Unfrozen hyperscalars and supersymmetry

Patrick Meessen[*]

Instituto de Física Teórica, Universidad Autónoma de Madrid, C-XVI
Cantoblanco, E-28049 Madrid, Spain

Key words supergravity, supersymmetric solutions, hypermultiplets.
PACS 04A25

We discuss the effect that non-constant hyperscalars have in supersymmetric solutions, discuss the equations governing such solutions and show some intriguing solutions.

Hypermultiplets are quite peculiar in that they couple to other supermultiplets only gravitationally; it is however a mistake to think that this dissociation is not accompagnied by somatization.

Since the coupling of the hypermultiplets to the other multiplets is only gravitationally, there is no obstruction whatsoever to considering supersymmetric solutions with frozen, *i.e.* constant hyperscalars, which accounts in part why they have been largely ignored in the literature. In fact, only recently [1, 2, 3] has the effect of defrosting the hyperscalars on the known supergravity solutions been investigated leading to some uncommon, yet with hindsight predictable, results: turning on the hyperscalars curves the space orthogonal to the Killing direction implied by preserved supersymmetry, we shall refer to this orthogonal space as the basespace, in such a way that one preserves 1, 2 or 4 supercharges. The technical reason for this lies in the fact that the hyperscalars induce a composite $\mathfrak{su}(2)$-connection in the covariant derivative acting on the Killing spinors. Since it is $\mathfrak{su}(2)$, the trace of the vector-bilinear is still a Killing vector, which can be either a timelike or a null vector.[1] The non-singlet bilinears are however charged under this composite $\mathfrak{su}(2)$, and therefore lead to deformations of the structures that appear in the unfrozen case.

As a concrete case, let us discuss the timelike case in $N = 1$ $d = 5$ supergravity coupled to n_v vector- and n_h hypermultiplets: in the case $n_h = 0$, the supersymmetric solutions in said class are determined by a 4-dimensional hyperKähler metric, some closed, selfdual 2-forms Θ^I $(I = 0, \ldots, n_v)$ and functions H_I satisfying [5]

$$\nabla^2_{(4)}\, H_I \;-\; \tfrac{1}{4}\, C_{IJK}\, \Theta^J \cdot \Theta^K \;=\; 0\,, \tag{1}$$

where the d'Alembertian is taken w.r.t. the 4-dimensional hyperKähler metric, and the C_{IJK} are the intersection numbers defining the Chern-Simons couplings between the vector fields. Furthermore, the solutions always preserve half or all the supersymmetries.

The above equations are equivalent to the equations of motion for the gauge fields and the fact that the Θs have to be closed is equivalent to the Bianchi identities for the gauge fieldstrengths. That this is the only information needed to specify the solution completely is largely due to supersymmetry.

If we then defrost the hyperscalars, the first thing one finds is that the 4-dimensional manifold is no longer hyperKähler! In fact one finds that the $\mathfrak{su}(2)$-part of the holonomy that vanished on the hyperKähler space, now has to be such as to cancel the, pull-back of the, $\mathfrak{su}(2)$ connection induced by the hyperscalars. The equations of motion concerning the gravity- and the vector

* E-mail: patrick.meessen@uam.es, Phone: +34 91497 7041, Fax: +34 91497 8557

1 Observe that in $N = (1,0)$ $d = 6$ sugra, due to the chiral nature of the spinors, only the null case occurs [4].

multiplets then lead to the same structure as in the frozen case, but using the near-hyperKähler metric in stead of the hyperKähler one.

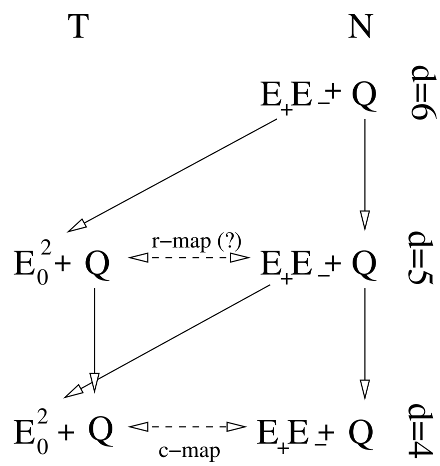
The second thing that pops up is a constraint on the hyperscalars that determines the hyperscalars to be a quaternionic map [6], *i.e.*

$$\sum_{r=1}^{3} \mathsf{J}^r{}_\mu{}^\nu\ \partial_\nu q^X\ \mathsf{J}^r_X{}^Y \ =\ \partial_\mu q^Y\ , \tag{2}$$

where J^r, *resp.* J^r, are the almost-quaternionic structures on the hypervariety and the 4-dimensional basespace. If the hyperscalars satisfy the above equation and the metric on the basespace is as above, then one can see that the solutions are generically 1/8-BPS and that the equations of motion for the hyperscalars are identically satisfied.

One should however observe that the problem is far from trivial: in order to determine the cancellation of the holonomies, we need to know the hyperscalars, but they in their turn are determined by Eq. (2) for which one already needs to know the metric on the basespace.

Some solutions to the above problem can be generated by using the existing relations between the various supergravities, which grosso modo are



In this sketch, E_0 (E_+) is the form related to the Timelike (Null) Killing vector, Q stands for the, metric on the, basespace, and solid lines indicate dimensional reduction over a spacelike direction in the space from which the line departs. More explicitly, consider the null case in $d = 6$: there we know that the general solution is of wave-type with a 4-dimensional metric on the wave-front, which is a 1-parameter family of near-hyperKähler metrics; the relevant parameter is the coordinate u in which the wave propagates. If we then want to consider dimensional reduction, we can either reduce over a circle in the basespace leading to a solution in the null case in $d = 5$, or we can reduce over a spacelike direction in the null cone, which then means that we must consider u-independent solutions and in fact means solving the same equations as discussed above. Starting from $d = 4$ makes more sense since in some cases the equations that need to be solved are easier.

One of the most enigmatic class of solutions found in [7] in the hyperless case with vector multiplets on the coset $\mathrm{Sl}(2,\mathbb{R})/\mathrm{U}(1)$, and generalised to the general case in [8], are the cosmic strings; these solutions lie in the null class and are characterised by vanishing vector fieldstrengths and the (anti-)holomorphic scalars in the vector multiplets are (anti-)holomorphic, *e.g.* $Z^i = Z^i(z)$; the spacetime metric is then given by

$$ds^2 \ =\ dt^2\ -\ dx^2\ -\ e^{-\mathcal{K}}\ dz\ d\bar{z}\ , \tag{3}$$

where $\mathcal{K}$ is the pull-back of the Kähler potential specifying the special geometry of the vector multiplets. A further, and for our purposes important, characteristic is that this class of solutions consistes of 1/2-BPS solutions and that the corresponding Killing spinor is constant. As the Killing spinor is constant, we are assured that (standard) dimensional reduction won't break supersymmetry any further, and since dimensional reduction is a fundamental ingredient in the supergravity implementation of the c-map, we are also assured that the c-mapped solution will be 1/2-BPS; the technical verification of this fact is spelled out in Ref. [1].

Seeing that this solution was created using the c-map one can ask oneself how generic it is: Indeed, since the c-map results in a restricted class of hypervarieties called *dual quaternionic manifolds*, the existence of cosmic strings for more general hypervarieties is à priori not guaranteed.

As was stated above, a generic solution with unfrozen hyperscalars is a 1/8-BPS solution and the, to the knowledge of the author, only solution known to solve the system was found by Jong, Kaya and Sezgin. In Ref. [2] they gave an explicit example with non-trivial and not-obviously-holomorphic hyperscalars taking values in the symmetric hypervariety $H_4 = \mathrm{SO}(4,1)/\mathrm{SO}(4)$.

By taking the metric on the coset to be

$$ds^2_{H4} = \frac{dq^X\ dq^X}{1\ -\ q^2} \quad (X = 1, \ldots, 4)\ , \tag{4}$$

and taking the metric on the basespace with coordinates x^m $(m = 1, \ldots, 4)$, to be conformally flat, one finds a 1/8 BPS, static, asymptotically flat, spherically symmetric, solution with unfrozen hyperscalars, but with trivial vector- and scalarfields, in the $SO(1,4)/SO(4)$ coset:

$$\begin{aligned} ds^2 &= dt^2 - \left(1 - \frac{1}{x^6}\right)^{2/3} dx^m dx^m\ , \\ q^X &= \delta^X{}_m \frac{x^m}{x^4}\ , \end{aligned} \tag{5}$$

A few comments on this solution are in order: first, concerning the statement that this solution be spherically symmetric, it is true that naively this is broken by the hyperscalars. But a spacetime SO(4) rotation can, due to the fact that the hyperscalars take values in the coset space $\mathrm{SO}(4,1)/\mathrm{SO}(4)$, always be compensated for by using a symmetry of the action.

Secondly, the region $x^2 = 1$ corresponds to a true curvature singularity which obviously is naked. Since there are no conserved charges in this system, the *no hair* conjecture suggests that black-hole type (*i.e.* spherically symmetric) solutions of this and similar systems will always be singular, but a more detailed study is needed to reach a final conclusion since they may be excluded by a mechanism like the one discussed in Ref. [9]. Furthermore, a higher-dimensional stringy interpretation of this, and similar solutions, is also needed as to interpret this singularity correctly.

As a further example let us now consider how solutions of minimal $N = 1, d = 5$ sugra[2] are deformed by the coupling to these hyperscalars. For the sake of simplicity we consider the simplest $(\Theta = 0)$, static solution which is determined by a single function K, which is harmonic w.r.t. the metric on the basespace. The supersymmetric solution can be written as

$$\begin{aligned} ds^2 &= K^{-2}\ dt^2\ -\ K\ \left(1 + \frac{\lambda}{x^6}\right)^{2/3}\ dx^m dx^m\ , \\ A &= -\sqrt{3}\ K^{-1}\ dt\,, \\ q^m &= \frac{x^m}{x^4}\ . \end{aligned} \tag{6}$$

[2] This is equivalent to taking $n_v = 0$ and $C_{111} = 1$.

If we consider the hyperscalars to be frozen and choose the harmonic function such that we end up with an asymptotically flat, spherically symmetric solution with positive mass, *i.e.*

$$K = 1 + \frac{|Q|}{x^2} \,, \tag{7}$$

we find the 5-dimensional Reissner-Nordström black hole [10], which has an event horizon at $x = 0$ that covers all singularities.

When the hyperscalars are unfrozen and we have the above base manifold, K, again determined by imposing asymptotic flatness and spherical symmetry, is given by

$$K \;=\; 1 \;+\; |Q| \;\frac{{}_2\mathrm{F}_1\left(\frac{1}{3},\frac{2}{3}\;;\;\frac{4}{3}\;;\;x^{-6}\right)}{x^2} \,, \tag{8}$$

where ${}_2F_1$ is a Gauß hypergeometric function. It is easy to see that $\lim_{x^2\to\infty} K \;=\; 1$ and that ${}_2\mathrm{F}_1\left(\frac{1}{3},\frac{2}{3}\;;\;\frac{4}{3}\;;\;x^{-6}\right)/x^2$ is a real, strictly positive and monotonically decreasing function on the interval $x^2 \in (1,\infty)$. The real question then is: what happens at $x^2 = 1$? Eq. [11, 15.1.20] gives a straightforward answer

$$ {}_2F_1\left(\,\tfrac{1}{3},\tfrac{2}{3};\tfrac{4}{3};1\,\right) \;=\; \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \;\sim\; 1.76664 \,, \tag{9}$$

which implies that there is a naked singularity at $x^2 = 1$.

The fact that we find a naked singularity in these asymptotically flat spacetimes is bothersome and one must ask oneself how generic this ocurrence of a naked singularity is. Should this behaviour be generic, then we are obliged to find out how string theory gets rid of them, the usual suspects being quantum and non-perturbative corrections. But then, it might not be a problem for string theory: the hypervariety $\mathrm{SO}(4,1)/\mathrm{SO}(4)$ is not directly obtainable by a string theory compactification,[3] and it seems highly unlikely that the embedding of $\mathrm{SO}(4,1)/\mathrm{SO}(4)$ into more general hypervarieties will respect the simple structure of the solution. Work along these directions is in progress.

Acknowledgements The author would like to thank J. Bellorín, A. Celi, M. Hübscher, T. Ortín, S. Vandoren and A. Van Proeyen for a lot of things. This work was financed in part by the *Fondo Social Europeo*, the *Comunidad Autónoma de Madrid* and the *European Union* and the author thanks them all heartily.

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