoulos D-term can cancel. In consequence, uplifting does not happen and there is no Minkowski (or de Sitter with small cosmological constant) minimum towards which the inflaton can slow-roll.

Another possibility would consist of changing the Kähler potential for the matter condensate M, which has been, up to now, taken to be canonical in (6.8), without introducing any extra fields. We looked at the possibility of considering a more string-motivated ansatz, namely

$$K = -3\log(T + \bar{T} - |M|^2) \ , \tag{6.22}$$

and we checked that the shape and position of the extrema were similar to those obtained using (6.8). Therefore this modification of the original model would not turn any components of T or M into a suitable inflaton.

Finally, a natural and simple extension is to consider just an additional neutral singlet, χ, which is not coupled to the SU(N) sector, except gravitationally (χ may be a superfield from another sector of the setup). The absence of terms in the superpotential coupling the singlet χ to the T and M fields implies that eq. (6.12) holds and, as in the previous model, there are no supersymmetric vacua. Furthermore let us assume, for simplicity, that χ has canonical Kähler potential, $\Delta K = |\chi|^2$, and a polynomial superpotential $\Delta W(\chi) = \sum \lambda_n \chi^n$. If χ is to play the role of the inflaton it is convenient (in order to implement a topological inflation mechanism) that the potential has degenerate vacua with different χ values. This condition is fulfilled, for example, when $\Delta W(\chi)$ possesses some discrete symmetry and the singlet takes a vacuum expectation value. Again for the sake of simplicity, we will assume that the model has a $\mathbb{Z}_2$ symmetry $\chi \to -\chi$. Then, the complete superpotential (in m_p units) reads

$$\begin{aligned} W &= W_0 + W_{n.p} + \Delta W(\chi^2) \\ &= W_0 + W_{n.p} + \lambda_2\chi^2 + \lambda_4\chi^4 + \lambda_6\chi^6 \ , \end{aligned} \tag{6.23}$$

with $W_{n.p}$ given by eq. (6.7). Higher other terms can be added but they get more and more irrelevant as long as $\langle\chi\rangle < m_p$. In summary, the model is characterised by the superpotential in (6.23) and the Kähler potential

$$K = -3\log(T+\bar{T}) + |M|^2 + |\chi|^2 \ . \tag{6.24}$$

The independent parameters are W_0 and λ_i. Besides, there are the parameters defining the gauge sector, though these should be around their natural values: $N \sim \mathcal{O}(10)$, $k_N \sim \mathcal{O}(1)$ and $q, \bar{q} \sim \mathcal{O}(1)$. Let us examine next the physics resulting for this simple extension of the initial setup, as far as inflation is concerned.

6.1.3.1 The potential

We will first study the structure of the extrema of the scalar potential, after the addition of the χ singlet. Notice that, due to the $\mathbb{Z}_2$ symmetry, any extremum in the modulus-condensate sector is still an extremum of the enlarged potential for $\chi = 0$. The stability of these extrema will depend on the parameter λ_2. In particular, for sufficiently small λ_2 the extrema become saddle points. This can be understood by fixing $\lambda_2 = 0$. Then, from eq. (6.10),

$$\Delta V = \left[V_{\rm F} + e^K|W|^2\right]_{\chi=0}|\chi|^2 \ + \ \mathcal{O}(|\chi|^4) \ . \tag{6.25}$$

The value of $V_{\rm F}$ at $\chi = 0$ is the same as in the original dS or Minkowski vacuum; thus, as explained in sect. 2, $V_{\rm F} < 0$. As a matter of fact, in these scenarios the $V_{\rm F}$ term in eq. (6.25) dominates over the second one within the brackets, giving rise to an instability in the singlet direction. It is clear that the slope of this instability is reduced once we switch on the mass term ($\lambda_2 \neq 0$) in eq. (6.23). Concerning the implementation of inflation, we can in principle use this saddle point as the source of eternal topological inflation. Then the original vacuum must be a de Sitter (not Minkowski) vacuum.

In this setup, playing with λ_2, a value for the η parameter consistent with slow-roll and observational data for the spectral index, $n_s \simeq 1 + 2\eta \simeq 0.95$, can be easily obtained although, admittedly, this represents a certain tuning of λ_2. Besides the initial saddle point, we need of course an actual minimum of the potential corresponding to the physical "quasi-Minkowski" vacuum, towards which the inflaton could roll. A minimum for large enough $|\chi|$ is actually quite easy to generate due to the quadratic (and higher order) terms in eq. (6.23). Then, the sign and size of λ_4, λ_6 can be chosen so that the minimum does correspond to a quasi-Minkowski vacuum, as desired. Of course this represents a new tuning and, in this case, a very severe one, though this is nothing but the usual fine-tuning to adjust the phenomenological cosmological constant. Finally, the values of W_0 and λ_i must be tuned in order to reproduce the size of the observed power-spectrum, $P(k) \sim 10^{-10}$ (more details will be given in the next subsection). In the analysed examples this requires V at the saddle point to be $\sim \mathcal{O}(10^{-16})$.

Let us present now one example where all these conditions are fulfilled. The modulus-condensate sector is the same that the one described in section 6.1.1 . The other parameters defining the superpotential are

$$W_0 = 0.4204 \quad , \quad \lambda_2 = -0.215 \quad , \quad \lambda_4 = -0.055 \quad \text{and} \quad \lambda_6 = -0.009 \; . \tag{6.26}$$

The potential derived from this model has:

- a saddle point at

$$T_{\rm R} = 18.6407 \quad , \quad \rho_{\rm M} = 0.03551 \quad , \quad \chi = 0 \; . \tag{6.27}$$

- a quasi-Minkowski minimum at

$$T_{\rm R} = 18.6554 \quad , \quad \rho_{\rm M} = 0.03549 \quad , \quad \chi = 0.0908 \; . \tag{6.28}$$

Notice that the $T_{\rm R}$ and $\rho_{\rm M}$ fields vary less than one per mil from one extremum to the other. Therefore we expect that the singlet will be the main inflaton component.

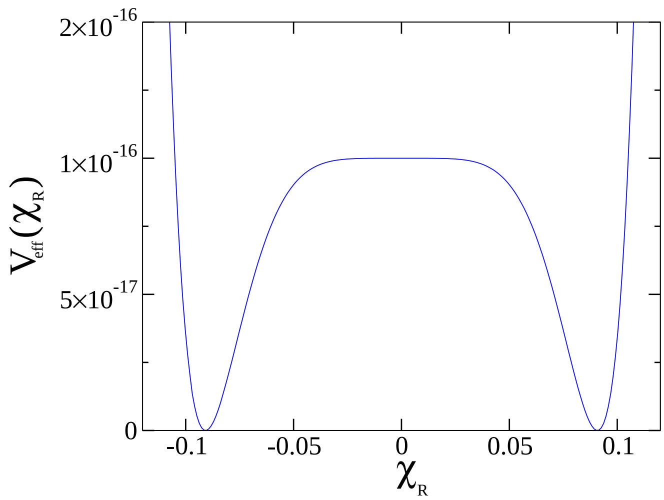

Figure 6.4: Effective potential as a function of $\chi_{\rm R}$.

The potential depends on six variables, $T_{\rm R}$, $T_{\rm I}$, $\rho_{\rm M}$, $\alpha_{\rm M}$, $\chi_{\rm R}$, $\chi_{\rm I}$ and it is, therefore, impossible to draw a picture to illustrate how these extrema are connected. However, some simplifications are possible. $T_{\rm I}$ and $\alpha_{\rm M}$ only enter the problem through the $\varphi_{n.p}$ combination, i.e. the phase of the non-perturbative superpotential given in eq. (6.14).

Actually, for positive W_0, the potential is minimised at $\varphi_{n.p} = \pi$, as in the case without singlet. That means that $-W_0$ and $W_{n.p}$ are aligned, partially cancelling each other. Analogously, the potential is minimised by taking $\chi_{\rm I} \to 0$. Therefore, we can integrate out all phases/imaginary parts, so that the potential depends on ($T_{\rm R}$, $\rho_{\rm M}$, $\chi_{\rm R}$). We just need one more step to illustrate the potential along which the inflaton ($\equiv$ singlet) evolves. If the inflaton scale and effective mass along the slow-roll are much smaller than the $T_{\rm R}$ and $\rho_{\rm M}$ masses we can also integrate out the latter fields. In fact, since the $T_{\rm R}$ and $\rho_{\rm M}$ masses are not far from m_p, this has to be the case if $|\eta| \ll 1$, as required for slow-roll to happen[6].

Consequently, we can define $T_{\rm R}(\chi_{\rm R})$ and $\rho_{\rm M}(\chi_{\rm R})$ through the extremisation conditions

$$\begin{aligned} \partial_{T_{\rm R}} V(T_{\rm R}\,,\, \rho_{\rm M}\,,\, \chi_{\rm R}) &= 0\;, \\ \partial_{\rho_{\rm M}} V(T_{\rm R}\,,\, \rho_{\rm M}\,,\, \chi_{\rm R}) &= 0\;, \end{aligned} \tag{6.29}$$

and finally write

$$V_{\rm eff}(\chi_{\rm R}) \equiv V(T_{\rm R}(\chi_{\rm R})\,,\, \rho_{\rm M}(\chi_{\rm R})\,,\, \chi_{\rm R})\;. \tag{6.30}$$

The resulting potential is shown in figure 6.4. It is of order $\mathcal{O}(10^{-16})$ in m_p units, which means that it involves mass scales of order $\mathcal{O}(10^{-4})$ [7]. The potential is almost flat in a wide region between $\chi = 0$ and the minimum. Given the relatively large VEV of $\chi_{\rm R}$, we consider this potential suitable for implementing topological inflation.

6.1.3.2 The moduli evolution

After analysing the potential we are ready to write and solve the equations describing the evolution of the matter and gravitational fields during the rolling from the saddle to the minimum. We will work in Planck units, $m_p = 1$.

It is convenient to use here real (rather than complex) matter fields, i.e. $\{\Phi_i\}_{i=1,\ldots,N} \to \{\phi_i\}_{i=1,\ldots,2N}$, where N denotes the number of complex scalar fields. Then the supergravity Lagrangian can be written as

$$|g|^{-1/2}\mathcal{L}_{\rm matter} = K_{ij}\, g^{\mu\nu}\partial_\mu\Phi^i\partial_\nu\bar{\Phi}^j - V = \frac{1}{2}\mathcal{G}_{ij}\, g^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j - V\;, \tag{6.31}$$

where V is given by eq. (6.9). The relation between K_{ij} and $\mathcal{G}_{ij}$ can be straightforwardly calculated.

We write the supergravity potential as a sum of the F and D parts $V = V_{\rm F} + V_{\rm D}$. Since the D part is associated with the anomalous U(1), it only involves the fields T and M.

[6] Incidentally, this also means that we can focus on the scalar power spectrum, since the isocurvature fluctuations are negligible, given the hierarchy of scales between the inflation and the other fields in the system.

[7] This is a couple of orders of magnitude below the threshold to produce gravity waves observable in future experiments [209].

Its expression was given in eq. (6.11). Using the anomaly cancellation condition in (6.5) we can rewrite it as

$$V_{\rm D} = \frac{3\pi}{8Nk_N T_{\rm R}} \frac{(q+\bar{q})^3}{q^3+\bar{q}^3} \left(\rho_M^2 + \frac{3}{4\pi k_N T_{\rm R}}\right)^2 . \tag{6.32}$$

The F part is given by eq. (6.10) and depends on the Kähler derivatives for the chiral superfields,

$$\begin{pmatrix} D_T W \\ M D_M W \\ \chi D_\chi W \end{pmatrix} = \begin{pmatrix} \frac{-3}{T+\bar{T}} - \frac{4\pi k_N}{N-1} & \frac{-3}{T+\bar{T}} & \frac{-3}{T+\bar{T}} & \frac{-3}{T+\bar{T}} & \frac{-3}{T+\bar{T}} \\ M\bar{M} - \frac{2}{N-1} & M\bar{M} & M\bar{M} & M\bar{M} & M\bar{M} \\ \chi\bar{\chi} & \chi\bar{\chi} & \chi\bar{\chi}+2 & \chi\bar{\chi}+4 & \chi\bar{\chi}+6 \end{pmatrix} \begin{pmatrix} W_{n.p} \\ W_0 \\ W_2 \\ W_4 \\ W_6 \end{pmatrix} , \tag{6.33}$$

where $W_{2,4,6}$ stand for the monomials proportional to $\chi^{2,4,6}$ respectively in eq. (6.23). Notice that the above matrix elements are real. As a consequence, the only relevant phases are the relative ones among the different terms appearing in the superpotential.

In our case, for the range of interest of $T_{\rm R}$, $\rho_{\rm M}$ and χ_R (i.e. from the saddle point to the minimum neighbourhood) the potential is minimal for $\varphi_{n.p} = \pi$ (we assume real W_0) and $\chi_I = 0$. Then if we evolve the fields from an initial configuration with all the phases at the minimal value, they will remain constant. We can then restrict ourselves to a reduced potential, $V_{\rm F}(T_{\rm R}, \rho_{\rm M}, \chi_{\rm R})$. In terms of these three relevant variables, we get

$$\begin{aligned} V_{\rm F} \;=\; & \frac{1}{8T_{\rm R}^3} e^{\rho_{\rm M}^2+\chi_{\rm R}^2} \\ & \left\{ 3\left(A - \frac{8\pi T_{\rm R} k_N}{3}\left(\frac{2}{\rho_{\rm M}^2}\right)^{\frac{1}{N-1}} e^{\frac{-4\pi k_N}{N-1}T_{\rm R}}\right)^2 + \rho_{\rm M}^2\left(A + \left(\frac{2}{\rho_{\rm M}^2}\right)^{\frac{N}{N-1}} e^{\frac{-4\pi k_N}{N-1}T_{\rm R}}\right)^2 \right. \\ & \left. + \chi_{\rm R}^2\left(A + 2\lambda_2 + 4\lambda_4\chi_{\rm R}^2 + 6\lambda_6\chi_{\rm R}^4\right)^2 - 3A^2 \right\} , \end{aligned} \tag{6.34}$$

where the function A is given by

$$A = W_0 - (N-1)\left(\frac{2}{\rho_{\rm M}^2}\right)^{\frac{1}{N-1}} e^{\frac{-4\pi k_N}{N-1}T_{\rm R}} + \lambda_2\chi_{\rm R}^2 + \lambda_4\chi_{\rm R}^4 + \lambda_6\chi_{\rm R}^6 , \tag{6.35}$$

and the case without the singlet is recovered by taking the limit $\chi \to 0$ in (6.33) and (6.34).

On the other hand, we parameterise the spacetime metric $g_{\mu\nu}$ as a Friedmann-Robertson-Walker metric given by

$$ds^2 = dt^2 - a(t)^2 dx_i dx^i . \tag{6.36}$$

Then, the field equations (for constant fields in space) deduced from the matter and the gravity Lagrangian are

$$\ddot{\phi}^i + 3\,H\,\dot{\phi}^i + \Gamma^i_{jk}\,\dot{\phi}^j\,\dot{\phi}^k + \mathcal{G}^{ij}\frac{\partial V}{\partial\phi^j} = 0 , \tag{6.37}$$

subject to the constraint

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\left[\frac{1}{2}\mathcal{G}_{ij}\,\dot{\phi}^i\dot{\phi}^j + V\right] \quad , \tag{6.38}$$

where dots denote time derivatives and, as usual, Γ^i_{jk} are the Christoffel symbols derived from the $\mathcal{G}_{ij}$ metric

$$\Gamma^i_{jk} = \frac{1}{2}\,\mathcal{G}^{im}\left[\frac{\partial \mathcal{G}_{mk}}{\partial \phi^j} + \frac{\partial \mathcal{G}_{jm}}{\partial \phi^k} - \frac{\partial \mathcal{G}_{jk}}{\partial \phi^m}\right] \quad , \tag{6.39}$$

with $\mathcal{G}^{ij}\mathcal{G}_{jk} = \delta^i_k$. In order to determine the evolution of the matter fields, we perform a change of independent variable from time to number of e-folds, N_e,

$$a(t) = e^{N_e(t)} \quad , \quad H = \frac{dN_e(t)}{dt} \quad . \tag{6.40}$$

Then, from (6.37) and (6.38), we can write the evolution equations for the matter fields as [8]

$$\phi^{i\prime\prime} + \left[1 - \frac{1}{6}\,\mathcal{G}_{jk}\,\phi^{j\prime}\phi^{k\prime}\right]\left[3\,\phi^{i\prime} + 3\,\mathcal{G}^{ij}\,\frac{1}{V}\left(\frac{\partial V}{\partial \phi^j}\right)\right] + \Gamma^i_{jk}\,\phi^{j\prime}\phi^{k\prime} = 0 \quad , \tag{6.41}$$

where prime means derivative respect to N_e. Note that in this way the scale factor is no longer present in the evolution equations.

In our case we have six real component fields, $\{\phi_i\} \equiv \{T_{\rm R}, T_{\rm I}, \rho_{\rm M}, \alpha_{\rm M}, \chi_{\rm R}, \chi_{\rm I}\}$. Although we have performed all the numerical computations with the complete set of ϕ fields, it is possible to decouple $\{T_{\rm I}, \alpha_{\rm M}, \chi_{\rm I}\}$ since, as argued in the previous subsection, they minimise the potential at well-defined values independent of the value of the other fields. So they rapidly fall into their minimising values and play no role in the evolution of the other fields.

Therefore, the matter Lagrangian in (6.31) for the three relevant fields, $\{T_{\rm R}, \rho_{\rm M}, \chi_{\rm R}\}$, has the form

$$|g|^{-1/2}\mathcal{L}_{\rm matter} = \left[\frac{3}{4T_{\rm R}^2}\,\partial_\mu T_{\rm R}\,\partial^\mu T_{\rm R} + \partial_\mu \rho_{\rm M}\,\partial^\mu \rho_{\rm M} + \partial_\mu \chi_{\rm R}\,\partial^\mu \chi_{\rm R} - V\right] \quad , \tag{6.42}$$

and the evolution equations in (6.41) applied to these fields are explicitly given by

$$\begin{aligned}
\chi_{\rm R}'' + \left[1 - \frac{1}{3}\,{\chi_{\rm R}'}^2 - \frac{1}{4T_{\rm R}^2}\,{T_{\rm R}'}^2 - \frac{1}{3}\,{\rho_{\rm M}'}^2\right]\left[3\,\chi_{\rm R}' + \frac{3}{2V}\left(\frac{\partial V}{\partial \chi_{\rm R}}\right)\right] &= 0 \quad , \\
T_{\rm R}'' + \left[1 - \frac{1}{3}\,{\chi_{\rm R}'}^2 - \frac{1}{4T_{\rm R}^2}\,{T_{\rm R}'}^2 - \frac{1}{3}\,{\rho_{\rm M}'}^2\right]\left[3\,T_{\rm R}' + \frac{2T_{\rm R}^2}{V}\left(\frac{\partial V}{\partial T_{\rm R}}\right)\right] &= \frac{{T_{\rm R}'}^2}{T_{\rm R}} \quad , \\
\rho_{\rm M}'' + \left[1 - \frac{1}{3}\,{\chi_{\rm R}'}^2 - \frac{1}{4T_{\rm R}^2}\,{T_{\rm R}'}^2 - \frac{1}{3}\,{\rho_{\rm M}'}^2\right]\left[3\,\rho_{\rm M}' + \frac{3}{2V}\left(\frac{\partial V}{\partial \rho_{\rm M}}\right)\right] &= 0 \quad .
\end{aligned} \tag{6.43}$$

[8] In the slow-roll approximation, eq. (6.41) simplifies to $\phi'^i + \mathcal{G}^{ij}\,\frac{1}{V}\frac{\partial V}{\partial \phi^j} = 0$.

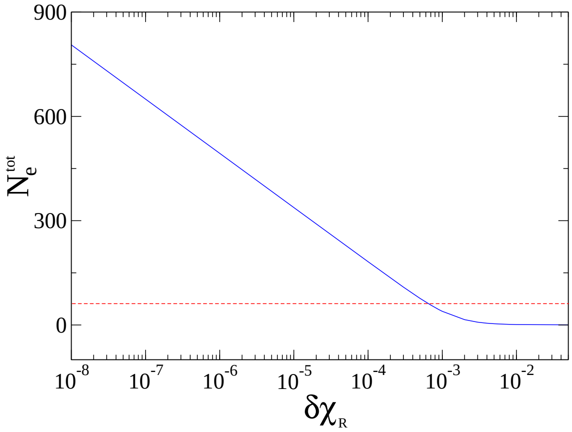

Figure 6.5: Plot of the total number of e-folds of inflation, $N_e^{\rm tot}$, as a function of the initial condition for the inflaton quoted as the shift, $\delta\chi_{\rm R}$, with respect to its value at the saddle point given by $\chi_{\rm R} = 0$. The dotted line indicates the 60 e-folds needed to make inflation successful.

As argued in the previous section, the inflaton corresponds essentially to the $\chi_{\rm R}$ field. The total number of e-folds, $N_e^{\rm tot}$, depends on the initial condition for $\chi_{\rm R}$. If, initially, $\chi = 0$, then $N_e^{\rm tot} \to \infty$. Otherwise $N_e^{\rm tot}$ depends on the initial shift, $\delta\chi_{\rm R}$. The figure 6.5 shows the dependence of $N_e^{\rm tot}$ on $\delta\chi_{\rm R}$ using the values of the parameters of the model given in the previous subsection. Note that $N_e^{\rm tot} \geq 50-60$, as phenomenologically required corresponds to $\delta\chi_{\rm R}|_{\rm initial} \lesssim 10^{-3}$. Recall here that, since we are using the saddle point at $\chi = 0$ as the origin of topological inflation, this means that all the initial conditions are realized in practice (they correspond to different spatial points inside the associated domain wall). The final stages of inflation are the same for all of them, the only difference being the total number of e-folds before the end of inflation. In consequence, any region of the domain wall with $\delta\chi_{\rm R}|_{\rm initial} \lesssim 10^{-3}$ gives appropriate inflation, with no tuning of initial conditions.

In order to show the evolution profiles for the singlet ($\chi_{\rm R}$), modulus ($T_{\rm R}$) and condensate ($\rho_{\rm M}$) we have taken $\delta\chi_{\rm R}|_{\rm initial} = 10^{-4}$, which corresponds to $N_e^{\rm tot} \simeq 180$, but we insist that the results are the same for any other initial condition (provided $N_e^{\rm tot} > 60$) since the last 60 e-folds take place in the same way. The corresponding profiles are shown in figure 6.6. At $N_e \sim 180$ the fields start to oscillate around their minimum values, signalling the end of inflation.

To determine the end of inflation more precisely we need to evaluate the slow-roll parameters. These are the ϵ parameter,

$$\epsilon = \frac{1}{2V^2}\mathcal{G}^{ij}\partial_i V \partial_j V = \frac{1}{2V^2}\mathcal{G}^{ij}\frac{\partial V}{\partial\phi^i}\frac{\partial V}{\partial\phi^j} \ , \tag{6.44}$$

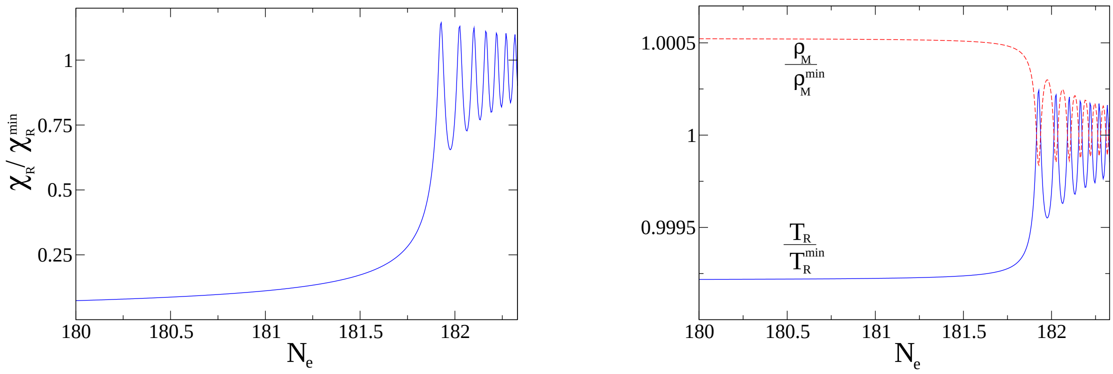

Figure 6.6: Cosmological evolution of the singlet $\chi_{\rm R}$ (left plot) and the $(T_{\rm R}, \rho_{\rm M})$ (right plot) fields, normalised to their values at the minimum, as a function of the number of e-folds, N_e.

and the η parameter defined as the most negative eigenvalue of the matrix [134]

$$\eta^i_j = \frac{1}{V}\mathcal{G}^{ik}\left(\partial_k\partial_j V - \Gamma^l_{kj}\partial_l V\right) \ . \tag{6.45}$$

The ϵ parameter remains $\mathcal{O}(10^{-9})$ along the evolution. The behaviour of η as a function of N_e is shown in figure 6.7, from which we infer that inflation lasts until $N_e \sim 180$.

The spectral index is defined to be

$$n_s = 1 + \frac{d\log(P(k))}{d\log(k)} \ , \tag{6.46}$$

where $P(k)$ is the power spectrum of scalar density perturbations [9] [134]

$$P(k) = \frac{1}{50\pi^2}\frac{H^4}{\mathcal{L}_{kin}} = \frac{1}{150\pi^2}\frac{V}{\left(\frac{1}{2}\mathcal{G}_{ij}\phi'^i\phi'^j\right) - \frac{1}{3}\left(\frac{1}{2}\mathcal{G}_{ij}\phi'^i\phi'^j\right)^2} \ , \tag{6.47}$$

which is shown, as $n_s - 1$, in figure 6.8. Recall that the window of allowed values for this parameter coming from the WMAP data corresponds to it being evaluated ~ 60 e-folds before the end of inflation.

In our case this would mean $N_e \sim 120$, which is the region shown in detail on the right hand side of figure 6.8. The spectral index in (6.46) was calculated using

$$n_s \simeq 1 + \frac{d\log(P_k(N_e))}{dN_e} \ , \tag{6.48}$$

since $d\log k \cong dN_e$ at horizon crossing. Note that $n_s \sim 0.95$, as required by the WMAP fit [27, 28]. Note also that $dn_s/d\log k \ll 1$, which is consistent with refs [27, 28].

[9] In the slow-roll approximation, $\frac{1}{2}\mathcal{G}_{ij}\phi'^i\phi'^j = \epsilon$, so we could compute the power spectrum as $P(k) = \frac{1}{150\pi^2}\frac{V}{\epsilon - \frac{1}{3}\epsilon^2} \simeq \frac{1}{150\pi^2}\frac{V}{\epsilon}$. Similarly, $n_s \simeq 1 + 2\eta - 6\epsilon + \cdots$. However, we have done the calculation without any approximation.

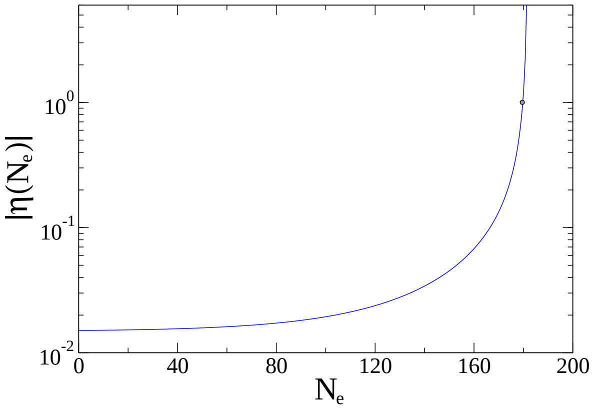

Figure 6.7: Absolute value of η as a function of N_e. The dot indicates $\eta = 1$.

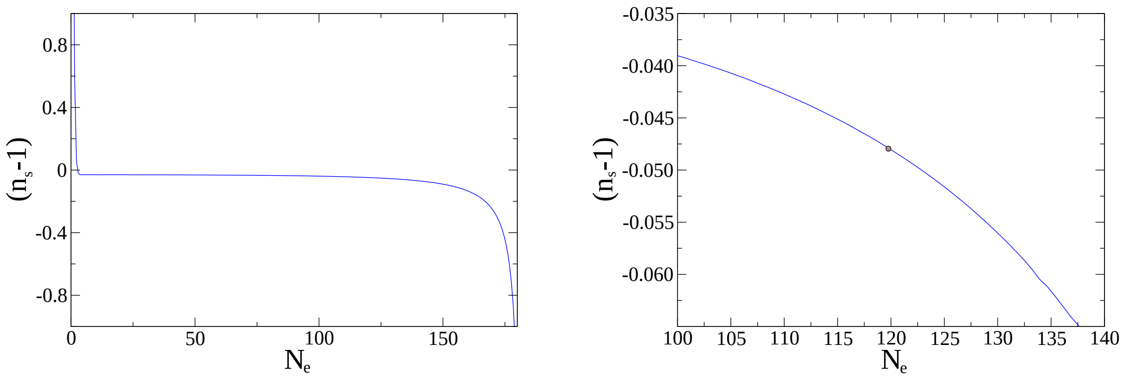

Figure 6.8: Left: n_s as a function of N_e for the whole range of the evolution of the inflaton. Right: zoom of the region where the value of n_s is provided by WMAP.

We have arranged the parameters so that the slope and normalisation of $P(k)$ (namely, $P(k) = 4 \times 10^{-10}$) around 60 e-folds before the end of inflation are satisfied at the same time.

Let us now compare our approach and results with some recent proposals which touch upon similar models. In ref. [210] the issue of decoupling moduli which are apparently irrelevant for inflation in the context of type IIB superstring theory compactified on a Calabi-Yau orientifold was addressed. Some of their methods apply to the large volume stabilisation proposed and developed in refs [94, 102, 211]. Inflation in that context has also been reviewed in refs [132, 135], where the evolution of the imaginary part of the relevant modulus was considered. They conclude that a minimum setup of three moduli is needed for inflation to work, and the stabilisation of two of those is assumed before starting the evolution of the third. Although their claim is that no fine-tuning is needed in these scenarios (as opposed to that of ref. [134]), we believe that there is an implicit tuning of the shape of the potential, encoded in the dynamics of the two moduli which are already stabilised.

As discussed above, the need of fine-tuning is also necessary in the scenario presented here. This point is sustained as well by the results in ref. [212], where an attempt was made to incorporate chaotic/new inflation within supergravity in a string context. The characteristics of those models are similar to ours, namely a modulus T_{R} and an inflaton Φ, the latter with polynomial self interactions, but the mechanism for stabilising T_{R} is different. The conclusions are, however, similar to ours, namely a substantial degree of fine-tuning is needed to arrange a successful inflation. This is unpleasant, although, as discussed in the paper, it is a generic fact of virtually all working examples of modular inflation. This may be an indication of a general disease in inflationary scenarios based on supergravities coming from strings, or just a consequence of the fact that inflationary model building in this context is still in its infancy. Given the obvious interest of such context, the fact that models can actually be built is a sign of progress and should encourage further work in the field. Finally, one can take a more optimistic (though less demanding) view within a landscape philosophy. Then, the apparent fine-tuning of our "local" universe would be a consequence of the need of inflation for the creation of large and lasting universes with matter, where the latter can evolve to life.

6.2 A few words on generalised fluxes and modular inflation

An exhaustive exploration of whether moduli stabilisation in a de Sitter vacuum and modular inflation stand any chance of taking place simultaneously in generalised flux models has not been carried out up to date.

Ever since modular inflation was ruled out (see ref. [13]) in the simplest type IIA supergravity models including gauge fluxes, O6-planes and D6-branes at any point in field space (the slow-roll ϵ parameter in (6.15) has a lower bound $\epsilon \geq \frac{27}{13}$ whenever $V > 0$), much effort has been devoted to explore more elaborated type II scenarios with non-vanishing geometric fluxes ω_{ab}^{c}. To this respect, a detailed analysis of type IIA toroidal orientifolds with metric fluxes was carried out in ref. [137], where only two IIA orientifolds based on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold were found to survive the no-go theorems on the existence of dS extrema ($V > 0$ and $\epsilon = 0$) derived therein. However, these models suffered from the η problem when it comes to implement slow-roll modular inflation.

The class of supergravity models based on coset spaces with non-vanishing geometric fluxes has also been deeply investigated in ref. [14] for type IIA compactifications on SU(3)-structure manifolds as well as in ref. [15] for type IIB orientifolds with SU(2)-structure. Concerning the former, only those models based on reductions upon the $\mathrm{SU}(2) \times \mathrm{SU}(2)$

manifold were not ruled out to accommodate for dS extrema. Unfortunately all the dS extrema found in ref. [14] turned out to be tachyonic with an η parameter being $\eta < -2$, then excluding slow-roll modular inflation starting at these points.

The above results are consistent with those in section 4.6.1 where we found that any Minkowski extremum (and its deformation to de Sitter) in a geometric type IIA isotropic flux model results unstable (tachyonic). Notice that the IIB orientifold models based on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ isotropic orbifold that we systematically analysed in chapters 3 and 4, are T-dual to type IIA compactifications on an SU(3)-structure space with non-geometric fluxes.

Recalling that de Sitter, almost Minkowski, stable vacua were found to exist (see section 4.5.4.2) in the IIB supergravity models based on an $\mathfrak{so}(3,1)$ B-field reduction, it would be interesting to know whether another dS extrema could live in the vicinity of one of these stable vacua. Indeed, a dS saddle (tachyonic) point also appears close to these Mkw/dS stable vacua in field space with a much larger energy. This saddle point is continuously connected to the Mkw/dS vacuum, hence providing us with a natural scenario in which to investigate the possibilities for slow-roll modular inflation to take place. We find that the evolution from the dS saddle point to the Mkw vacuum also suffers from the standard η problem with $|\eta| \sim \mathcal{O}(10)$. This can be overcome by lifting the Mkw vacuum to a dS one (see figure 4.9). As we approach the critical value $\delta\theta_{\xi}^{*}$, beyond which the dS vacuum disappears, the η parameter at the saddle point tends to zero. However, and simultaneously to this process, the saddle point and the dS vacuum merge together[10], and inflation is not likely to take place. This agrees with the previous results in refs [14, 137] derived in the absence of non-geometric fluxes. Nevertheless, nothing prevents inflation from taking place far away from the region in the moduli space we have looked into.

In this sense, most of the work dealing with inflationary scenarios based on generalised flux compactifications including non-geometric fluxes remains to be done.

[10]This behaviour was previously found in the context of the modulus-condensate setup as it was depicted in figure 6.3.

Overview and Final Remarks

Fluxes have played an important role in String Theory research since the second String Theory revolution in the mid 1990's. Orbifolds and their later extension, orientifolds, provide explicit constructions of spaces which are intimately linked to Calabi Yau manifolds but allow for specific calculation of the dynamics of the space and the fields which live within them. By constructing $\mathcal{N}=1$ orientifolds from type II superstring compactifications, important properties and dynamics of the space can be investigated in a background which is easier to describe than that of the Calabi Yaus.

Along the pages of this thesis we have gone for a stroll through the cosmological aspects of moduli stabilisation in type II orientifold theories including generalised fluxes and branes. As regards the String Phenomenology challenge of stabilising the moduli fields in a de Sitter (almost Minkowski) vacuum, as required by the current cosmological data, one of the main recent developments has been the study of generalised flux backgrounds. In addition to the ordinary gauge form fluxes, generalised fluxes were proposed in type II theories to restore the invariance of the effective supergravity models under duality transformations. For instance, certain tensor fluxes referred to as non-geometric fluxes were introduced to restore T-duality between IIA and IIB theories as well as type IIB S-duality at the effective level.

In this respect, the generalised flux models arising from $\mathcal{N}=1$ orientifolds of type II compactifications, have become of principal interest. Specifically, those based on the $\mathbb{T}^6/(\mathbb{Z}_2\times\mathbb{Z}_2)$ orbifold, were found to be the most promising effective models where to look for de Sitter vacua as well as for inflation. These are precisely the $\mathcal{N}=1$ supergravity models in four dimensions we have been looking into. In particular, we have worked on the following subjects:

1. Classification of generalised flux backgrounds and their relation to gauged supergravities and the moduli stabilisation problem.
2. Prospects for inflation in scenarios including fluxes and non-perturbative effects.

As a starting point we focused on the T-duality invariant IIB orientifold models with O3/O7-planes, which incorporate just a non-geometric Q tensor flux besides the NS-NS $\bar{H}_3$ and the R-R $\bar{F}_3$ gauge form fluxes. In these models, the Q-flux defines a six-dimensional Q-algebra spanned by the vectors coming from the reduction of the B-field. Using this property and the orbifold symmetries under the assumption of isotropic fluxes, we identified the set of allowed Q-algebras and classified the non-geometric Q-flux backgrounds they induced. We encountered five Q-algebras compatible with the symmetries, each one leading to a characteristic flux-induced superpotential. Next we found the most general fluxes solving the Jacobi identities coming from the 12-dimensional $(\bar{H}_3, Q)$-flux algebra and left the flux-induced tadpoles for the C_4 and C_8 R-R gauge potentials as free variables, thereby enabling us to study which values were allowed for each Q-algebra.

We succeeded in finding families of three moduli (S, T, U) supersymmetric AdS$_4$ vacua with all moduli stabilised at small string coupling g_s. Their general properties were also discussed: these vacua typically exist in all models defined by the inequivalent Q-algebras, provided that arbitrary values of the flux-induced R-R tadpoles are allowed. Although in type IIB orientifolds with only R-R and NS-NS fluxes there is a non-trivial induced tadpole that must be cancelled by O3-planes or wrapped D7-branes, including non-geometric Q-fluxes can require other types of sources. For instance, similar to well understood AdS$_4$ models in type IIA, the induced flux-tadpoles might vanish implying that sources can be avoided. We also encountered examples in which sources of positive R-R charge are sufficient to cancel the tadpoles. As one might expect, these latter exotic vacua occur in models built using Q-fluxes satisfying the non-compact $\mathfrak{so}(3,1)$ Q-algebra. Such solutions might be ruled out once a deeper understanding of non-geometric fluxes has been developed. Another novel outcome was the appearance of multiple vacua for special sets of fluxes. However, they generically have $g_s > 1$ unless the net number of O3/D3 or O7/D7 sources needed to cancel the tadpoles is large. We also discussed briefly the issues of axionic shift symmetries and cancellation of Freed-Witten anomalies.

The next step was to investigate the interplay between the supergravity flux algebra and the moduli dynamics in the previous set of T-duality invariant IIB orientifold models including just the non-geometric Q-flux. In particular, the existence of non-supersymmetric dS/Mkw extrema. In this IIB language, we performed a complete classification of the allowed twelve-dimensional flux algebras and then studied no-go theorems, formulated in a type IIA language, on the existence of dS/Mkw extrema. By deriving a dictionary between the sources of potential energy in types IIA and IIB, we were able to combine algebra results and no-go theorems. The outcome was a systematic procedure for identifying phenomenologically viable models where dS/Mkw extrema might exist. We presented a complete classification of the allowed algebras and the viability of their resulting scalar potential, and we pointed at the models which stood any chance of producing a fully stable vacuum. Moreover, the most promising scenarios in terms of finding such vacua turned out to be those involving non-geometric fluxes. Once we reached this stage, it was a matter of performing a dedicated search for minima of the potentials that survive the no-go theorem.

We would like to stress here that the search for phenomenologically viable vacua in the context of String Theory will not succeed if it is understood as the brute force task of searching for minima in a high order degree polynomial potential. We have shown that combining apparently disconnected pieces of research, such as the classification of the allowed flux algebras in type IIB with the existence of no-go theorems for the presence of Mkw/dS vacua in type IIA, gives us the key to perform a systematic search for the most promising potentials. This ruled out almost half of the possible scenarios and provided us with simplified expressions for the remaining, potentially viable ones.

With the set of interesting models narrowed down, we carried out a complete and systematic analysis of the Mkw extrema of the supergravity potential for the promising cases previously hinted. We found that only one choice of B-field reduction, that based on the $\mathfrak{so}(3,1)$ Q-algebra, gives rise to minima with all moduli stabilised at a Minkowski vacuum. These solutions can also be deformed continuously to either de Sitter or anti de Sitter by a slight variation of the relevant parameters. Supersymmetry is broken by all moduli, at a scale which is, as expected, large for values of the fluxes of order one. Our systematic search showed that all the B-field reductions (but the $\mathfrak{nil}$ based one) produce Minkowski extrema with all but one direction stabilised. These tachyonic solutions show a specific pattern, as they always interpolate between singular points of the parameter space where one or several moduli go to either zero or infinity. We have also shown the breakdown of the potential energy contributions in the language of type IIA, in order to compare our results to those examples put forward in the context of the no-go theorems. In this way it is obvious that the solutions with stable Minkowski vacua require non-geometric flux contributions to the scalar potential.

After having studied the T-duality invariant orientifold models, we went one step further and investigated how to extend them to include a new ingredient, a non-geometric P-flux which appears when considering invariance of the IIB effective models also under S-duality transformations. We built upon our previous results and succeeded in implementing this P-flux as deformations of the Q-algebra by an element of its second cohomology class. After that, the new Jacobi identities involving the non-geometric P-flux were reinterpreted as integrability and cohomology conditions over the deformation. The problem of solving the integrability condition forced the non-geometric P-flux to define another six-dimensional P-algebra compatible with the orbifold symmetries, in analogy with the Q-algebra. Even though both algebras could be chosen independently, their embeddings into isotropic Q and P fluxes were restricted by the cohomology condition.

At this stage, algebraic geometry techniques were required. We made extensive use of the free software *Singular* to compute all solutions to the cohomology condition, breaking it into several families or branches with different implications. Different branches of solutions to the integrability and cohomology conditions were interpreted geometrically, as root alignments between the non-geometric flux-induced polynomials entering the effective superpotential. Additional Jacobi identities of the twelve-dimensional flux algebra involving also gauge fluxes $(\bar{F}_3, \bar{H}_3)$ remained a linear system. This fact allowed us to split non-geometric (Q, P)-background fluxes into what we referred to as type A and B configurations. The type B configurations were found to be those for which a non vanishing flux-induced C_8' tadpole might be generated. Using these methods, supersymmetric solutions turned out to be easily and systematically computable. We presented a simple AdS_4 solution with all moduli stabilised and for which the fluxes do not induce tadpoles, as well as several supersymmetric Minkowski solutions. Because of the importance of the latter from the phenomenological point of view, we further presented some families of supersymmetric Minkowski solutions and identified examples already found in the literature within our construction. Finally we further discussed the lifting of these T- and S-duality invariant orientifold models to $\mathcal{N} = 4$ gauged supergravities as well as pointed to the electric-magnetic flux algebra they would give rise to.

The point to be highlighted is that we have succeeded in connecting properties of the vacua to the underlying flux algebra. This could help towards extending the description of non-geometric fluxes beyond the effective action limit. At present one of the most challenging problems in need of new insights is precisely to formulate String Theory on general backgrounds at the microscopic level.

In the home stretch of the thesis we changed to explore cosmological aspects of moduli fixing in type IIB models. In particular we analysed simple IIB proposals considering moduli stabilisation as a seed for inflation. We started by studying in detail the simplest setup including gaugino condensation with gauge group $\mathrm{SU}(N)$ due to stacks of D7-branes and a Fayet-Iliopoulos contribution to the D part of the scalar potential coming from the anomalous U(1) factor that typically appears. Squark meson condensates M are formed and a non-perturbative term, involving the Kähler *modulus* T, appears in the superpotential. We studied the dynamics of this coupled system and we found that neither T-driven nor M-driven modular inflation could be accommodated within this setup. Then we moved to its minimal extension, that of including an additional self-interacting neutral field χ. The resultant model incorporated eternal χ-driven topological inflation with a non-supersymmetric Minkowski vacuum and could be arranged to comply with all the WMAP constraints. Next we looked into generalised fluxes as far as modular inflationary models was concerned. However, we found a more pessimistic picture due to the persistence of the η-problem when it comes to implement slow-roll inflation within generalised flux scenarios of moduli stabilisation.

We are optimistic that present efforts in theoretical and experimental physics will shed light upon the challenge of linking strings and low energy physics in the near future.

Visión General y Comentarios Finales

Desde la segunda revolución que tuvo lugar en Teoría de Cuerdas a mediados de los 90's, el estudio de las *compactificaciones* con flujos de fondo ha jugado un papel importante en la investigación en Teoría de Cuerdas. Los *orbifolios* y, más tarde, su extensión a *orientifolios*, suponen un manera de construir espacios que están íntimamente ligados con espacios de Calabi-Yau pero que a su vez permiten calcular de forma explícita su dinámica y la de los campos que habitan en ellos.

A lo largo de esta tesis hemos explorado algunos aspectos cosmológicos relacionados con la estabilización de *moduli* en *orientifolios* de tipo II teniendo en cuenta la presencia de flujos generalizados y de *branas*. El estudio de los flujos generalizados ha significado un gran avance en el área de la Fenomenología de Cuerdas debido a su importancia a la hora de estabilizar los *moduli* en un vacío de Sitter (aproximadamente Minkowski) como requieren los datos cosmológicos actuales. Estos flujos generalizados se introdujeron en las teorías de tipo II (junto con los flujos *gauge* ordinarios) con la intención de restablecer la invariancia de los modelos de supergravedad efectivos bajo transformaciones de dualidad. Por ejemplo, determinados flujos tensoriales conocidos como flujos no geométricos se introdujeron para restablecer T-dualidad entre las teorías IIA y IIB al igual que S-dualidad en la teoría IIB a nivel efectivo.

Los modelos de supergravedad $\mathcal{N} = 1$ basados en *orientifolios* de tipo II han resultado ser un buen lugar donde explorar estas cuestiones. Concretamente, aquéllos construidos sobre el *orbifolio* $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ son especialmente interesantes a la hora de buscar vacíos de Sitter e *inflación*. Estos modelos de supergravedad son los que hemos investigado en esta tesis. En particular, hemos indagado en los dos siguientes aspectos:

1. Clasificación de fondos de flujos generalizados así como su relación con supergravedades *gaugeadas* y con el problema de la estabilización de *moduli*.

2. Posibilidad de producir *inflación* en escenarios de cuerdas basados en flujos generalizados y efectos no perturbativos.

Como punto de partida nos centramos en los modelos *orientifolios* de tipo IIB con O3/O7-*planos* invariantes bajo transformaciones de T-dualidad los cuales incorporan un tensor de flujo no geométrico Q además de los flujos *gauge* de NS-NS $\bar{H}_3$ y de R-R $\bar{F}_3$. En estos modelos, el flujo Q determina una Q-álgebra seis-dimensional generada por los vectores que provienen de la reducción del campo B. Utilizando esta propiedad y las simetrías del *orbifolio* bajo la restricción de flujos isótropos, identificamos el conjunto de Q-álgebras permitidas y clasificamos los flujos no geométricos que éstas inducían. Encontramos cinco Q-álgebras compatibles con las simetrías cada una de las cuales da lugar a un superpotencial característico inducido por los flujos. A continuación encontramos las configuraciones de flujos $(\bar{H}_3, Q)$ más generales que cumplen las identidades de Jacobi asociadas al álgebra completa y dejamos los *tadpoles* inducidos para los campos de R-R C_4 y C_8 como parámetros libres.

En estos modelos isótropos de supergravedad encontramos familias de vacíos supersimétricos AdS_4 con los tres *moduli* (S, T, U) estabilizados a valores pequeños de la constante de acoplo de la cuerda g_s. También discutimos sus propiedades generales: estos vacíos existen típicamente en todos los modelos definidos por las Q-álgebras inequivalentes en tanto en cuanto los tadpoles inducidos por los flujos de R-R puedan tomar valores arbitrarios. Aunque en los modelos *orientifolios* de tipo IIB con flujos *gauge* aparece un *tadpole* que ha de ser cancelado mediante O3-*planos* y D7-*branas*, la inclusión de flujos no geométricos Q puede requerir la presencia de otros tipos de fuentes. Por ejemplo, los *tadpoles* inducidos por los flujos pueden anularse, lo cual es compatible con no tener fuentes, al igual que ocurre en algunos vacíos AdS_4 en modelos basados en la teoría de tipo IIA. También encontramos ejemplos en los que tener fuentes con carga de R-R positiva es suficiente para cancelar los *tadpoles* inducidos por los flujos. Como uno podría esperar, estos vacíos exóticos ocurren en modelos basados en la Q-álgebra no compacta $\mathfrak{so}(3,1)$. No obstante, estas soluciones podrían ser descartadas tras entender de manera más fundamental el origen de los flujos no geométricos. Otro resultado nuevo fue la coexistencia de múltiples vacíos para determinadas configuraciones de flujos. Sin embargo, estos vacíos dan lugar a valores no perturbativos de la constante de acoplo de la cuerda, $g_s > 1$, a menos que el número neto de fuentes O3/D3 y O7/D7 necesarias para cancelar los *tadpoles* inducidos por los flujos sea grande. También discutimos brevemente algunas cuestiones sobre la simetría asociada a translaciones de los *axiones* y la cancelación de las anomalías de Freed-Witten.

El siguiente paso consistió en investigar la interrelación entre el álgebra de los flujos y la dinámica de los *moduli* en los modelos *orientifolios* de tipo IIB invariantes bajo transformaciones de T-dualidad, los cuales incluyen el flujo no geométrico Q. En particular, la existencia de extremos dS/Mkw. En este lenguaje de la teoría IIB, llevamos a cabo una clasificación completa de las álgebras doce-dimensionales permitidas y estudiamos teoremas de imposibilidad, formulados en el lenguaje de la teoría IIA, relacionados con la existencia de extremos dS/Mkw. Fuimos capaces de combinar resultados sobre álgebras y teoremas de imposibilidad gracias a una correspondencia entre las fuentes de energía potencial en las teorías IIB y IIA, lo que significó un procedimiento sistemático para identificar modelos viables fenomenológicamente en los que pudieran existir extremos dS/Mkw. Presentamos una clasificación completa de las álgebras permitidas así como de la viabilidad de sus superpotenciales característicos a la hora de producir vacíos totalmente estables. Además, los modelos más prometedores para este propósito resultaron ser aquéllos que también involucran flujos no geométricos cuando se formulan en el lenguaje de la teoría IIA. Llegados a este punto, el objetivo era llevar a cabo una búsqueda exhaustiva de mínimos del potencial escalar para los modelos que sobrevivían a los teoremas de imposibilidad.

Es importante resaltar que la búsqueda de vacíos viables fenomenológicamente en el contexto de Teoría de Cuerdas no resulta eficiente si se plantea como una exploración de mínimos de un potencial polinómico de order alto mediante la fuerza bruta. Hemos visto que combinar piezas aparentemente desconectadas, tales como la clasificación de álgebras de flujos en la teoría IIB y los teoremas de imposibilidad relacionados con la existencia de vacíos Mkw/dS en la teoría IIA, nos da la clave para llevar a cabo una identificación sistemática de los modelos interesantes. Esto descartó casi la mitad de los modelos posibles y simplificó los restantes.

Tras reducir notablemente el conjunto de modelos interesantes, llevamos a cabo un análisis sistemático y completo de los extremos Mkw de sus potenciales escalares. Encontramos que únicamente los modelos de supergravedad construidos sobre una reducción del campo B basada en la Q-álgebra $\mathfrak{so}(3,1)$ dan lugar a mínimos Minkowski con todos los *moduli* estabilizados. Estas soluciones se pueden deformar continuamente para producir vacíos de Sitter o anti de Sitter mediante una variación pequeña de los parámetros relevantes. Supersimetría se rompe espontáneamente a una escala de energías muy alta (para flujos de orden uno) debido al valor esperado en el vacío de todos los *moduli*. En nuestra búsqueda sistemática de vacíos encontramos que todas las reducciones del campo B, excepto la basada en la Q-álgebra $\mathfrak{nil}$, dan lugar a modelos de supergravedad que albergan extremos Minkowski con todas las direcciones, a excepción de una, estabilizadas. Estas soluciones *taquiónicas* siguen un patrón específico ya que siempre interpolan entre puntos singulares del espacio de parámetros en los que uno o varios de los *moduli* van a cero o a infinito. También presentamos una descomposición del potencial escalar en sus diferentes contribuciones interpretadas en el lenguaje de la teoría IIA con la intención de comparar nuestros resultados con los ejemplos propuestos en el contexto de los teoremas de imposibilidad. De esta manera pudimos ver cómo los modelos de supergravedad que dan lugar a vacíos Minkowski tienen contribuciones al potencial escalar que vienen necesariamente de flujos no geométricos.

Después de estudiar los modelos *orientifolios* invariantes bajo transformaciones de T-dualidad, dimos un paso adelante e investigamos cómo extenderlos para incluir un ingrediente nuevo: un flujo no-geométrico P el cual aparece al considerar también la invariancia de los modelos efectivos de la teoría IIB bajo transformaciones de S-dualidad. Trabajando sobre nuestros resultados anteriores logramos implementar el flujo P como deformaciones de la Q-álgebra mediante un elemento de su segunda clase de cohomología. De esta manera, las nuevas identidades de Jacobi que involucran el flujo no geométrico P se reinterpretan como condiciones de integrabilidad y de cohomología sobre la deformación. Para satisfacer las condiciones de integrabilidad, el flujo P ha de definir una P-álgebra seis-dimensional compatible con las simetrías del *orbifolio* de forma análoga a como ocurría con la Q-álgebra. A pesar de que ambas álgebras se podían elegir de manera independiente, la forma en la cual se embebían dentro de los flujos Q y P resultaban restringidas por las condiciones de cohomología.

Para obtener las distintas ramas de la solución a las condiciones de cohomología tuvimos que hacer uso extenso de técnicas de Geometría Algebraica. Utilizando el programa *Singular* obtuvimos las diferentes ramas de la solución y vimos que escoger una rama u otra se traducía en diferentes alineamientos entre las raíces de los polinomios inducidos por los flujos no geométricos en el superpotencial efectivo. Las identidades de Jacobi restantes asociadas al álgebra completa, las cuales también involucraban los flujos *gauge* $(\bar{F}_3, \bar{H}_3)$, daban lugar a un sistema de ecuaciones lineales sencillo. Esto nos permitió dividir las configuraciones de los flujos no geométricos (Q, P) en dos tipo a los que denominamos tipos A y B. Las configuraciones de tipo B resultaron ser aquellas en las que era posible generar un *tadpole* para la forma de R-R C_8'. Haciendo uso de estos métodos pudimos estudiar de forma sistemática la estabilización de los *moduli* de la *compactificación* en vacíos supersimétricos. Presentamos una solución AdS_4 supersimétrica sencilla con todos los *moduli* estabilizados en la cual los flujos de fondo no generaban *tadpoles*. También presentamos varias familias de vacíos supersimétricos Minkowski debido a su importancia desde un punto de vista fenomenológico e identificamos en nuestra construcción algunos de los ejemplos existentes en la literatura. Finalmente discutimos estos modelos de supergravedad invariantes bajo transformaciones de T-dualidad y S-dualidad en términos de supergravedades *gaugeadas* $\mathcal{N} = 4$ y apuntamos hacia el álgebra de flujos (eléctricos y magnéticos) a la que darían lugar.

Un aspecto destacable es que logramos conectar propiedades de los vacíos de *moduli* con el álgebra de los flujos de fondo que les subyace. Esto podría ayudar a extender la descripción de los flujos no geométricos más allá de la acción efectiva. Por el momento, describir los fondos de flujos generalizados a nivel microscópico representa un problema abierto en Teoría de Cuerdas.

En la recta final de la tesis, hemos explorado aspectos cosmológicos de la estabilización de *moduli* en modelos de supergravedad de tipo IIB. En particular, hemos analizado propuestas sencillas que consideran la estabilización de *moduli* como semilla para generar *inflación*. Comenzamos estudiando en detalle el escenario más sencillo que incorpora un condensado de *gauginos* con grupo *gauge* $SU(N)$ asociado a un conjunto de D7-*branas* coincidentes y una contribución de Fayet-Iliopoulos a la parte D del potencial escalar debida a un factor $U(1)$ anómalo que genéricamente aparece en estos modelos. En este escenario se forman mesones M compuestos de condensados de s-quarks y aparece un término no perturbativo que involucra el *modulus* de Kähler T en el superpotencial. Hemos estudiado la dinámica de este sistema acoplado y hemos concluido que *inflación* modular guiada por los campos M o T no tiene lugar en estos modelos. Esto nos llevó a estudiar la mínima extensión del escenario: incluir un campo neutro χ con autointeracción. El modelo resultante incorporaba *inflación* topológica (eterna) guiada por el campo χ, estabilización en un vacío Minkowski no supersimétrico y se ajustaba a todas las medidas experimentales realizadas por WMAP. A continuación investigamos la posibilidad de generar inflación en modelos con flujos generalizados. Esta opción se antoja complicada ya que los modelos parecían sufrir un problema η crónico cuando se trataba de implementar inflación, en la aproximación de rodar lento, basada en el proceso de estabilización de los *moduli*.

No obstante, somos optimistas en el reto que supone conectar las cuerdas con la física de bajas energías en un futuro próximo gracias al esfuerzo conjunto en Física Teórica y Experimental.

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A.

Appendices

Appendix A

Massless Spectrum of Type II Superstrings

In this appendix we compute the supergravity massless spectrum of type II superstrings freely moving[1] in a ten-dimensional Minkowski spacetime $\mathbb{M}_{1,9}$. As a propagating one-dimensional object, the embedding of a string into a ten-dimensional Minkowski spacetime $\mathbb{M}_{1,9}$ is encoded in the bosonic fields $X^M(\sigma^\alpha)$ with $M = 0,\ldots,9$ and $\alpha = 0,1$. The coordinate $\sigma^0 \equiv \tau$ is the worldsheet time coordinate whereas $\sigma^1 \equiv \sigma$ denotes the coordinate along the string $0 \leq \sigma < \pi$. Furthermore, because of type II superstrings are closed strings, these bosonic fields will satisfy the closed-string boundary conditions

$$X^M(\tau,\sigma+\pi) = X^M(\tau,\sigma) \ , \tag{A.2}$$

Introducing the so-called *light-cone* spacetime coordinates, i.e., transverse coordinates $X^i = X^i$ with $i = 1,\ldots,8$ and null coordinates $\sqrt{2}\,X^\pm = \left(X^0 \pm X^9\right)$, and fixing the *light-cone gauge*[2], makes possible to eliminate the string excitations along the null coordinates $X^\pm$. In this gauge the theory will describe excitations along the transverse coordinates X^i. These coordinates, labelled by the index i, transform as a vector $\mathbf{8_v}$ under the SO(8) transverse group of rotations.

In the Green-Schwarz (GS) formalism in the *light-cone* gauge [1], the action of a type II superstring propagating in a flat background geometry is given by

$$S_{\text{l.c}} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\ \partial_\alpha X^i\,\partial^\alpha X^i + \frac{i}{\pi}\int d^2\sigma\ \left(S^a\,\partial_+ S^a + \tilde{S}^{a\,(\dot{a})}\,\partial_-\tilde{S}^{a\,(\dot{a})}\right) \ , \tag{A.3}$$

[1] For a freely moving open/closed string, a flat 2d worldsheet metric

$$h_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \ , \tag{A.1}$$

can be chosen for the worldsheet topology.

[2] This gauge turns out with the advantage that all states in the spectrum will be physical states.

where $\partial_\pm \equiv \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$ and the parameter α' relates to the string length l_s as

$$l_s^2 = 2\,\alpha' \ . \tag{A.4}$$

This GS formalism is supersymmetric in ten-dimensional Minkowski spacetime: the above worldsheet action of eq.(A.3) involves not only the X^i fields describing bosonic string excitations along the transverse spacial coordinates, but also fermionic ones S^a (right-moving) and $\tilde{S}^{a\,(\dot{a})}$ (left-moving) with either the same (IIB superstring) or opposite (IIA superstring) chirality in eight dimensions. The index a refers to transforming in the spinorial representation $\mathbf{8_s}$ of SO(8), while that of $\dot{a}$ refers to transforming in its conjugate representation $\mathbf{8_c}$ of SO(8).

After quantising this free field theory imposing periodic boundary conditions in both bosonic and fermionic worldsheet fields, they have the mode expansion of

$$\begin{aligned} X_R^i &= \frac{x^i}{2} + \frac{p^i}{2}\,\sigma^- + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\,\alpha_n^i\,e^{-i2n\sigma^-} \quad , \qquad S^a = \sum_{-\infty}^{\infty} S_n^a\,e^{-i2n\sigma^-} \quad , \\ X_L^i &= \frac{x^i}{2} + \frac{p^i}{2}\,\sigma^+ + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\,\tilde{\alpha}_n^i\,e^{-i2n\sigma^+} \quad , \quad \tilde{S}^{a\,(\dot{a})} = \sum_{-\infty}^{\infty} \tilde{S}_n^{a\,(\dot{a})}\,e^{-i2n\sigma^+} \quad , \end{aligned} \tag{A.5}$$

where $\sigma^\pm = \tau \pm \sigma$ and we have set $\alpha' = \frac{1}{2}$. We have also the splitting $X^i = X_R^i + X_L^i$ for the bosonic fields into right-moving (R) and left-moving (L) parts respectively. The mass-shell condition is given by

$$\alpha'\,M^2 = 2\,(N + \tilde{N}) \ , \tag{A.6}$$

with $N \equiv \sum_{n=1}^{\infty} \alpha_{-n}^i\alpha_n^i + n\,S_{-n}^a S_n^a$ and $\tilde{N} \equiv \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i + n\,\tilde{S}_{-n}^{a\,(\dot{a})}\tilde{S}_n^{a\,(\dot{a})}$ accounting for the right- and left-moving oscillator excitations, respectively.

The massless states in the spectrum of type II superstrings will be then given by the tensor product of the right-moving S_0^a and left-moving fermionic zero modes $\tilde{S}_0^{a\,(\dot{a})}$ since they commute with the mass operator of eq.(A.6). Since both sectors can be treated analogously, let us describe the right-moving one generated by S_0^a. The common fermionic relations $\left\{S_0^a, S_0^b\right\} = \delta^{ab}$ can be expressed in terms of fermionic oscillators $\sqrt{2}\,b_m = \left(S_0^{2m-1} + i\,S_0^{2m}\right)$ now satisfying

$$\left\{b_m, b_n^\dagger\right\} = \delta_{mn} \quad , \quad \{b_m, b_n\} = \left\{b_m^\dagger, b_n^\dagger\right\} = 0 \quad , \quad \text{where} \quad m = 1, \ldots, 4 \ . \tag{A.7}$$

This becomes equivalent to fix an SO(8) $\supset$ SU(4) $\times$ U(1) embedding of the transverse rotational symmetry. Under this symmetry, the $\{b_m\}$ transform in the fundamental representation $\mathbf{4}$ of SO(4) with $\frac{1}{2}$ units of U(1) charge, i.e. $\mathbf{4}\,(\frac{1}{2})$, whereas $\left\{b_m^\dagger\right\}$ transform in the conjugate representation.

Different states are then built by successively applying creation operators $\left\{b_m^\dagger\right\}$ upon the vacuum $|0\rangle$ state which is annihilated by any of the $\{b_m\}$ operators. They form a 16-dimensional representation of SU(4) × U(1) in which we can distinguish between bosonic states with an even number of creation operators

STATE	SU(4) × U(1)
$\|0\rangle$	$\mathbf{1}\,(1)$
$b_m^\dagger\, b_n^\dagger\,\|0\rangle$	$\mathbf{6}\,(0)$
$b_m^\dagger\, b_n^\dagger\, b_p^\dagger\, b_q^\dagger\,\|0\rangle$	$\mathbf{1}\,(-1)$

$\Rightarrow$ Neveu-Schwarz sector , (A.8)

and fermionic states having an odd number of them

STATE	SU(4) × U(1)
$b_m^\dagger\,\|0\rangle$	$\bar{\mathbf{4}}\,(\frac{1}{2})$
$b_m^\dagger\, b_n^\dagger\, b_p^\dagger\,\|0\rangle$	$\mathbf{4}\,(-\frac{1}{2})$

$\Rightarrow$ Ramond sector . (A.9)

In the Neveu-Schwarz-Ramond formalism, the former states come from the Neveu-Schwarz (NS) sector while the latter come from the Ramond (R) sector. Furthermore, because of the decomposition under SU(4) × U(1) of the $\mathbf{8_v}$, $\mathbf{8_s}$ and $\mathbf{8_c}$ representations

$$\begin{aligned} \mathbf{8_v} &= \mathbf{1}\,(1) + \mathbf{6}\,(0) + \mathbf{1}\,(-1) \\ \mathbf{8_s} &= \mathbf{4}\,(\tfrac{1}{2}) + \bar{\mathbf{4}}\,(-\tfrac{1}{2}) \\ \mathbf{8_c} &= \mathbf{4}\,(-\tfrac{1}{2}) + \bar{\mathbf{4}}\,(\tfrac{1}{2})\ , \end{aligned} \tag{A.10}$$

it becomes explicit from (A.8) and (A.9) that states in the NS sector transform as a vector $\mathbf{8_v}$ while those in the R sector transform in the $\mathbf{8_c}$ spinorial representation. Then, the states in the right-moving sector furnishes a $(\mathbf{8_v} \oplus \mathbf{8_c})$ representation of SO(8). If we had started with the type IIA left-moving sector generated by $\tilde{S}^{\dot{a}}$, we would have obtained the same vector representation $\mathbf{8_v}$ for the states in the NS sector whereas those in the R sector would transform in the $\mathbf{8_s}$ spinorial representation. Hence furnishing this time a $(\mathbf{8_v} \oplus \mathbf{8_s})$ representation of SO(8).

The massless ground states for type II superstrings come then from combining the right-moving and left-moving sectors, i.e. $16 \times 16 = 256$ states,

$$\begin{aligned} \text{IIB}: \quad (\mathbf{8_v} \oplus \mathbf{8_c}) \otimes (\mathbf{8_v} \oplus \mathbf{8_c}) &= (\,|i\rangle \oplus |\dot{a}\rangle\,) \otimes (\,|j\rangle \oplus |\dot{b}\rangle\,) \\ \text{IIA}: \quad (\mathbf{8_v} \oplus \mathbf{8_c}) \otimes (\mathbf{8_v} \oplus \mathbf{8_s}) &= (\,|i\rangle \oplus |\dot{a}\rangle\,) \otimes (\,|j\rangle \oplus |b\rangle\,) \end{aligned} \tag{A.11}$$

giving rise to bosonic fields in the NS-NS and R-R sectors together with fermionic fields in the NS-R and R-NS ones.

Bosonic fields

The 64 bosonic states $|i\rangle \otimes |j\rangle$ of the spectrum coming from the NS-NS sector are the same in both IIB and IIA superstrings, hence becoming a universal type II bosonic sector.

Because of the decomposition

$$\text{NS-NS sector}: \qquad (\mathbf{8_v} \otimes \mathbf{8_v}) = \mathbf{35_v} \oplus \mathbf{28_v} \oplus \mathbf{1} \;\; \rightarrow \;\; g \oplus B_2 \oplus \varphi \;, \tag{A.12}$$

this sector can be described in terms of a symmetric traceless, antisymmetric and scalar tensors representing the graviton g, the 2-form B_2 and the scalar dilaton ϕ fields, respectively.

In contrast, the 64 bosonic states coming from the R-R sector are no longer the same for the IIB and IIA superstrings. In the case of the type IIB theory, they correspond to states $|\dot{a}\rangle \otimes |\dot{b}\rangle$ built from the tensor product of two spinors of the same chirality. This tensor product has the decomposition of

$$\text{IIB} \quad \text{R-R sector}: \qquad (\mathbf{8_c} \otimes \mathbf{8_c}) = \mathbf{35_c} \oplus \mathbf{28_c} \oplus \mathbf{1} \;\; \rightarrow \;\; C_4 \oplus C_2 \oplus C_0 \;, \tag{A.13}$$

representing a fourth-rank antisymmetric self-dual tensor form C_4, a 2-form C_2 and a scalar C_0 respectively. In the type IIA theory, R-R states $|\dot{a}\rangle \otimes |b\rangle$ come from the tensor product of two spinors with opposite chiralities. This time, the tensor product decomposition reads

$$\text{IIA} \quad \text{R-R sector}: \qquad (\mathbf{8_c} \otimes \mathbf{8_s}) = \mathbf{56_v} \oplus \mathbf{8_v} \;\; \rightarrow \;\; C_3 \oplus C_1 \;, \tag{A.14}$$

hence describing a 3-form C_3 and a 1-form C_1 respectively. The type IIA particle content, unlike that of the IIB theory, is also obtained by dimensional reduction of $d = 11$ supergravity on a circle [2].

Fermionic fields

Fermions in the particle spectrum of type II superstrings arise from the NS-R and R-NS sectors. These 128 states crucially depend on whether right-moving and left-moving supermultiplets have the same chirality or opposite ones. For the chiral type IIB theory, fermionic states $|i\rangle \otimes |\dot{a}\rangle$ and $|\dot{a}\rangle \otimes |i\rangle$ stem from the decompositions

$$\text{IIB}: \quad \begin{aligned} (\mathbf{8_v} \otimes \mathbf{8_c}) &= \mathbf{8_s} \oplus \mathbf{56_s} \;\; \rightarrow \;\; \chi_1 \oplus \psi_1 \\ (\mathbf{8_c} \otimes \mathbf{8_v}) &= \mathbf{8_s} \oplus \mathbf{56_s} \;\; \rightarrow \;\; \chi_2 \oplus \psi_2 \;, \end{aligned} \tag{A.15}$$

resulting in two spin-$\frac{1}{2}$ fermions χ's of the same chirality together with two gravitinos ψ's (spin-$\frac{3}{2}$ fermions) also with the same chirality. In the case of the non-chiral type IIA theory, the spectrum of fermionic states $|i\rangle \otimes |a\rangle$ and $|\dot{a}\rangle \otimes |i\rangle$ coming from the decomposition

$$\text{IIA}: \quad \begin{aligned} (\mathbf{8_v} \otimes \mathbf{8_s}) &= \mathbf{8_c} \oplus \mathbf{56_c} \;\; \rightarrow \;\; \chi_1 \oplus \psi_1 \\ (\mathbf{8_c} \otimes \mathbf{8_v}) &= \mathbf{8_s} \oplus \mathbf{56_s} \;\; \rightarrow \;\; \chi_2 \oplus \psi_2 \;, \end{aligned} \tag{A.16}$$

consists of two spin-$\frac{1}{2}$ fermions χ's of opposite chiralities and two gravitinos ψ's (spin-$\frac{3}{2}$ fermions) also with opposite chiralities.

The above massless spectra of type IIB and IIA superstring theories perfectly fit to the field content of the chiral and non-chiral $\mathcal{N} = 2$, $d = 10$ supergravities, respectively [213].

Appendix B

Parameterised R-R fluxes in T-dual flux models

In this appendix we give the explicit expressions for the original R-R fluxes a_A in terms of the axionic shifts (ξ_s, ξ_t) and the tadpole parameters (ξ_3, ξ_7) or (λ_2, λ_3), depending on the Q-algebra. For the semidirect sum $\mathfrak{su}(2) \oplus_{\mathbb{Z}_3} \mathfrak{u}(1)^3 \sim \mathfrak{iso}(3)$ and the nilpotent $\mathfrak{nil}$ algebras there is another auxiliary variable λ_1 as explained in section 3.2.1. In all cases there is a non-singular rotation matrix from the a_A's to the new variables.

In principle the ξ's and λ's are just real constants but the resulting a_A fluxes must be integers. The exact nature of these parameters can be elucidated starting with the non-geometric Q-fluxes of each gauge subalgebra $\mathfrak{g}_{gauge}$. For example, following the discussion at the end of section 3.1.1.1, for the $\mathfrak{su}(2)^2$ case when $\epsilon_1 \epsilon_2 = 0$ it transpires that $(\xi_3, \xi_7, \xi_s, \xi_t) \in \mathbb{Q}$.

There is a universal structure in the R-R fluxes that is worth noticing. For all Q-algebras the dependence on the axionic shift parameters (ξ_s, ξ_t) is of the form

$$\begin{aligned} a_0 &= -b_0\,\xi_s + 3\,c_0\,\xi_t\, + \cdots \\ a_1 &= -b_1\,\xi_s - (2\,c_1 - \tilde{c}_1)\,\xi_t\, + \cdots \\ a_2 &= -b_2\,\xi_s - (2\,c_2 - \tilde{c}_2)\,\xi_t\, + \cdots \\ a_3 &= -b_3\,\xi_s + 3\,c_3\,\xi_t\, + \cdots \end{aligned} \tag{B.1}$$

where $\cdots$ stands for extra terms depending on the tadpole parameters. This structure reflects the invariance of the theory under real shifts $S \to S - \xi_s$ and $T \to T - \xi_t$ of the linear moduli fields entering the T-duality invariant superpotential in (2.31).

Next we present the the full expression of the original R-R fluxes a_A in terms of the new flux variables for each of the Q-algebras.

B.1 The $\mathfrak{su}(2)^2$ case

$$
\begin{aligned}
a_0 &= \delta^3(\epsilon_1\xi_3+\epsilon_2\xi_s)+\beta^3(\epsilon_1\xi_s-\epsilon_2\xi_3)+3\delta\beta^2(\xi_t-\xi_7)+3\beta\delta^2(\xi_t+\xi_7)\ ,\\
a_1 &= -\gamma\delta^2(\epsilon_1\xi_3+\epsilon_2\xi_s)-\alpha\beta^2(\epsilon_1\xi_s-\epsilon_2\xi_3)-\beta(\beta\gamma+2\alpha\delta)(\xi_t-\xi_7)-\delta(\alpha\delta+2\beta\gamma)(\xi_t+\xi_7)\ ,\\
a_2 &= \delta\gamma^2(\epsilon_1\xi_3+\epsilon_2\xi_s)+\beta\alpha^2(\epsilon_1\xi_s-\epsilon_2\xi_3)+\alpha(\alpha\delta+2\beta\gamma)(\xi_t-\xi_7)+\gamma(\beta\gamma+2\alpha\delta)(\xi_t+\xi_7)\ ,\\
a_3 &= -\gamma^3(\epsilon_1\xi_3+\epsilon_2\xi_s)-\alpha^3(\epsilon_1\xi_s-\epsilon_2\xi_3)-3\gamma\alpha^2(\xi_t-\xi_7)-3\alpha\gamma^2(\xi_t+\xi_7)\ .
\end{aligned}
$$

B.2 The $\mathfrak{so}(3,1)$ case

$$
\begin{aligned}
a_0 &= \delta(\delta^2-3\beta^2)(\epsilon_1\xi_3+\epsilon_2\xi_s)+\beta(\beta^2-3\delta^2)(\epsilon_1\xi_s-\epsilon_2\xi_3)-3(\beta^2+\delta^2)(\beta\xi_t-\delta\xi_7)\ ,\\
a_1 &= (\gamma\beta^2+2\alpha\beta\delta-\gamma\delta^2)(\epsilon_1\xi_3+\epsilon_2\xi_s)+(\alpha\delta^2+2\beta\gamma\delta-\alpha\beta^2)(\epsilon_1\xi_s-\epsilon_2\xi_3)\\
&+ (\beta^2+\delta^2)(\alpha\xi_t-\gamma\xi_7)+2(\alpha\beta+\gamma\delta)(\beta\xi_t-\delta\xi_7)\ ,\\
a_2 &= (\delta\gamma^2-2\alpha\beta\gamma-\delta\alpha^2)(\epsilon_1\xi_3+\epsilon_2\xi_s)+(\beta\alpha^2-2\alpha\gamma\delta-\beta\gamma^2)(\epsilon_1\xi_s-\epsilon_2\xi_3)\\
&- 2(\alpha\beta+\gamma\delta)(\alpha\xi_t-\gamma\xi_7)-(\alpha^2+\gamma^2)(\beta\xi_t-\delta\xi_7)\ ,\\
a_3 &= -\gamma(\gamma^2-3\alpha^2)(\epsilon_1\xi_3+\epsilon_2\xi_s)-\alpha(\alpha^2-3\gamma^2)(\epsilon_1\xi_s-\epsilon_2\xi_3)+3(\alpha^2+\gamma^2)(\alpha\xi_t-\gamma\xi_7)\ .
\end{aligned}
$$

B.3 The $\mathfrak{su}(2)+\mathfrak{u}(1)^3$ case

$$
\begin{aligned}
a_0 &= \delta^3(\epsilon_1\xi_3+\epsilon_2\xi_s)+\beta^3(\epsilon_1\xi_s-\epsilon_2\xi_3)+3\beta\delta^2\xi_t-3\delta\beta^2\xi_7\ ,\\
a_1 &= -\gamma\delta^2(\epsilon_1\xi_3+\epsilon_2\xi_s)-\alpha\beta^2(\epsilon_1\xi_s-\epsilon_2\xi_3)-\delta(\alpha\delta+2\beta\gamma)\xi_t+\beta(\beta\gamma+2\alpha\delta)\xi_7\ ,\\
a_2 &= \delta\gamma^2(\epsilon_1\xi_3+\epsilon_2\xi_s)+\beta\alpha^2(\epsilon_1\xi_s-\epsilon_2\xi_3)+\gamma(\beta\gamma+2\alpha\delta)\xi_t-\alpha(\alpha\delta+2\beta\gamma)\xi_7\ ,\\
a_3 &= -\gamma^3(\epsilon_1\xi_3+\epsilon_2\xi_s)-\alpha^3(\epsilon_1\xi_s-\epsilon_2\xi_3)-3\alpha\gamma^2\xi_t+3\gamma\alpha^2\xi_7\ .
\end{aligned}
$$

B.4 The $\mathfrak{su}(2)\oplus_{\mathbb{Z}_3}\mathfrak{u}(1)^3$ case

$$
\begin{aligned}
a_0 &= \delta^3(\epsilon_2\xi_s+3\xi_t)+\beta\delta^2(\epsilon_1\xi_s-3\xi_t+3\lambda_1)+3\delta\beta^2\lambda_2+\beta^3\lambda_3\ ,\\
a_1 &= -\gamma\delta^2(\epsilon_2\xi_s+3\xi_t)-\tfrac{1}{3}\delta(\alpha\delta+2\beta\gamma)(\epsilon_1\xi_s-3\xi_t+3\lambda_1)-\beta(\beta\gamma+2\alpha\delta)\lambda_2-\alpha\beta^2\lambda_3\ ,\\
a_2 &= \delta\gamma^2(\epsilon_2\xi_s+3\xi_t)+\tfrac{1}{3}\gamma(\beta\gamma+2\alpha\delta)(\epsilon_1\xi_s-3\xi_t+3\lambda_1)+\alpha(\alpha\delta+2\beta\gamma)\lambda_2+\beta\alpha^2\lambda_3\ ,\\
a_3 &= -\gamma^3(\epsilon_2\xi_s+3\xi_t)-\alpha\gamma^2(\epsilon_1\xi_s-3\xi_t+3\lambda_1)-3\gamma\alpha^2\lambda_2-\alpha^3\lambda_3\ .
\end{aligned}
$$

B.5 The $\mathfrak{nil}$ case

$$
\begin{aligned}
a_0 &= \delta^3(\epsilon_2\xi_s+3\xi_t)+\gamma\delta^2(\epsilon_1\xi_s+3\lambda_1)+3\delta\gamma^2\lambda_2+\gamma^3\lambda_3\ ,\\
a_1 &= -\gamma\delta^2(\epsilon_2\xi_s+3\xi_t)+\tfrac{1}{3}\delta(\delta^2-2\gamma^2)(\epsilon_1\xi_s+3\lambda_1)-\gamma(\gamma^2-2\delta^2)\lambda_2+\delta\gamma^2\lambda_3\ ,\\
a_2 &= \delta\gamma^2(\epsilon_2\xi_s+3\xi_t)+\tfrac{1}{3}\gamma(\gamma^2-2\delta^2)(\epsilon_1\xi_s+3\lambda_1)+\delta(\delta^2-2\gamma^2)\lambda_2+\gamma\delta^2\lambda_3\ ,\\
a_3 &= -\gamma^3(\epsilon_2\xi_s+3\xi_t)+\delta\gamma^2(\epsilon_1\xi_s+3\lambda_1)-3\gamma\delta^2\lambda_2+\delta^3\lambda_3\ .
\end{aligned}
$$

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