Newman-Penrose approach to $N = 2$ $D = 4$ sugra

This is a set of complementary notes to (`hep-th/0603099`), and deals with some Newman-Penrose approach to the classification of supersymmetric solutions to $N = 2$ $d = 4$ supergravity coupled to vector multiplets. Since these are just some notes, caringly referred to as *Piece of Toilet Paper*, there really is no sense in complaining/whining/bitching about the references.

The $N = 2$ action coupled to n_V vector multiplets and $n_H = 0$ hypermultiplets, truncated to the bosonic part, is given by [1][1]

$$\mathcal{S} \;=\; \int_4 \left[\tfrac{1}{2}R \;+\; \mathcal{G}_{i\bar{\jmath}}\partial_a Z^i \partial^a \overline{Z}^{\bar{\jmath}} \;+\; \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}F^\Lambda \wedge *F^\Sigma \;-\; \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}F^\Lambda \wedge F^\Sigma\right] \;, \tag{1}$$

from which the equations of motion can be seen to be

$$R_{ab} \;=\; -2\mathcal{G}_{i\bar{\imath}}\partial_{(a}Z^i\partial_{b)}\overline{Z}^{\bar{\imath}} - 4\mathrm{Im}(\mathcal{N})_{\Lambda\Sigma}\left[F^\Lambda_{ac}F^{\Sigma\, c}_{b} - \tfrac{1}{4}g_{ab}F^\Lambda_{cd}F^{\Sigma cd}\right] \;, \tag{2}$$

$$0 \;=\; d\left[\mathrm{Im}(\mathcal{N})_{\Lambda\Sigma} \;*F^\Sigma \;-\; \mathrm{Re}(\mathcal{N})_{\Lambda\Sigma}\; F^\Sigma\right] \;, \tag{3}$$

$$0 \;=\; \mathfrak{D}_a\partial^a Z^i \;-\; i\mathcal{G}^{i\bar{\imath}}\left[\partial_{\bar{\imath}}\overline{\mathcal{N}}_{\Lambda\Sigma}F^{\Lambda -}_{ab}F^{\Sigma -|ab} - \partial_{\bar{\imath}}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda +}_{ab}F^{\Sigma +|ab}\right] \;. \tag{4}$$

and in the same setting we have the supersymmetry equations

$$\delta\Psi_{\aleph a} \;=\; \mathfrak{D}_a\epsilon_\aleph \;+\; T^-_{ab}\varepsilon_{\aleph\beth}\gamma^b\,\epsilon^\beth \;, \tag{5}$$

$$\delta\lambda^{i\aleph} \;=\; i\,\nabla\!\!\!\!/\, Z^i\,\epsilon^\aleph \;+\; 2G\!\!\!/^{\,i-}\epsilon_\beth\,\varepsilon^{\aleph\beth} \;, \tag{6}$$

where we have defined $\mathfrak{D}\epsilon_\aleph = \nabla\epsilon_\aleph \;+\; i/2\mathcal{Q}\epsilon_\aleph$, where of course we also have $2i\;\mathcal{Q} = dZ^i\partial_i\mathcal{K} - d\overline{Z}^{\bar{\imath}}\partial_{\bar{\imath}}\mathcal{K}$, with $\mathcal{K}$ the Kähler potential. This also means that $\mathcal{G}_{i\bar{\imath}} = \partial_i\partial_{\bar{\imath}}\mathcal{K}$.

1 Special Kähler geometry

The formal starting point for the definition of a Special Kähler manifold, lies in the definition of a Kähler-Hodge manifold. A KH-manifold is a complex line bundle over a Kähler manifold $\mathcal{M}$, such that the first, and only, Chern class of the line bundle equals the Kähler form. This then implies that the exponential of the Kähler potential can be used as a metric on the Line bundle. Furthermore, the connection on the line bundle is

[1] $\mathrm{Im}(\mathcal{N})$ is negative and this fixes the signs.

$\mathcal{Q} = (2i)^{-1}(dz^i\partial_i\mathcal{K} - d\bar{z}^{\bar{\imath}}\partial_{\bar{\imath}}\mathcal{K})$. Let us denote the line bundle by $L^1 \to \mathcal{M}$, where the superscript is there to indicate that the covariant derivative is $\mathfrak{D} = \nabla + i\mathcal{Q}$

In order to make things worse, consider then a flat $2(n+1)$ vector bundle $E \to \mathcal{M}$ with structure group $Sp(n+1;\mathbb{R})$, and take a section $\mathcal{V}$ of the product bundle $E\otimes L^1 \to \mathcal{M}$ and its complex conjugate $\overline{\mathcal{V}}$, which formally is a section of the bundle $E\otimes L^{-1}\to\mathcal{M}$, but who cares? Anyway, a special Kähler manifold, then is a bundle $E\otimes L^1 \to \mathcal{M}$, for which there exists a section $\mathcal{V}$ such that

$$\mathcal{V} \;=\; \left(\begin{array}{c} \mathcal{L}^\Lambda \\ \mathcal{M}_\Sigma \end{array}\right) \;\to\; \left\{\begin{array}{lcl} \langle \mathcal{V} \mid \overline{\mathcal{V}}\rangle & \equiv & \overline{\mathcal{L}}^\Lambda \mathcal{M}_\Lambda \;-\; \mathcal{L}^\Lambda\overline{\mathcal{M}}_\Lambda \;=\; -i \\ \mathfrak{D}_{\bar{\imath}}\mathcal{V} & = & 0\;, \\ \langle \mathfrak{D}_i\mathcal{V} \mid \mathcal{V}\rangle & = & 0\;. \end{array}\right. \tag{7}$$

If we then define

$$\mathcal{U}_i \;\equiv\; \mathfrak{D}_i\mathcal{V} \;=\; \left(\begin{array}{c} f_i^\Lambda \\ h_{\Sigma\ i} \end{array}\right) \;,\; \overline{\mathcal{U}}_{\bar{\imath}} \;=\; \overline{\mathcal{U}_i}\;, \tag{8}$$

then it follows from the basic definitions that

$$\begin{array}{lclclcl} \mathfrak{D}_{\bar{\imath}}\,\mathcal{U}_i & = & \mathcal{G}_{i\bar{\imath}}\,\mathcal{V} & , & \langle \mathcal{U}_i \mid \overline{\mathcal{U}}_{\bar{\imath}}\rangle & = & i\mathcal{G}_{i\bar{\imath}}\;, \\ \langle \mathcal{U}_i \mid \overline{\mathcal{V}}\rangle & = & 0 & , & \langle \mathcal{U}_i \mid \mathcal{V}\rangle & = & 0\;. \end{array} \tag{9}$$

Let us have a look at $\langle \mathfrak{D}_i\mathcal{U}_j \mid \mathcal{V}\rangle = -\langle\, \mathcal{U}_j \mid \mathcal{U}_i\rangle$, where we have made use of the third constraint. As one can see the r.h.s. is antisymmetric in i and j, whereas the l.h.s. is symmetric (This you can see by either a brute force calculation or by looking at App. (B.3) and thinking a bit). This then means that $\langle \mathfrak{D}_i\mathcal{U}_j \mid \mathcal{V}\rangle = \langle \mathcal{U}_j \mid \mathcal{U}_i\rangle = 0$. The importance of this last equation is that if we group together $\mathcal{E}_\Lambda = (\mathcal{V},\mathcal{U}_i)$, then we can see that $\langle \mathcal{E}_\Sigma \mid \overline{\mathcal{E}}_\Lambda\rangle$ is a non-degenerate matrix, which allows us to construct an identity operator for the symplectic indices, such that for a given section of $\mathcal{A} \ni \Gamma\,(E,\mathcal{M})$ we have

$$\mathcal{A} \;=\; i\langle \mathcal{A} \mid \overline{\mathcal{V}}\rangle\;\mathcal{V} \;-\; i\langle \mathcal{A} \mid \mathcal{V}\rangle\;\overline{\mathcal{V}} \;+\; i\langle \mathcal{A} \mid \mathcal{U}_i\rangle\mathcal{G}^{i\bar{\imath}}\;\overline{\mathcal{U}}_{\bar{\imath}} \;-\; i\langle \mathcal{A} \mid \overline{\mathcal{U}}_{\bar{\imath}}\rangle\mathcal{G}^{i\bar{\imath}}\mathcal{U}_i\;. \tag{10}$$

We saw that $\mathfrak{D}_i\mathcal{U}_j$ is symmetric in i and j, but what more can be said about it? As one can easily see, the innerproduct with $\overline{\mathcal{V}}$ and $\overline{\mathcal{U}}_{\bar{\imath}}$ vanishes due to the basic properties. Let us then define the weight $(2,-2)$ object

$$\mathcal{C}_{ijk} \;\equiv\; \langle \mathfrak{D}_i\,\mathcal{U}_j \mid \mathcal{U}_k\rangle \;\;\to\;\; \mathfrak{D}_i\,\mathcal{U}_j \;=\; i\mathcal{C}_{ijk}\mathcal{G}^{k\bar{l}}\overline{\mathcal{U}}_{\bar{l}}\;, \tag{11}$$

the last equation being a consequence of Eq. (10). Since the $\mathcal{U}$´s are orthogonal, however, one can see that $\mathcal{C}$ is completely symmetric in its 3 indices, and 2 small calculations show that

$$\mathfrak{D}_{\bar{\imath}}\,\mathcal{C}_{jkl} \;=\; 0 \;\;,\;\; \mathfrak{D}_{[i}\,\mathcal{C}_{j]kl} \;=\; 0\;. \tag{12}$$

Observe that these equations imply the existence of a function $\mathcal{S}$, such that

$$\mathcal{C}_{ijk} \;=\; \mathfrak{D}_i\mathfrak{D}_j\mathfrak{D}_k\;\mathcal{S}\;, \tag{13}$$

and reportedly one finds [2] $\mathcal{S}\sim\mathcal{L}^{\Lambda}\mathrm{Im}(\mathcal{N})_{\Lambda\Sigma}\mathcal{L}^{\Sigma}$.

The last equation that can be derived without introducing the monodromy matrix, and seemingly the one that is the reason why special geometry is used in $N=2$ $D=4$ sugra, is the Riemann tensor for the Levi-Cività connection on $\mathcal{M}$. This can be calculated because we know the relation between $\mathfrak{D}$ and ∇ and we have $[\nabla_a,\nabla_b]X_c = R_{abcd}X^d$. Anyway, the result is

$$R_{i\bar{\jmath}k\bar{l}} \;=\; \mathcal{G}_{i\bar{\jmath}}\,\mathcal{G}_{k\bar{l}} \;+\; \mathcal{G}_{i\bar{l}}\,\mathcal{G}_{k\bar{\jmath}} \;-\; \mathcal{C}_{ikm}\,\overline{\mathcal{C}}_{\bar{\jmath}\bar{l}\bar{m}}\,\mathcal{G}^{m\bar{m}}\;. \tag{14}$$

Let us then introduce the concept of a monodromy matrix $\mathcal{N}$, which can be defined through the relations

$$\mathcal{M}_{\Lambda} \;=\; \mathcal{N}_{\Lambda\Sigma}\,\mathcal{L}^{\Sigma} \;\;,\;\; h_{\Lambda i} \;=\; \overline{\mathcal{N}}_{\Lambda\Sigma}\,f_i^{\Sigma}\;, \tag{15}$$

The relations of $\langle\mathcal{U}_i\mid\overline{\mathcal{V}}\rangle=0$ then implies that $\mathcal{N}$ is symmetric, which then automatically trivializes $\langle\mathcal{U}_i\mid\mathcal{U}_j\rangle=0$.

From the other basic properties in (9) we find

$$-\tfrac{1}{2} \;=\; \mathcal{L}^{\Lambda}\;\mathrm{Im}(\mathcal{N})_{\Lambda\Sigma}\bar{\mathcal{L}}^{\Sigma}\;, \tag{16}$$

$$0 \;=\; \mathcal{L}^{\Lambda}\;\mathrm{Im}(\mathcal{N})_{\Lambda\Sigma}\;f_i^{\Sigma}\;, \tag{17}$$

$$-\tfrac{1}{2}\mathcal{G}_{i\bar{\imath}} \;=\; f_i^{\Lambda}\;\mathrm{Im}(\mathcal{N})_{\Lambda\Sigma}\;\bar{f}_{\bar{\imath}}^{\Sigma}\;. \tag{18}$$

Further identities that can be derived are

$$(\partial_i\mathcal{N}_{\Lambda\Sigma})\;\mathcal{L}^{\Sigma} \;=\; -2i\mathrm{Im}(\mathcal{N})_{\Lambda\Sigma}\;f_i^{\Sigma}\;, \tag{19}$$

$$\partial_i\overline{N}_{\Lambda\Sigma}\;f_j^{\Sigma} \;=\; -2\mathcal{C}_{ijk}\mathcal{G}^{k\bar{k}}\mathrm{Im}(\mathcal{N})_{\Lambda\Sigma}\;\bar{f}_{\bar{k}}^{\Sigma}\;, \tag{20}$$

$$\mathcal{C}_{ijk} \;=\; f_i^{\Lambda}f_j^{\Sigma}\;\partial_k\overline{N}_{\Lambda\Sigma}\;, \tag{21}$$

$$\mathcal{L}^{\Sigma}\partial_{\bar{\imath}}\mathcal{N}_{\Lambda\Sigma} \;=\; 0\;, \tag{22}$$

$$\partial_{\bar{\imath}}\overline{\mathcal{N}}_{\Lambda\Sigma}\;f_i^{\Sigma} \;=\; 2i\mathcal{G}_{i\bar{\imath}}\mathrm{Im}(\mathcal{N})_{\Lambda\Sigma}\;\mathcal{L}^{\Sigma}\;, \tag{23}$$

An important identity that one can derive, is given by

$$U^{\Lambda\Sigma} \;\equiv\; f_i^{\Lambda}\mathcal{G}^{i\bar{\imath}}\bar{f}_{\bar{\imath}}^{\Sigma} \;=\; -\tfrac{1}{2}\mathrm{Im}(\mathcal{N})^{-1|\Lambda\Sigma} \;-\; \overline{\mathcal{L}}^{\Lambda}\mathcal{L}^{\Sigma}\;, \tag{24}$$

so that $\overline{U^{\Lambda\Sigma}} = U^{\Sigma\Lambda}$. The graviphoton projector is defined by

$$\mathcal{T}_\Lambda \;=\; 2i\mathcal{L}^\Sigma \;\mathrm{Im}\,(\mathcal{N})_{\Sigma\Lambda} \;\;, \tag{25}$$

Let us construct the $(n+1)\times(n+1)$-matrices $M = (\mathcal{M}_\Lambda, \bar{h}_{\Lambda\;\bar{\imath}})$ and $L = (\mathcal{L}^\Lambda, \bar{f}_{\bar{\imath}}^\Lambda)$. With it we can write the defining relations for the monodromy matrix as $M_{\Lambda\Sigma} = \mathcal{N}_{\Lambda\Omega}L^{\Omega}{}_{\Sigma}$, a system which we can easily solve by putting $\mathcal{N} = ML^{-1}$, where L^{-1} is the inverse of L. Formally one finds

$$L^{-1} \;=\; -2\begin{pmatrix} \mathcal{L}^\Lambda \mathrm{Im}(\mathcal{N})_{\Lambda\Sigma} \\ \mathcal{G}^{\bar{\imath}i}\; f_i^\Lambda \mathrm{Im}(\mathcal{N})_{\Lambda\Sigma}\end{pmatrix} \;,\; \equiv \;-\; \begin{pmatrix} i\overline{\mathcal{T}}_\Sigma \\ 2 f_\Sigma^{\bar{\imath}} \end{pmatrix} \;, \tag{26}$$

which is a recursive argument, but will be useful in order to derive more interesting results.

The first identity that can be derived is

$$\mathcal{G}^{i\bar{\imath}}\partial_{\bar{\imath}}\overline{\mathcal{N}}_{\Lambda\Sigma} \;=\; -2\left(\bar{f}^i_\Lambda \mathcal{T}_\Sigma \;+\; \mathcal{T}_\Lambda \bar{f}^i_\Sigma\right) \;, \tag{27}$$

and the, for the moment, last is

$$\partial_{\bar{\imath}}\mathcal{N}_{\Lambda\Sigma} \;=\; 4\;\overline{\mathcal{C}}_{\bar{\imath}\bar{\jmath}\bar{k}}\; f^{\bar{\jmath}}_\Lambda\; f^{\bar{k}}_\Sigma \;. \tag{28}$$

1.1 Prepotential: Existence and more formulae

Let us start by introducing the explicitly holomorphic section $\Omega = e^{-\mathcal{K}/2}\mathcal{V}$, which allows us to rewrite the system (7) as

$$\Omega \;=\; \begin{pmatrix} \mathcal{X}^\Lambda \\ \mathcal{F}_\Sigma \end{pmatrix} \;\;\rightarrow\;\; \begin{cases} \langle\Omega\mid\overline{\Omega}\rangle & \equiv\;\; \overline{\mathcal{X}}^\Lambda\mathcal{F}_\Lambda \;-\; \mathcal{X}^\Lambda\overline{\mathcal{F}}_\Lambda \;=\; -i\; e^{-\mathcal{K}} \\ \partial_{\bar{\imath}}\Omega & =\;\; 0\;, \\ \langle\partial_i\Omega\mid\Omega\rangle & =\;\; 0\;. \end{cases} \tag{29}$$

If we now assume that $\mathcal{F}_\Lambda$ depends on Z^i through the $\mathcal{X}$'s, then from the last equation we can derive that

$$\partial_i\mathcal{X}^\Lambda\left[2\mathcal{F}_\Lambda \;-\; \partial_\Lambda\left(\mathcal{X}^\Sigma\mathcal{F}_\Sigma\right)\right] \;=\; 0\;. \tag{30}$$

If $\partial_i\mathcal{X}^\Lambda$ is invertible as a $n\times(n+1)$ matrix, then we must conclude that

$$\mathcal{F}_\Lambda \;=\; \partial_\Lambda\mathcal{F}(\mathcal{X})\;, \tag{31}$$

where $\mathcal{F}$ is a homogeneous function of degree 2: The *prepotential*.

Making use of the prepotential and the definitions (15), we can then calculate

$$\mathcal{N}_{\Lambda\Sigma} \;=\; \overline{\mathcal{F}}_{\Lambda\Sigma} \;+\; 2i\frac{\mathrm{Im}(\mathcal{F})_{\Lambda\Lambda'}\mathcal{X}^{\Lambda'}\mathrm{Im}(\mathcal{F})_{\Sigma\Sigma'}\mathcal{X}^{\Sigma'}}{\mathcal{X}^{\Omega}\mathrm{Im}(\mathcal{F})_{\Omega\Omega'}\mathcal{X}^{\Omega'}} \;, \tag{32}$$

which even though not beautiful, at least is manifestly symmetric. Having the explicit form of $\mathcal{N}$, we can derive an explicit representation for $\mathcal{C}$ by applying Eq. (22): one finds

$$\mathcal{C}_{ijk} \;=\; e^{\mathcal{K}}\; \partial_i\mathcal{X}^{\Lambda}\; \partial_j\mathcal{X}^{\Sigma}\; \partial_k\mathcal{X}^{\Omega}\; \mathcal{F}_{\Lambda\Sigma\Omega} \;, \tag{33}$$

so that the prepotential really determines all structures in special geometry.

A last remark has to be made about the existence of a prepotential: clearly, given a holomorphic section Ω a prepotential need not exist. It was shown in [3], however, that one can always apply an $Sp(n+1,\mathbb{R})$ transformation such that a prepotential exists. Clearly the $N=2$ sugra action is not invariant under the full $Sp(n+1,\mathbb{R})$, but the equations of motion and the supersymmetry equations are. This means that for the purpose of this article we can always, even if this is not done, impose the existence of a prepotential.

2 NP translation of the susy variations

Seeing the fact that we are dealing with chiral spinors with $\gamma_5\epsilon_{\aleph} = \epsilon_{\aleph}$ and $\gamma_5\epsilon^{\aleph} = -\epsilon^{\aleph}$, we will make the identification

$$\epsilon^1 = \left(\begin{array}{c}\alpha_A\\ 0\end{array}\right) \;,\; \epsilon^2 = \left(\begin{array}{c}\beta_A\\ 0\end{array}\right) \;,\; \epsilon_1 = \left(\begin{array}{c}0\\ \bar{\alpha}_{\bar{A}}\end{array}\right) \;,\; \epsilon^1 = \left(\begin{array}{c}0\\ \bar{\beta}_{\bar{A}}\end{array}\right) \;. \tag{34}$$

If we then also define

$$T^-_{ab} \;\equiv\; \bar{\phi}_{\bar{A}\bar{B}}\varepsilon_{AB} \;=\; \mathcal{T}_{\Lambda}\; \bar{\phi}^{\Lambda}_{\bar{A}\bar{B}}\varepsilon_{AB} \;, \tag{35}$$

$$G^{i-}_{ab} \;\equiv\; \bar{\psi}^{i}_{\bar{A}\bar{B}}\varepsilon_{AB} \;=\; -\mathcal{G}^{i\bar{\imath}}\bar{f}^{\Lambda}_{\bar{\imath}}\mathrm{Im}\,(\mathcal{N})_{\Lambda\Sigma}\,\bar{\phi}^{\Sigma}_{\bar{A}\bar{B}}\varepsilon_{AB} \;, \tag{36}$$

we can translate the susy variations (5) to

$$\nabla_a\bar{\alpha}_{\bar{B}} \;=\; -\tfrac{i}{2}\mathcal{Q}_a\bar{\alpha}_{\bar{B}} \;+\; \sqrt{2}\bar{\phi}_{\bar{A}\bar{B}}\;\beta_A \;, \tag{37}$$

$$\nabla_a\bar{\beta}_{\bar{B}} \;=\; -\tfrac{i}{2}\mathcal{Q}_a\bar{\beta}_{\bar{B}} \;-\; \sqrt{2}\bar{\phi}_{\bar{A}\bar{B}}\;\alpha_A \;, \tag{38}$$

$$\sqrt{2}i\;\nabla_a Z^i\;\alpha^A \;=\; 4\;\bar{\psi}^i_{\bar{A}\bar{B}}\;\bar{\beta}^{\bar{B}} \;, \tag{39}$$

$$\sqrt{2}i\;\nabla_a Z^i\;\beta^A \;=\; -4\;\bar{\psi}^i_{\bar{A}\bar{B}}\;\bar{\alpha}^{\bar{B}} \;. \tag{40}$$

Similar equations can be obtained from the above by complex conjugation.

By making use of the relation (24), we can can obtain

$$\bar{\phi}^{\Lambda}_{\bar{A}\bar{B}} \;=\; 2\; f^{\Lambda}_{i}\; \bar{\psi}^{i}_{\bar{A}\bar{B}} \;+\; i\bar{\mathcal{L}}^{\Lambda}\; \bar{\phi}_{\bar{A}\bar{B}}\;, \tag{41}$$

which allows us to calculate F^{Λ}.

The Bianchi identity and the equation of motion for the gauge fields can be expressed in the NP language as

$$\begin{aligned} \nabla_{A\bar{B}}\phi^{\Lambda\ A}{}_{B} &= \nabla_{B\bar{A}}\bar{\phi}^{\Lambda\ \bar{A}}{}_{\bar{B}}\;, \quad (42)\\ \nabla_{A\bar{B}}\left(\mathcal{N}_{\Sigma\Lambda}\phi^{\Lambda\ A}{}_{B}\right) &= \nabla_{B\bar{A}}\left(\overline{\mathcal{N}}_{\Sigma\Lambda}\bar{\phi}^{\Lambda\ \bar{A}}{}_{\bar{B}}\right)\;, \quad (43) \end{aligned}$$

3 Non-degenerate case

In this section we will contemplate the possibility that

$$\alpha_{A}\beta^{A} \;=\; V \;\neq\; 0 \quad ; \quad \alpha_{A}\beta_{B} \;-\; \alpha_{B}\beta_{A} \;=\; V\varepsilon_{AB}\;. \tag{44}$$

This allows us to define the unnormalized tetrad

$$\left.\begin{aligned} L_{a} &= \alpha_{A}\bar{\alpha}_{\bar{A}} \\ N_{a} &= \beta_{A}\bar{\beta}_{\bar{B}} \end{aligned}\right\} \;\rightarrow\; L_{a}N^{a} = V\overline{V}\;, \qquad \left.\begin{aligned} M_{a} &= \alpha_{A}\bar{\beta}_{\bar{A}} \\ \bar{M}_{a} &= \beta_{A}\bar{\alpha}_{\bar{B}} \end{aligned}\right\} \;\rightarrow\; M_{a}\bar{M}^{a} = -V\overline{V}\;. \tag{45}$$

It is also worth-while to introduce the combinations

$$\begin{array}{llllllllll} T &=& \frac{1}{\sqrt{2}}\left(L \;+\; N\right) &,& L &=& \frac{1}{\sqrt{2}}\left(T \;+\; X\right) &,& T^{2} &=& V\bar{V}\;, \\ X &=& \frac{1}{\sqrt{2}}\left(L \;-\; N\right) &,& N &=& \frac{1}{\sqrt{2}}\left(T \;-\; X\right) &,& X^{2} &=& -V\bar{V}\;, \\ Y &=& \frac{1}{\sqrt{2}}\left(M \;+\; \bar{M}\right) &,& M &=& \frac{1}{\sqrt{2}}\left(Y \;+\; iZ\right) &,& Y^{2} &=& -V\bar{V}\;, \\ Z &=& \frac{1}{\sqrt{2}i}\left(M \;-\; \bar{M}\right) &,& \bar{M} &=& \frac{1}{\sqrt{2}}\left(X \;-\; iZ\right) &,& Z^{2} &=& -V\bar{V}\;. \end{array} \tag{46}$$

The non-degeneracy of the tetrad also implies that

$$\begin{aligned} g_{ab} &= \left(V\bar{V}\right)^{-1}\;\left[L_{a}N_{b} \;+\; N_{a}L_{b} \;-\; M_{a}\bar{M}_{b} \;-\; \bar{M}_{a}M_{b}\right] \\ &= \left(V\bar{V}\right)^{-1}\;\left[T_{a}T_{b} \;-\; X_{a}X_{b} \;-\; Y_{a}Y_{b} \;-\; Z_{a}Z_{b}\right]\;. \end{aligned} \tag{47}$$

Armed with these definitions we can calculate[2]

$$\nabla_{a}T_{b} \;=\; -V\bar{\phi}_{\bar{A}\bar{B}}\varepsilon_{AB} \;-\; \bar{V}\;\overline{\bar{\phi}_{\bar{A}\bar{B}}}\;\varepsilon_{\bar{A}\bar{B}}\;, \tag{48}$$

[2] The occurrence of $\overline{\bar{\phi}_{\bar{A}\bar{B}}}$ might seem a bit odd, but is done on purpose in order not to create confusion with $\mathcal{T}_{\Lambda}\phi^{\Lambda}_{AB}$.

$$
\begin{aligned}
\nabla_a X_b &= 2\bar{\phi}_{\bar{A}\bar{B}}\ \alpha_{(A}\beta_{B)}\ +\ 2\ \overline{\bar{\phi}_{\bar{A}\bar{B}}}\ \bar{\alpha}_{(\bar{A}}\bar{\beta}_{\bar{B})}\ , && (49)\\
\nabla_a M_b &= \sqrt{2}\ \overline{\bar{\phi}_{\bar{A}\bar{B}}}\ \bar{\beta}_{\bar{A}}\bar{\beta}_{\bar{B}}\ -\ \sqrt{2}\bar{\phi}_{\bar{A}\bar{B}}\ \alpha_A\alpha_B\ . && (50)
\end{aligned}
$$

The first of these equations has the typical expansion of a real 2-form, and as such we find that $\nabla_{(a}T_{b)} = 0$: $T^a\partial_a$ is a Killing vector. On the other hand, the last two are clearly invariant under the substitution $a \leftrightarrow b$, so that $dX = dM = 0$. So, after a complex conjugation and a glance at (46), then says that $dX = dY = dZ = 0$, which enables us to introduce the coordinate representation

$$
\begin{array}{lcllllcl}
T_a d\xi^a &=& V\bar{V}\,(dt+\omega_m dx^m) &,& T^a\partial_a &=& \partial_t\ , \\
X_a d\xi^a &=& dx^1 &,& X^a\partial_a &=& -V\bar{V}\,[\partial_1\ -\ \omega_1\partial_t]\ , \\
Y_a d\xi^a &=& dx^2 &,& Y^a\partial_a &=& -V\bar{V}\,[\partial_2\ -\ \omega_2\partial_t]\ , \\
Z_a d\xi^a &=& dx^3 &,& Z^a\partial_a &=& -V\bar{V}\,[\partial_3\ -\ \omega_3\partial_t]\ .
\end{array} \quad (51)
$$

This obviously means that the metric has the archetypical *conforma-stationary form*:

$$
ds^2\ =\ V\bar{V}\,(dt\ +\ \omega)^2 - \left(V\bar{V}\right)^{-1} d\vec{x}^2\ . \quad (52)
$$

Using Eqs. (37,38), we can calculate the differential of V only to find

$$
\mathfrak{D}_a\bar{V}\ =\ 2\ \bar{\phi}_{\bar{A}\bar{B}}\ T_A{}^{\bar{B}}\ . \quad (53)
$$

This relation can be inverted, which gives the result that

$$
\bar{\phi}_{\bar{A}\bar{B}}\ =\ -\left(V\bar{V}\right)^{-1}\ T^A{}_{\bar{B}}\ \mathfrak{D}_a\bar{V}\ , \quad (54)
$$

which is not only a fundamental part for the determination of F^Λ, but also for giving a restriction on ω. Indeed, calculating dT through its coordinate representation and by Eqs. (48,54) we find that (We choose the convention for ε_{mnp} such that $\varepsilon_{123} = 1$)

$$
V\bar{V}\,(\partial_m\omega_n\ -\ \partial_n\omega_m)\ =\ i\varepsilon_{mnp}\left[d\log\left(V/\bar{V}\right)\ -\ 2i\mathcal{Q}\right]_p\ . \quad (55)
$$

In order to completely fix F^Λ, we need to know $\bar{\psi}^i$ and this can be obtained by multiplying Eq. (39) with β^B and subtracting from this Eq. (40) times α^B. The result of this minor calculational challenge is

$$
\bar{\psi}^i_{\bar{A}\bar{B}}\ =\ \tfrac{i}{2\bar{V}}\ \nabla_a Z^i\ T^A{}_{\bar{B}}\ . \quad (56)
$$

One might wonder why the expressions (54) and (56) lack the necessary symmetricness in $\bar{A}$ and $\bar{B}$; Imposing this symmetricness implies that Z^i, $\bar{Z}^{\bar{\imath}}$,

V and $\bar{V}$ are t-independent, which, seeing that it is the direction associated to a Killing vector, should not be too big a surprise.

We are now in a position to calculate F^Λ: using Eq. (41) together with Eqs. (54,56) we can see that

$$F^\Lambda \;=\; \mathrm{Im}(\mathcal{S}^\Lambda)\wedge(dt+\omega) \;-\; \tfrac{1}{2V\bar{V}}\varepsilon_{mnp}dx^{mn}\mathrm{Re}(\mathcal{S}^\Lambda)_p \;, \tag{57}$$

where for convenience we have defined

$$\mathcal{S}^\Lambda \;=\; \bar{\mathcal{L}}^\Lambda\mathfrak{D}\bar{V} \;-\; Vf_i^\Lambda dZ^i \;=\; \bar{\mathcal{L}}^\Lambda\mathfrak{D}\bar{V} \;-\; V\mathfrak{D}\mathcal{L}^\Lambda \;, \tag{58}$$

where in the last step we made use of $\mathfrak{D}\mathcal{L}^\Lambda = d\mathcal{L}^\Lambda + i\mathcal{Q}\mathcal{L}^\Lambda$, the fact that $\mathcal{L}^\Lambda$ only depends on x through the scalars Z^i and $\bar{Z}^{\bar{\imath}}$ and that it is covariantly holomorphic. This last observation, then allows us to see that

$$\mathrm{Im}(\mathcal{S}^\Lambda) \;=\; -\,d\mathrm{Im}(V\mathcal{L}^\Lambda)\;, \tag{59}$$

which allows us to distill the temporal component of the gauge fields.

Calculating then the Bianchi identity $dF^\Lambda = 0$, and using Eqs. (55), one finds that

$$\vec{\partial}\left[(V\bar{V})^{-1}\mathrm{Re}(\vec{\mathcal{S}}^\Lambda)\right] \;=\; \tfrac{2\sqrt{2}i}{V\bar{V}}\vec{\partial}\mathrm{Im}(V\mathcal{L}^\Lambda)\cdot\left\{\vec{\partial}\log\left(\bar{V}/V\right) \;+\; 2i\vec{\mathcal{Q}}\right\}\;. \tag{60}$$

We can do a wee bit better by massaging expression (57) to obtain

$$F^\Lambda \;=\; d\left[-\mathrm{Im}(V\mathcal{L}^\Lambda)\,(dt \;+\; \omega)\right] \;+\; \tfrac{1}{2}dx^{mn}\;\varepsilon_{mnp}\partial_p\mathrm{Re}(\bar{\mathcal{L}}^\Lambda/V)\;. \tag{61}$$

This equation for F^Λ means that we can forget about Eq. (60), since now there is a far compacter and nicer expression we can derive, namely

$$\vec{\partial}^2\;\mathrm{Re}(\bar{\mathcal{L}}^\Lambda/V) \;=\; 0\;, \tag{62}$$

or in other words: the $(n+1)$ functions $\mathrm{Re}(\bar{\mathcal{L}}^\Lambda/V)$ are harmonic on $\mathbb{R}^3$!!

Let us then consider the EOM for the gauge fields: as one can see, one can rewrite Eq. (3) to be

$$dH_\Lambda \;\equiv\; d\left[\overline{\mathcal{N}}_{\Lambda\Sigma}\;F^{\Sigma -} \;+\; \mathcal{N}_{\Lambda\Sigma}\;F^{\Sigma +}\right] \;=\; 0\;. \tag{63}$$

This form of the equation is particularly useful if we couple it to Eq. (41) and use the definition of the monodromy matrix (15), to find

$$H^-_{\Lambda\;ab} \;=\; \left(2h_{\Lambda i}\;\bar{\psi}^i_{\bar{A}\bar{B}} \;+\; i\bar{M}_\Lambda\;\bar{\phi}_{\bar{A}\bar{B}}\right)\varepsilon_{AB}\;. \tag{64}$$

Now observe that this is just the same equation as defining F^{Λ} once we make the appropriate substitutions. This then at once allows us to conclude that the equations of motion for the gauge field are equivalent to the statement

$$\vec{\partial}^2\ \mathrm{Re}(\bar{M}_{\Lambda}/V)\ =\ 0\ . \tag{65}$$

Now this is quite surprising, and I am not quite sure what to make of it, but let me point out that

$$\mathrm{Re}(\mathcal{L}^{\Lambda}/\bar{V})\mathrm{Im}(V\mathcal{M}_{\Lambda})\ -\ \mathrm{Re}(\mathcal{M}_{\Lambda}/\bar{V})\mathrm{Im}(V\mathcal{L}^{\Lambda})\ =\ -\tfrac{1}{2}\ . \tag{66}$$

Also armed with this knowledge and a small look at [4], we can then rewrite Eq. (55) as

$$\partial_{[m}\omega_{n]}\ =\ 2\varepsilon_{mnp}\left[\mathrm{Re}\left(\bar{\mathcal{M}}_{\Lambda}/V\right)\partial_p\mathrm{Re}\left(\bar{\mathcal{L}}^{\Lambda}/V\right)\ -\ \mathrm{Re}\left(\bar{\mathcal{L}}^{\Lambda}/V\right)\partial_p\mathrm{Re}\left(\bar{\mathcal{M}}_{\Lambda}/V\right)\right]\ , \tag{67}$$

so that the integrability condition is automatically satisfied. A more compact version of the above formula is then found by introducing the real, harmonic symplectic section $\mathcal{R}$ by

$$\partial_{[m}\omega_{n]}\ =\ 2\varepsilon_{mnp}\langle\ \mathcal{R}\mid\partial_p\mathcal{R}\rangle\ \ \leftrightarrow\ \ \mathcal{R}\ =\ \left(\begin{array}{c}\mathrm{Re}\left(\bar{\mathcal{L}}^{\Lambda}/V\right)\\ \mathrm{Re}\left(\bar{\mathcal{M}}_{\Lambda}/V\right)\end{array}\right)\ . \tag{68}$$

With the knowledge at hand we can rewrite the equations of motion for the scalars, Eq. (4), to

$$0\ =\ \vec{\partial}^2 Z^i+\Gamma_{jk}{}^i\vec{\partial}Z^j\vec{\partial}Z^k\ +\ 2\vec{\mathfrak{D}}\log\bar{V}\ \vec{\partial}Z^i\ +\ i\tfrac{\bar{V}}{V}\mathcal{G}^{i\bar{\imath}}\bar{\mathcal{C}}_{\bar{\imath}\bar{\jmath}\bar{k}}\vec{\partial}\bar{Z}^{\bar{\jmath}}\vec{\partial}\bar{Z}^{\bar{k}}\ . \tag{69}$$

Now this is a bugger to check, so why not calculate $\mathfrak{D}_c\mathfrak{D}_a Z^i$ directly from the gaugino variation? In that case we will need

$$\mathfrak{D}G^{i-}\ =\ \tfrac{i}{2}\mathcal{G}^{i\bar{\imath}}\bar{f}^{\Lambda}_{\bar{\imath}}\ d\mathcal{N}_{\Lambda\Sigma}\wedge F^{\Sigma+}\ . \tag{70}$$

which one can derive using the Bianchi identities, the equations of motion for the gauge fields and the properties of Special geometry. Translating this identity to the NP conventions, one arrives at

$$\mathfrak{D}_{A\bar{B}}\bar{\psi}^{i}_{\bar{A}}{}^{\bar{B}}\ =\ -\tfrac{i}{2}\mathcal{G}^{i\bar{\jmath}}\bar{f}^{\Lambda}_{\bar{\jmath}}\ \partial_{B\bar{A}}\mathcal{N}_{\Lambda\Sigma}\ \phi^{\Sigma B}_{A}\ . \tag{71}$$

With this knowledge we can calculate $\mathfrak{D}_c\partial_a Z^i\alpha^A$ and $\mathfrak{D}_c\partial_a Z^i\beta^A$ from Eqs. (39,40). Subtracting these equations from each other, multiplying with g^{ca} and using Eq. (71), we find

$$\begin{aligned}\mathfrak{D}_c\partial^c Z^i\ &=\ 8i\ \bar{\psi}^i_{\bar{A}\bar{B}}\bar{\phi}^{\bar{A}\bar{B}}\ +\ 2\bar{f}^{i\Lambda}\partial_{\bar{\jmath}}\mathcal{N}_{\Lambda\Sigma}\phi^{\Sigma A}_{C}\ V^{-1}\partial_a\bar{Z}^{\bar{\jmath}}T^{C\bar{A}}\\ &=\ 8i\ \bar{\psi}^i_{\bar{A}\bar{B}}\bar{\phi}^{\bar{A}\bar{B}}\ -\ 2i\mathcal{G}^{i\bar{\jmath}}\partial_{\bar{\jmath}}\mathcal{N}_{\Lambda\Sigma}\phi^{\Lambda}_{AC}\phi^{\Sigma AC}\\ &=\ 2i\mathcal{G}^{i\bar{\jmath}}\left[\partial_{\bar{\jmath}}\overline{\mathcal{N}}_{\Lambda\Sigma}\ \bar{\phi}^{\Lambda}_{\bar{A}\bar{B}}\bar{\phi}^{\Lambda\bar{A}\bar{B}}-\partial_{\bar{\jmath}}\mathcal{N}_{\Lambda\Sigma}\ \phi^{\Lambda}_{AC}\phi^{\Sigma AC}\right]\ ,\end{aligned} \tag{72}$$

which is nothing but the equation of motion of the scalars, Eq. (4), in NP language. This then means that if we have an Ansatz satisfying the supersymmetry equations that satisfy the Bianchi identity and the equations of motion for the gauge fields, then the equation of motion of the scalars is identically satisfied. This is, seeing the quite horrible form of Eq. (69), quite a relief.

Another useful identity can be derived in the same manner as above, namely

$$\mathfrak{D}\left(\mathcal{T}_\Lambda F^{\Lambda -}\right) \;=\; 2i\mathcal{G}_{i\bar{\imath}}\, dZ^i \wedge \overline{G^{i-}} \quad\leftrightarrow\quad \mathfrak{D}_B{}^{\bar{B}}\bar{\phi}_{\bar{C}\bar{B}} \;=\; 2i\mathcal{G}_{i\bar{\imath}}\,\nabla_{C\bar{C}} Z^i\; \psi^{\bar{\imath}}_B{}^C \;, \tag{73}$$

in the derivation of which we, once again, assumed that the Bianchi identity and the equations of motion are satisfied. This is of great use if we want to calculate the integrability condition for the gravitino. As an example, it is easy to see that one can determine the Ricci scalar by

$$\tfrac{1}{4}R\;\epsilon_{\bar{C}} \;=\; \nabla_{B\bar{C}}\nabla^{B\bar{B}}\epsilon_{\bar{B}} \;+\; \nabla_B{}^{\bar{B}}\nabla^B{}_{\bar{C}}\epsilon_{\bar{B}} \;, \tag{74}$$

where ϵ is either $\bar{\alpha}$ or $\bar{\beta}$. A small calculation using Eq. (73) and the defining relations Eqs. (37–40), then shows that

$$R \;=\; -2\partial_c Z^i \partial^c \bar{Z}^{\bar{\imath}}\;\mathcal{G}_{i\bar{\imath}} \;, \tag{75}$$

which is nothing but the contracted version of Eq. (2). Once again, this means that the trace-part of the equation of motion is identically satisfied if we have Ansatz that solves the supersymmetry variations, the Bianchi identity and the equations of motion for the gauge fields. Of course, we still need to find out whether this also holds for the full system of equations of motion.

It is also not hard to see that the integrability condition for the gravitini actually leads to the desired result: writing

$$R_{ab} \;\equiv\; R_{AB\bar{A}\bar{B}} \;=\; \Phi_{AB\bar{A}\bar{B}} \;+\; \tfrac{1}{4}R\;\varepsilon_{AB}\varepsilon_{\bar{A}\bar{B}} \;, \tag{76}$$

where the last step is the break down of R_{ab} into irreducibles. It is then straightforward to see that

$$\tfrac{1}{2}R_{AB\bar{A}}{}^{\bar{B}}\epsilon_{\bar{B}} \;=\; \nabla_{A\bar{A}}\nabla_B{}^{\bar{B}}\epsilon_{\bar{B}} \;-\; \nabla_B{}^{\bar{B}}\nabla_{A\bar{A}}\epsilon_{\bar{B}} \;, \tag{77}$$

which follows immediately from $2\gamma^b\left[\nabla_a, \nabla_b\right]\epsilon \;=\; R_{ab}\gamma^b\epsilon$. Calculating the l.h.s. through the supersymmetry equations, making use of Eqs. (56,73) and the identity $2T_{A\bar{A}}T^{A\bar{B}} = V\bar{V}\delta_{\bar{A}}{}^{\bar{B}}$, one can see that a solution of the supersymmetry equations that also solves the Bianchi identity and the equations of motion for the gauge fields, must satisfy

$$0 \;=\; R_{ab} \;+\; 2\partial_{(a}Z^{i}\partial_{b)}\bar{Z}^{\bar{\jmath}}\;\mathcal{G}_{i\bar{\jmath}} \;-\; 8\mathrm{Im}(\mathcal{N})_{\Lambda\Sigma}\;\bar{\phi}^{\Lambda}_{\bar{A}\bar{B}}\;\phi^{\Sigma}_{AB}\;, \tag{78}$$

which is of course nothing but the Einstein equation (Eq. (2)).

3.1 Summary of the degenerate case and Max. Susy Sols.

From the above discussion it is clear that given an Ansatz for a solution that solves the supersymmetry equations, the equations of motion for the metric and the scalars are identically satisfied iff the Ansatz solves the Bianchi identity and the equations of motion for the gauge fields. Since in the degenerate case we have solved for the most general Ansatz solving the supersymmetry equations, the only real constraints from the equations of motions are Eqs. (62,65), which combined state that

$$\mathcal{R} \;=\; \mathrm{Re}\left(\mathcal{V}/\bar{V}\right)\;, \tag{79}$$

where $\mathcal{R}$ is a symplectic vector of real, harmonic functions on $\mathbb{R}^3$. W.r.t. the metric in Eq. (52), the gauge fields then take on the form dictated by Eq. (61) and the "dragging factor" is determined by Eq. (68). In order to then completely fix the solution, we must find an expression for $V\bar{V}$.

In fact, $V\bar{V}$ is determined by the normalization of the section $\mathcal{V}$, see Eqs. (7), to be

$$1/V\bar{V} \;=\; i\langle\;\mathcal{V}/\bar{V}\;\mid\;\overline{\mathcal{V}}/V\;\rangle\;, \tag{80}$$

which means that we must invert Eq. (79). As was aptly observed in [4], Eq. (79) is nothing but the equation determining the stabilization of the moduli [5], but where the charges are substituted by harmonic functions.

Generically the solutions in this class preserve half of the available supersymmetries. This can be seen by rewriting Eq. (56) to $2\bar{V}G^{i-}_{ab} = i(dZ^i\wedge T)^-_{ab}$ and plugging this expression into the gaugino variation (6). The result of this operation is

$$i\not{\nabla}Z^{i}\left(\;\epsilon^{\aleph}\;+\;\bar{V}^{-1}\not{T}\;\epsilon_{\beth}\varepsilon^{\aleph\beth}\right)\;=\;0\;, \tag{81}$$

which is consistent due to $\epsilon^{*}_{\aleph} = \epsilon^{\aleph}$ and Eq. (46).

In order to have a chance of having 4+ solutions, however, then means that the scalars are constant and that all Special Geometry attributes are constants over spacetime. This implies that all 4+ solutions of $D=4$ $N=2$

Sugra coupled to vector multiplets are embeddings of minimal $D = 4$ $N = 2$ Sugra, and especially the 3 vacua: Minkowski, Robinson-Bertotti and KG4 [6]. The question then arises whether there are solutions that preserve 5, 6 or 7 supersymmetries.[3] Arguments of generalized holonomy [7], however, show that such solutions do not exist.

4 Degenerate case

In this case we have that

$$\alpha_A \beta^A \ = \ 0 \ \ \rightarrow \ \ \beta \sim \bar{V} \alpha \ , \tag{82}$$

where $\bar{V}$ is an arbitrary, complex function.[4]

Substituting the degeneracy condition in Eqs. (39,40) and subtracting we find that

$$0 \ = \ 4 \left(V\bar{V} \ + \ 1 \right) \bar{\psi}^i_{\bar{A}\bar{B}} \ \bar{\alpha}^{\bar{B}} \ , \tag{83}$$

which together with the symmetricness of $\bar{\psi}^i_{\bar{A}\bar{B}}$ implies

$$\bar{\psi}^i_{\bar{A}\bar{B}} \ = \ \bar{\psi}^i \bar{\alpha}_{\bar{A}} \bar{\alpha}_{\bar{B}} \ \ \rightarrow \ \ \alpha^A \nabla_{A\bar{A}} Z^i \ = \ 0 \ . \tag{84}$$

From the gravitino variations we can derive that

$$\nabla_a V \, \bar{\alpha}_{\bar{B}} \ = \ -\sqrt{2} \left(V\bar{V} + 1 \right) \ \bar{\phi}_{\bar{A}\bar{B}} \ \alpha_A \ , \tag{85}$$

from which one can derive $\bar{\phi}_{\bar{A}\bar{B}} = \bar{\phi} \ \bar{\alpha}_{\bar{A}} \bar{\alpha}_{\bar{B}}$, by contraction with $\bar{\alpha}^{\bar{B}}$. The vector $l_a = \alpha_A \bar{\alpha}_{\bar{A}}$ satisfies $dl = 0$, so that $l = du$, and then it follows that

$$\nabla_a V \ = \ -\sqrt{2} \left(V\bar{V} + 1 \right) \ \bar{\phi} \ l_a \ , \tag{86}$$

meaning that V and $\bar{\phi}$ are functions of u only. This means that we can define a spinor $O_A = Y(u)\alpha_A$ such that we can get rid of the gauge field in the expression for its covariant derivative. Since it is the gauge field contribution that generates the twisting of l, the new $L_a = O_A \bar{O}_{\bar{A}}$ will be null, exact and twist-free, whence covariantly constant.

---

[3] In lack of a better place to put this: Observe that the near-horizon limit of Schwarzschild-Taub-NUT is not supersymmetric, nor is it a solution. Neither is its asymptotic limit a solution. Yep, Taub-NUT is one hell of a strange solution.

[4] We take the case $\bar{V} \neq 0$, since the case that $\bar{V} = 0$ will lead to the system (87–91) with $\bar{\phi}_{\bar{A}\bar{B}} = 0$.

All of this allows for the introduction of the normalized tetrad, $O_A I^A = 1$, which then satisfies

$$
\begin{aligned}
\nabla_a \bar{O}_{\bar{B}} &= -\tfrac{i}{2}\mathcal{Q}_a \bar{O}_{\bar{B}} \; , && (87)\\
\nabla_a \bar{I}_{\bar{B}} &= \tfrac{i}{2}\mathcal{Q}_a \bar{I}_{\bar{B}} \; + \; \bar{P}_a \; \bar{O}_{\bar{B}} \; , && (88)\\
0 &= O^A \nabla_{A\bar{A}} \; Z^i \; , && (89)\\
\bar{\phi}_{\bar{A}\bar{B}} &= \bar{\phi}(u) \; \bar{O}_{\bar{A}} \bar{O}_{\bar{B}} \; , && (90)\\
\bar{\psi}^i_{\bar{A}\bar{B}} &= \bar{\psi}^i \; \bar{O}_{\bar{A}} \bar{O}_{\bar{B}} \; , && (91)
\end{aligned}
$$

where we have introduced coordinates[5] u and v defined by $L_a d\xi^a = du$ and $L^a\partial_a = \partial_v$. Also observe that $\bar{\psi}^i$ is not restricted to depend only on u.

Eq. (89) is of special interest, since it not only states that $\partial_v Z^i = 0$, but also that $\delta Z^i = 0$, stating clearly that Z^i can at most depend on two coordinates: u is the obvious one, and we will call the other one z. It is then clear that in the degenerate case we can always set up a coordinate representation for the tetrad such that[6]

$$
\begin{array}{lclcclcl}
L &=& du &,& D &=& \partial_v \; , \\
N &=& dv + H du + A d\bar{z} + \bar{A} dz &,& \Delta &=& \partial_u - H\partial_v \; , \\
M &=& -\bar{\omega}^{-1} \; dz &,& \delta &=& \omega\left(\partial_{\bar{z}} - A\partial_v\right) \; , \\
\bar{M} &=& -\omega^{-1} \; d\bar{z} &,& \bar{\delta} &=& \bar{\omega}\left(\partial_z - \bar{A}\partial_v\right) \; ,
\end{array}
\qquad (92)
$$

where one should observe that we discarded possible du terms in M, $\bar{M}$ and Δ since they lead to redefinitions of A, $\bar{A}$ and H once we consider the metric. Furthermore, since ∂_v is a Killing vector, all the functions that appear are necessarily v-independent.

It is then easy to give a coordinate expression for the field strengths:

$$
F \;=\; (F^\Lambda, H_\Lambda)^T \;=\; \left(2\mathcal{U}_i \; \bar{\psi}^i + i\bar{\mathcal{V}} \; \bar{\phi}\right) \; L \wedge \bar{M} \; + \; \text{C.C.} \; . \qquad (93)
$$

We also have that

$$
\mathrm{Im}(\mathcal{N})_{\Lambda\Sigma} \; T^{(\Lambda\Sigma)}_{ab} \;=\; 4\left(\bar{\psi}^i \mathcal{G}_{i\bar{\imath}} \psi^{\bar{\imath}} \; + \; \tfrac{1}{4}\phi\bar{\phi}\right) \; L_a L_b \; , \qquad (94)
$$

which will come in handy whilst discussing the equations of motion.

By making use of Eq. (41) and the various results in this section, we can rewrite the Bianchi identities and the equations of motion for the gauge fields as

[5] Even though the coordinate u used in these expressions is not the coordinate u used before, we'll use it anyway and hope that this practice will not lead to confusion.

[6] This coordinate representation can be set up independently of whether Z depends on z or not, since it is the maximal form compatible with a covariantly constant null vector.

Let us rewrite Eq. (93) by introducing the explicit explicit expressions for the tetrad in Eq. (92). This then leads to

$$F \;=\; -\omega^{-1}\left(2\mathcal{U}_i\;\bar{\psi}^i + i\bar{\mathcal{V}}\;\bar{\phi}\right)\;du\wedge d\bar{z}\;+\;\text{C.C.}\;. \tag{95}$$

The combination of the Bianchi and the equation of motion are restated as $dF=0$, so that a tiny calculation yields

$$\partial_z\left[\omega^{-1}\left(2\mathcal{U}_i\;\bar{\psi}^i\;+\;i\bar{\mathcal{V}}\;\bar{\phi}\right)\right]\;=\;\partial_{\bar{z}}\left[\bar{\omega}^{-1}\left(2\bar{\mathcal{U}}_{\bar{\imath}}\;\psi^{\bar{\imath}}\;-\;i\mathcal{V}\;\phi\right)\right]\;. \tag{96}$$

Let us define[7] $\bar{\Psi}^i=\bar{\psi}^i/\omega$, then using Eqs (11) and the fact that $\partial_z\left(e^{-\mathcal{K}/2}\bar{\mathcal{V}}\right)=\partial_z\Omega=0$, we can expand the above equation to be

$$\begin{aligned} 0 \;&=\; 2\left[\mathfrak{D}_z\overline{\Psi}^i + i\partial_{\bar{z}}\bar{Z}^{\bar{\jmath}}\;\mathcal{G}^{i\bar{\imath}}\;\bar{\mathcal{C}}_{\bar{\imath}\bar{\jmath}\bar{k}}\;\Psi^{\bar{k}}\right]\;\mathcal{U}_i\;+\;c.c. \\ &\quad + ie^{-\mathcal{K}/2}\partial_z\left(e^{\mathcal{K}/2}/\omega\right)\;\bar{\mathcal{V}}\;\bar{\phi}(u)\;+\;c.c. \end{aligned} \tag{97}$$

Using the orthogonality of the symplectic sections, see Eq. (9), we see that

$$\mathfrak{D}_z\overline{\Psi}^i\;=\;-i\partial_{\bar{z}}\bar{Z}^{\bar{\jmath}}\;\mathcal{G}^{i\bar{\imath}}\;\bar{\mathcal{C}}_{\bar{\imath}\bar{\jmath}\bar{k}}\;\Psi^{\bar{k}}\;, \tag{98}$$

and that $\partial_z(e^{\mathcal{K}/2}/\omega)\bar{\phi}=0$. So in particular, if $\bar{\phi}\neq 0$ then we must have $\omega=e^{\mathcal{K}/2}B(u,\bar{z})$.

Following Tod, we then distinguish between two cases: the Easy/degenerate-degenerate case when the Z only depend on u, and the Hard/non-degenerate-degenerate case, when at least one Z has a dependence on z. Before analyzing these two cases, let us first have a look at the (gaug/dilat/vector)ino variations:

4.1 Analysis of the Killing spinor equation

Before plunging into the marvelous world of solutions, let us analyse the Killing spinor equation with what we already know.

Let us start with Eq. (6): this turns out to be

$$0\;=\;i\Gamma^+\Delta Z^i\epsilon^{\aleph}\;-\;i\Gamma\;\bar{\delta}Z^i\epsilon^{\aleph}\;-\;2\bar{\psi}^i\;\Gamma^+\bar{\Gamma}\;\epsilon_{\beth}\varepsilon^{\aleph\beth}\;. \tag{99}$$

where we have introduced a base for the Γ-matrices such that $\{\Gamma^+,\Gamma^-\}=2$ and $\{\Gamma,\bar{\Gamma}\}=-2$. In fact $\Gamma=\Gamma^{\omega}$ but it is just less confusing to use Γ. Also, since $\sqrt{2}\Gamma=\Gamma^2+i\Gamma^3$ and $\sqrt{2}\bar{\Gamma}=\Gamma^2-i\Gamma^3$ we have that $B\Gamma B^{-1}=-\bar{\Gamma}^*$, $B\bar{\Gamma}B^{-1}=-\Gamma^*$ and also $B\Gamma^{\pm}B^{-1}=-\Gamma^{\pm *}$. All of this must seem rather

[7] Observe that $\bar{\Psi}^i$ is a weight $(-1,1)$ object.

strange to the Majorana-oriented mind, but it is all perfectly normal for a Weyl representation.

The first thing to see is that of course $\bar{\Gamma}\epsilon_{\cdot} = B\,(\Gamma\epsilon^{\cdot})^{*}$. If we couple this to $\Gamma\epsilon^{\cdot} = -\Gamma^{02}\Gamma^{+}\epsilon^{\cdot}$, then we see that one either has to impose $\Gamma^{+}\epsilon^{\cdot} = 0$, which then also implies $\Gamma\epsilon^{\cdot} = \bar{\Gamma}\epsilon_{\cdot} = 0$, or that the Z's are constants and $\bar{\psi}^{i} = 0$. As is obvious, the possibilities in the latter case are the ones of minimal $N = 2$ $d = 4$ and were discussed by Tod, and do contain the Kowalski-Gliman wave.

Anyway, we like history and will therefore copy Tod in considering 2 cases: Cases in which the scalar only depend on u, to which we'll refer as Easy cases, and the general case which we'll call hard cases.

4.2 Easy cases!

This means that no Z^{i} has a dependence on z, whence Z^{i} and $\bar{Z}^{j}$ are functions of u only. The equation of motion for the metric in the $z\bar{z}$ direction then indicates that, using the terminology of Appendix (B.2), $\partial_{z}\partial_{\bar{z}}U = 0$, which then implies that we can get rid of U by coordinate transformations that do not change the set-up, whence we take $U = 0$. Plugging this result into the equation of motion for the metric in the uz direction we find that $\partial_{z}(\partial_{z}A - \partial_{\bar{z}}\bar{A}) = 0$ and coupling this to its complex conjugate, we find that

$$\partial_{z}A - \partial_{\bar{z}}\bar{A} \;=\; 2iS(u)\;. \tag{100}$$

By making the coordinate transformation $z \to z\;e^{iS(u)}$, we can transform $S(u)$ away, after which a simple $v \to v + \ldots$ redefinition, is enough to get rid of A altogether.

As far as the other equations of motions are concerned, it is easy to see that the equations of motion for the scalars are trivially satisfied as $g^{uu} = 0$. Furthermore, since all the Special Geometry objects only depend on u, we can rewrite Eq. (96) to

$$\mathcal{U}_{i}\;\partial_{z}\bar{\psi}^{i} \;=\; \bar{\mathcal{U}}_{\bar{\imath}}\;\partial_{\bar{z}}\psi^{\bar{\imath}}\;, \tag{101}$$

Contracting this equation with $\mathcal{U}_{j}$, then states

$$i\mathcal{G}_{j\bar{\imath}}\;\partial_{\bar{z}}\psi^{\bar{\imath}} \;=\; 0 \quad\to\quad \psi^{\bar{\imath}} \;=\; \psi^{\bar{\imath}}(u,z)\;, \tag{102}$$

and likewise for $\bar{\psi}^{i}$.

Returning then to the remaining equation of motion, we see that the wave profile H is determined by

$$H \;=\; \left[\mathcal{G}_{i\bar{\imath}}\dot{Z}^{i}\dot{\overline{Z}}^{\bar{\imath}} \;+\; 2\;\bar{\phi}\phi\right]\;z\bar{z} \;+\; 8\Upsilon^{\bar{\imath}}\mathcal{G}_{i\bar{\imath}}\bar{\Upsilon}^{i}\;, \tag{103}$$

where the Υ´s are such that $\partial_z \Upsilon^{\bar{\imath}} = \psi^{\bar{\imath}}$.

So concluding, the Easy cases comprise of pp-waves for which the wave profile need not be quadratic in the transverse coordinates.

Since in this case we have that $U = A = \bar{A} = 0$ and that $\mathcal{Q}_\omega = \mathcal{Q}_{\bar{\omega}} = 0$, the analysis of the gravitino equation is straightforward: First of all, as is usual for a pp-wave the Killing spinors cannot depend on v as one can see from $\delta\Psi_{\aleph -} = 0$. Also, $\delta\Psi_{\aleph\ \omega} = 0$ states that $\epsilon_\aleph = \epsilon_\aleph(\bar{z}, u)$ and hence also $\epsilon^\aleph = \epsilon^\aleph(z, u)$. The variation in the $\bar{\omega}$-direction then reads

$$\partial_{\bar{z}} \epsilon_\aleph \ + \ \bar{\phi}(u)\Gamma^{+} \epsilon^{\beth} \varepsilon_{\aleph\beth} \ = \ \partial_{\bar{z}} \epsilon_\aleph \ = \ 0 \ , \tag{104}$$

so that the Killing spinors depend on u only.

The remaining gravitino equation, after using the various results found up to now, reads

$$0 \ = \ \partial_u \epsilon_\aleph \ - \ \bar{\phi}(u)\ \overline{\Gamma} \epsilon^{\beth} \varepsilon_{\aleph\beth} \ , \tag{105}$$

which in principle can be solved.

4.3 Hard cases

A first result can be obtained by applying Eq. (133) to Z^i: making use of the known connection coefficients we can derive[8]

$$0 \ = \ \left(\delta - \tfrac{1}{2}\delta\mathcal{K}\right)\ \bar{\delta} Z^i \ \ \rightarrow \ \ \partial_{\bar{z}} \left(e^{-\mathcal{K}/2} \bar{\omega} \partial_z Z^i \right) \ = \ 0 \ , \tag{106}$$

which clearly states that $\omega = \exp\left(\mathcal{K}/2 + C(u, \bar{z})\right)$. From the point of view of the metric, however, one can get rid of C by a coordinate transformation, so that we will take $C = 0$. W.r.t. the normalization of the metric in Appendix (B.2), one sees that $U = -\mathcal{K}/2$, and the Einstein equation in the $z\bar{z}$ direction is easily checked by looking at Eq. (142).[9]

The equation of motion for the metric in the uz-direction and its complex conjugate then states that

$$\tfrac{1}{2} e^{\mathcal{K}} \left(\partial_z A - \partial_{\bar{z}} \bar{A} \right) \ = \ i\mathcal{Q}_u \ . \tag{107}$$

Let us analyze the gravitino equation at this point since it might give some extra input. As one can see from the cases, the common factor is that

[8]This is of course also derived in Eq. (98).

[9] Let us point out a typo in Tod's article [8]: The metric has the part $\bar{M} \otimes M + M \otimes \bar{M}$, which together with [8, (A.9–10)] indicate that the metric factor should be $e^{-2\phi}\omega\bar{\omega}$ and not Eq. [8, (A.17)], which is just the inverse.

one has to impose $\Gamma\epsilon^{\aleph} = \bar{\Gamma}\epsilon_{\aleph} = 0$. Doing this in $\delta\Psi_{\aleph\ \omega}$ and using the fact that $2\Gamma\bar{\Gamma} = [\Gamma, \bar{\Gamma}] - 2$, we see that

$$0 \;=\; \mathfrak{D}_{\omega}\epsilon_{\aleph} \;=\; \theta_{\omega}\epsilon_{\aleph} + \tfrac{1}{2}\theta_{\omega}U\ \epsilon_{\aleph} + \tfrac{i}{2}\mathcal{Q}_{\omega}\epsilon_{\aleph} \;=\; \theta_{\omega}\epsilon_{\aleph}\ , \tag{108}$$

where we made use of the fact that $U = -\mathcal{K}/2$ and that $2i\mathcal{Q}_{\omega} = \theta_{\omega}\mathcal{K}$. So once again we find $\epsilon_{\aleph} = \epsilon_{\aleph}(\bar{z}, u)$ and $\epsilon^{\aleph} = \epsilon^{\aleph}(z, u)$.

The covariant derivative in the $\bar{\omega}$ direction in its full glory reads

$$\mathfrak{D}_{\bar{\omega}}\epsilon_{\aleph} \;=\; \theta_{\bar{\omega}}\epsilon_{\aleph} + \tfrac{1}{2}\left(\theta_{+}U + i\mathcal{Q}_{+}\right)\Gamma^{+}\Gamma\epsilon_{\aleph} - \tfrac{1}{2}\theta_{\bar{\omega}}U\left(\Gamma\bar{\Gamma} + 1\right)\epsilon_{\aleph} - \tfrac{1}{4}\theta_{\bar{\omega}}\mathcal{K}\ \epsilon_{\aleph}\ , \tag{109}$$

so that the covariant derivative in the $\bar{\omega}$-direction is just the ordinary partial derivative. The variation in the $\bar{\omega}$-direction then simply becomes

$$0 \;=\; \theta_{\bar{\omega}}\epsilon_{\aleph} \;+\; \bar{\phi}\Gamma^{+}\epsilon^{\beth}\varepsilon_{\aleph\beth} \;=\; \theta_{\bar{\omega}}\epsilon_{\aleph}\ , \tag{110}$$

so that, once again, the Killing spinor depends only on u.

One thing should be observed though: the fact that the Killing spinor depends only on u is due to the fact that we chose to get rid of $C(u, \bar{z})$ in the solution of Eq. (106). Seeing however that to introduce it is equivalent to a Kähler transformation and that $\epsilon.$ has a definite weight under Kähler transformations, it can be reinstated with great ease.

The gravitino variation in the u-direction then reads

$$0 \;=\; \partial_{u}\epsilon_{\aleph} \;-\; \tfrac{1}{2}\left(e^{-U}\theta_{+}\bar{A} - \theta_{\omega}H\right)\Gamma^{+}\Gamma\epsilon_{\aleph} \;-\; \bar{\phi}(u)\bar{\Gamma}\ \epsilon^{\beth}\varepsilon_{\aleph\beth}\,, \tag{111}$$

which leads to Eq. (105).

There is only one equation of motion remaining, namely the Einstein equation in the uu direction. This is however not too enlightening nor too beautiful so we'll abstain from giving it here. Let us however point out that, since we can choose $\partial_{z}A + \partial_{\bar{z}}\bar{A}$ the way we like, basically since we have the freedom to $v \rightarrow v + h(u, z, \bar{z})$, we can take it such that

$$\partial_{z}A \;=\; -\partial_{u}Z^{i}\ \partial_{i}e^{-\mathcal{K}}\ , \tag{112}$$

leading to a small simplification of the Einstein equation in the UU direction.

A Conventions et cetera

Since we shall be using Penrose's methods in part of this PoTP, it is probably a reasonably good idea to spell a few things out: Penrose's method is heavily based on the isomorphism $\mathfrak{so}(1,3) \sim \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. This isomorphism means that one can label a representation of the Lorentz group by two half-integers (j,k). It is of course also true that $\mathfrak{so}(4) \sim \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and the difference lies in the action of conjugacy: in the Lorentzian case we have that complex conjugation acts on a representation as the map $* : (j,k) \to (k,j)$. In this notation a vector, which is an irrep, is then seen to be $(1/2,1/2)$ and a Weyl spinor is, depending on the chirality, either $(1/2,0)$ or $(0,1/2)$. Using $\mathfrak{su}(2)$-representation theory one can then also see that $(1/2,0) \otimes (0,1/2) = (1/2,1/2)$, or rather, one can see a vector as the product of two Weyl spinors of different chirality. The last thing we then need is the isomorphism $\mathfrak{su}(2) \sim \mathfrak{usp}(1)$, which means that a Weyl spinor can be seen as an $\mathfrak{usp}(1)$-vector. $\mathfrak{usp}(1)$ by definition leaves invariant the 2-dimensional symplectic form and it is this symplectic form that we are going use throughout. Let us take a base for the $(1/2,0)$ rep. by the vectors α_A and β_A, normalized to $\alpha_A \varepsilon^{AB} \beta_B \equiv \alpha_A \beta^A = 1$, where $\varepsilon^{AB} = -\varepsilon^{BA}$ and $\varepsilon^{12} = 1$. There was a rapid step, so we will spell it out: $\alpha^A = \varepsilon^{AB} \alpha_B$ and $\alpha_A = \alpha^B \varepsilon_{BA}$, so that we take $\varepsilon^{AB} \varepsilon_{CB} = \varepsilon_{BC} \varepsilon^{BA} = \delta_C{}^A$. Observe that this means that $\varepsilon_{12} = 1$, which is not (??) a typical PRC way of writing things but matches with [9], which we are following in order to make life a tad less complicated.

Using said isomorphisms, we can write an $\mathfrak{so}(1,3)$ spinor in the Weyl representation as $\Psi^T = (\psi_A, \chi_{\bar{A}})$, where $A, \bar{A} = 1,2$ and ψ sits in the $(1/2,0)$ rep. and χ in the $(0,1/2)$. Switching between representations is done by complex conjugation, which, since the Weyl spinor can be a complex vector, acts as $(\psi_A)^* = \overline{\psi}_{\bar{A}}$, where one puts a $^-$ to indicate that it now transforms in the $(0,1/2)$-irrep. In the same notation a vector, when decomposed w.r.t. the symplectic base is written as $V_{A\bar{A}}$, which is evidently real when $\overline{V}_{\bar{A}A} = V_{A\bar{A}}$. A Majorana spinor is a spinor which is invariant under the $*$-operation and hence can be parameterized in this Weyl base as $\Psi^T_{Maj} = (\psi_A, \overline{\psi}_{\bar{A}})$. An important question is about the representation of the 2-tensors: Let us have a look at the $\mathfrak{so}(1,3)$ representation theory. A product of 2 vectors gives $(1/2,1/2) \otimes (1/2,1/2) = (1,1) \oplus (1,0) \oplus (0,1) \oplus (0,0)$. The last one is the trace part, and $(1,0) \oplus (0,1)$ is the 2-form contribution and the $(1,1)$ is a traceless symmetric 2-tensor. The 2-tensor we want an expression for is the Minkowski metric η_{ab}, and since this is invariant it occurs in the $(0,0)$ part of the above expression. The occurrence of the 0 is significant since due to the $\mathfrak{su}(2)$ rules the 1 is associated to the symmetric part in the Clebsch-Gordon series $1/2 \otimes 1/2 = 1 \oplus 0$, whereas the zero is associated to the antisymmetric part. This then means that $\eta_{A\bar{A}B\bar{B}} = \eta_{[AB][\bar{A}\bar{B}]}$, but since we are in 2-dimensions, everything that is anti-symmetric in 2 $\mathfrak{usp}(1)$-indices is proportional to ε, whence we must have $\eta_{ab} = \varepsilon_{AB} \varepsilon_{\bar{A}\bar{B}}$, where a constant of proportionality can be seen to be 1. As was remarked before, the 2-form contribution is $(1,0)\oplus(0,1)$ and therefore we have that $B_{ab} = \phi_{AB}\varepsilon_{\bar{A}\bar{B}} + \chi_{\bar{A}\bar{B}}\varepsilon_{AB}$, with ϕ and χ symmetric 2-tensors of $\mathfrak{su}(2)$. Furthermore, if B is to be a real 2-form, then we have that $\chi_{\bar{A}\bar{B}} = (\phi_{AB})^* \equiv \overline{\phi}_{\bar{A},\bar{B}}$.

From the above examples it should be clear that in the decomposition to 2-spinor only the symplectic forms and symmetric tensors can appear. And also that every n-tensor has its translation into the 2-component spinor language.

The convention for the metric is the mostly minus one, and the Clifford algebra is generated by the γ-matrices, satisfying $\{\gamma_a, \gamma_b\} = 2\eta_{ab}\mathrm{I}_4$. We also define $\gamma_{ab} \equiv \frac{1}{2}[\gamma_a, \gamma_b]$, and in order to make contact with Penrose and Rindler [10], we take the explicit representation

$$[\gamma_p]_\alpha{}^\beta \;=\; \sqrt{2}\begin{pmatrix} 0 & \varepsilon_{PA}\varepsilon_{\bar{P}}{}^{\bar{B}} \\ -\varepsilon_{\bar{P}\bar{A}}\varepsilon_P{}^B & \end{pmatrix}\;. \tag{113}$$

Since we are dealing with stuff in the Weyl representation we shall also need

$$\gamma_5 \;\equiv\; -i\gamma_{0123} \;=\; \begin{pmatrix} -\varepsilon_A{}^B & 0 \\ 0 & \varepsilon_{\bar{A}}{}^{\bar{B}} \end{pmatrix}\;. \tag{114}$$

A.1 Short dictionary

In this section we shall put together the needed translations and conventions.

Indices and meaning:

Type	Associated structure
$a,\ b,\ \ldots$	Tangent space
$m, n, \ldots$	Flat $\mathbb{R}^3$-indices
$\alpha, \beta, \ldots$	4-d spinor indices
$A, B, \ldots;\ \bar{A}, \bar{B}, \ldots$	$\mathfrak{usp}(1)$/NP indices
$i, j, \ldots;\ \bar{\imath}, \bar{\jmath}, \ldots$	holomorphic and anti-holomorphic. There are n of them.
$\Lambda, \Sigma, \ldots$	$\mathfrak{sp}(n+1)$ indices
$\aleph, \beth, \ldots$	$N = 2$ spinor indices

Due to the fact that we are in a Weyl representation, there is a matrix B such that $B\Gamma_a B^{-1} = -\Gamma_a^*$. In the Majorana representation we would take $B = 1$ but not so here: In fact given the conventions in [1], it is just $B = C\Gamma_0$, which is in fact a usual relation for a unitary representation of the Clifford algebra. In the explicit NP representation you can see that

$$B \;=\; \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}\;. \tag{115}$$

All of this means that we need to define $\epsilon^{\aleph} = B\epsilon^*_{\aleph}$, which is indeed one of those things that we are going to use in certain parts of this PoTP.

To NP and back again:

$$\begin{aligned} \varepsilon^{12} &= 1\,, & (116)\\ \varepsilon^{CA}\varepsilon_{CB} &= \delta_B{}^A\,, & (117) \end{aligned}$$

$$\begin{aligned} \xi^A &= \varepsilon^{AB}\,\xi_B\,, & (118)\\ \xi_A &= \xi^B\,\varepsilon_{BA}\,, & (119) \end{aligned}$$

$$\begin{aligned} \eta &= \mathrm{diag}\,(+,-,-,-)\,, & (120)\\ \eta_{ab} &= \varepsilon_{AB}\varepsilon_{\bar{A}\bar{B}}\,, & (121) \end{aligned}$$

$$\begin{aligned} \epsilon_{0123} &= 1\,, & (122)\\ \epsilon_{abcd} &= i\,(\varepsilon_{AB}\varepsilon_{CD}\varepsilon_{\bar{A}\bar{C}}\varepsilon_{\bar{B}\bar{D}} \;-\; \varepsilon_{AC}\varepsilon_{BD}\varepsilon_{\bar{A}\bar{B}}\varepsilon_{\bar{C}\bar{D}})\,, & (123) \end{aligned}$$

$$[\gamma_c]_\alpha{}^\beta = \sqrt{2}\begin{pmatrix} 0 & \varepsilon_{CA}\varepsilon_{\bar{C}}{}^{\bar{B}} \\ -\varepsilon_{\bar{C}\bar{A}}\varepsilon_C{}^B & \end{pmatrix}\,, \qquad (124)$$

$$\begin{aligned} [\gamma_5]_\alpha{}^\beta &= -i\,[\gamma_{0123}]_\alpha{}^\beta \\ &= \begin{pmatrix} -\varepsilon_A{}^B & 0 \\ 0 & \varepsilon_{\bar{A}}{}^{\bar{B}} \end{pmatrix}\,, & (125) \end{aligned}$$

$$\begin{aligned} F^{\pm} &= \tfrac{1}{2}\,(F \pm i\;*F) \quad ; \quad *F_{ab} = \tfrac{1}{2}\varepsilon_{abcd}F^{cd}\,, & (126)\\ F^-_{ab} &= \bar{\phi}_{\bar{A}\bar{B}}\varepsilon_{AB}\,, \quad ; \quad F^+ = \overline{F^-}\,, & (127)\\ T^{(\Lambda\Sigma)}_{ab} &= F^{(\Lambda}_{ac}F_b^{\Sigma)c} \;-\; \tfrac{1}{4}g_{ab}F^{(\Lambda}_{cd}F^{\Sigma)cd} & (128)\\ &= -2\bar{\phi}^{(\Lambda}_{\bar{A}\bar{B}}\;\phi^{\Sigma)}_{AB}\,. & (129) \end{aligned}$$

A.1.1 Newman-Penrose formalism

These are the rules for the commutators of the directional derivatives acting on scalars [9]:

$$\begin{aligned} [\Delta, D] &= (\gamma+\bar{\gamma})\;D \;+\; (\epsilon+\bar{\epsilon})\;\Delta \;-\; (\bar{\tau}+\pi)\;\delta \;-\; (\tau+\bar{\pi})\;\bar{\delta}\,, & (130)\\ [\,\delta, D] &= (\bar{\alpha}+\beta-\bar{\pi})\;D \;+\; \kappa\;\Delta \;-\; (\bar{\rho}+\epsilon-\bar{\epsilon})\;\delta \;-\; \sigma\;\bar{\delta}\,, & (131)\\ [\,\delta, \Delta] &= -\bar{\nu}\;D \;+\; (\tau-\bar{\alpha}-\beta)\;\Delta \;+\; (\mu-\gamma+\bar{\gamma})\;\delta \;+\; \bar{\lambda}\;\bar{\delta}\,, & (132)\\ \left[\,\bar{\delta}, \delta\,\right] &= (\bar{\mu}-\mu)\;D \;+\; (\bar{\rho}-\rho)\;\Delta \;+\; (\alpha-\bar{\beta})\;\delta \;+\; (\beta-\bar{\alpha})\;\bar{\delta}\,. & (133) \end{aligned}$$

B Some useful geometrical data

This Appendix is intended to give the necessary connections, curvatures et cetera for the metrics that will turn up in this PoTP.

B.1 Conforma-stationary spacetimes

In this small section we'll present the curvatures of a metric of the type

$$ds^2 \;=\; e^{2U}\,(dt \;+\; A)^2 \;-\; e^{-2U}d\vec{x}^2 \;, \tag{134}$$

where U and ω depend only on x. A Vierbein can be defined by

$$\begin{array}{lllclll} e^0 &=& e^U\,(dt \;+\; A) & , & \theta_0 &=& e^{-U}\partial_t \;, \\[2ex] e^i &=& e^{-U}\;dx^i & , & \theta_i &=& e^U\,(\partial_i \;-\; A_i\partial_t) \;, \end{array} \tag{135}$$

which allows us to calculate the Levi-Cività connection to be

$$\omega_{i0} \;=\; e^U\partial_i U\;e^0 \;-\; \tfrac{1}{2}e^{3U}\Omega_{ij}\;e^j \;, \tag{136}$$

$$\omega_{ij} \;=\; -\tfrac{1}{2}e^{3U}\Omega_{ij}\;e^0 \;-\; 2e^U\;\partial_{[i}U\;e_{j]} \;, \tag{137}$$

where we have defined $\Omega_{ij}dx^{ij} = 2dA$.

You can find the rest of the expressions in [11].

B.2 Wave-like metrics

Let us take $\eta_{+-} = -\eta_{\omega\bar{\omega}} = 1$ and take the Vierbein to be

$$\begin{array}{lllclll} e^+ &=& du & , & \theta_+ &=& \partial_u \;-\; H\partial_v \;, \\ e^- &=& dv + Hdu + Ad\bar{z} + \bar{A}dz & , & \theta_- &=& \partial_v \;, \\ e^\omega &=& e^U dz & , & \theta_\omega &=& e^{-U}(\partial_z - \bar{A}\partial_v) \;, \\ e^{\bar{\omega}} &=& e^U d\bar{z} & , & \theta_{\bar{\omega}} &=& e^{-U}(\partial_{\bar{z}} - A\partial_v) \;, \end{array} \tag{138}$$

The spin connection then is

$$\begin{array}{lll} \omega_{+\bar{\omega}} &=& \left(e^{-U}\theta_+ A - \theta_{\bar{\omega}}H\right)\;e^+ \;-\; \left(\theta_+ U - \tfrac{1}{2}e^{-U}(\theta_\omega A - \theta_{\bar{\omega}}\bar{A})\right)\;e^\omega \;, \\ \omega_{+\omega} &=& \left(e^{-U}\theta_+ \bar{A} - \theta_\omega H\right)\;e^+ \;-\; \left(\theta_+ U + \tfrac{1}{2}e^{-U}(\theta_\omega A - \theta_{\bar{\omega}}\bar{A})\right)\;e^{\bar{\omega}} \;, \\ \omega_{\omega\bar{\omega}} &=& \tfrac{1}{2}e^{-U}\left(\theta_\omega A - \theta_{\bar{\omega}}\bar{A}\right)\;e^+ \;-\; \theta_\omega U\;e^\omega \;+\; \theta_{\bar{\omega}}U\;e^{\bar{\omega}} \;. \end{array} \tag{139}$$

It is then straightforward to calculate the Ricci curvature:

$$\begin{aligned} R_{++} &= -2e^{-2U}\partial_z\partial_{\bar{z}}H + 2e^{-U}\partial_u(e^U\partial_u U) + \tfrac{1}{2}e^{-4U}\left(\partial_z A - \partial_{\bar{z}}\bar{A}\right)^2 \\ &\quad + e^{-2U}\partial_u\left(\partial_z A + \partial_{\bar{z}}\bar{A}\right) \ , \qquad (140) \\ R_{+\omega} &= e^{-U}\partial_z\left[\partial_u U \ + \ \tfrac{1}{2}e^{-2U}(\partial_z A - \partial_{\bar{z}}\bar{A})\right] \ , \qquad (141) \\ R_{\omega\bar{\omega}} &= 2e^{-2U}\partial_z\partial_{\bar{z}}U \ . \qquad (142) \end{aligned}$$

Observe that $\partial_z A + \partial_{\bar{z}}\bar{A}$ can always be taken to be zero, by a small change of coordinates $v \to v + \dots$

B.3 Kähler spaces, connections and curvatures

Let it suffice to say that on a Kähler space, which is a complex manifold, there exist complex coordinates z^i and $\bar{z}^{\bar{\imath}} = \overline{z^i}$ and a function $\mathcal{K}$ such that the line element is $ds^2 = 2\mathcal{G}_{i\bar{\imath}}\ dz^i d\bar{z}^{\bar{\imath}}$, with $\mathcal{G}_{i\bar{\imath}} = \partial_i\partial_{\bar{\imath}}\mathcal{K}$. The function $\mathcal{K}$ is called the *Kähler potential.*[10]

A straightforward calculation of the Levi-Cività connection then shows that

$$\Gamma_{jk}{}^{i} \ = \ \mathcal{G}^{i\bar{\imath}}\partial_j\mathcal{G}_{\bar{\imath}k} \ \ , \ \ \Gamma_{\bar{\jmath}\bar{k}}{}^{\bar{\imath}} \ = \ \mathcal{G}^{\bar{\imath}i}\partial_{\bar{\jmath}}\mathcal{G}_{\bar{k}i} \ . \qquad (143)$$

This then also allows us to calculate the Riemann curvature, which is not that enlightening, so we will only state that only $R_{i\bar{\imath}j\bar{\jmath}}$ is non-vanishing. The Ricci curvature is quite simple and reads

$$R_{i\bar{\imath}} \ = \ \partial_i\partial_{\bar{\imath}}\left(\tfrac{1}{2}\log\det\mathcal{G}\right) \ . \qquad (144)$$

C KG4 and the Weyl base

This section is sitting here because it is nice to have it. As is perhaps not that well-known, $N = 2$ $D = 4$ sugra admits to a maximally supersymmetric pp-wave, called KG4, that of course can be derived from the Robinson-Bertotti solution by a Penrose limit. But of course Kowalski-Glikman found the solution before anybody thought of doing the PL, so there you go..

Anyway, in the Weyl notation that is used in these notes, we can embed KG4 by taking all moduli to be constant, $U = A = \bar{A} = \bar{\psi}^i = 0$, and take $\bar{\phi}(u) = \bar{\lambda}$. Then in order for this to be a solution we must have $H = 2\lambda\bar{\lambda}\ z\bar{z}$

[10] Derivatives like $\partial_i\partial_j\mathcal{K}$ can in general not be zero, since if they were, it would imply that the Christoffel symbols and their derivatives would vanish identically. This would mean that the Riemann tensor would vanish identically, so that one would be talking about $\mathbb{C}^n$.

as one can easily see from Eq. (103). The susy variations in this case then take on the form

$$\begin{aligned}
0 &= \partial_v \epsilon_{\aleph} \ , && (145)\\
0 &= \partial_z \epsilon_{\aleph} \ , && (146)\\
0 &= \partial_{\bar{z}} \epsilon_{\aleph} \ + \ \bar{\lambda}\Gamma^{+}\epsilon^{\beth}\varepsilon_{\aleph\beth} && (147)\\
0 &= \partial_u \epsilon_{\aleph} \ + \ \bar{z}H_0\Gamma^{+}\Gamma\epsilon_{\aleph} \ + \ zH_0\Gamma^{+}\bar{\Gamma}\epsilon_{\aleph} \ - \ \bar{\lambda}\ \bar{\Gamma}\epsilon^{\beth}\varepsilon_{\aleph\beth} \ , && (148)
\end{aligned}$$

where we have introduced the abbreviation $H_0 = \lambda\bar{\lambda}$ So, from Eq. (145,146) we see that $\epsilon_{\aleph}$ only depends on $\bar{z}$ and u, so that due to the identification $\epsilon^{\aleph} = B\epsilon^{*}_{\aleph}$, $\epsilon^{\aleph}$ depends only on z and u. Eq. (147) can then be solved by a u-dependent Weyl spinor $\chi_{\aleph}$ such that

$$\epsilon_{\aleph} \ = \ \chi_{\aleph} - \bar{z}\bar{\lambda}\Gamma^{+}\chi^{\beth}\varepsilon_{\aleph\beth} \ \rightarrow \ \epsilon^{\aleph} \ = \ \chi^{\aleph} \ + \ z\lambda\Gamma^{+}\chi_{\beth}\varepsilon^{\aleph\beth} \ . \tag{149}$$

If we then plug the above expression into Eq. (148) we see that the z-dependent terms vanish identically, whereas the $\bar{z}$-independent term, which therefore have to cancel by themselves, are

$$\partial_u\epsilon_{\aleph} \ = \ \bar{\lambda}\ \bar{\Gamma}\ \epsilon^{\beth}\varepsilon_{\aleph\beth} \ \longrightarrow \ \partial_u\epsilon^{\aleph} \ = \ -\lambda\ \Gamma\ \epsilon_{\beth}\varepsilon^{\aleph\beth} \ , \tag{150}$$

which is integrable as one can see by going to the Majorana representation. After plugging the last expression of Eq. (150) into the remaining terms one sees that Eq. (148) is satisfied identically without having to impose any restriction on the spinor $\chi_{\aleph}$.

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