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Anti-Mach type metrics and superalgebras in M-theory

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The plan of this PoTP is the following: In the Section (1) we will review the technical details of the contruction of 16+ Hpp-waves and the various constraints that come popping up and will discuss some solutions, especially the so-called time-dependent Hpp-waves: They are distinguished from the more familiar Cahen-Wallach Hpp-waves by the fact that they are not symmetric spaces, rather they are naturally reductive naturally reductuve spaces. In Section (2) we will make use of the general prescription for attributing a superalgebra to a supersymmetric solution, see *e.g* [16], in order to derive the generic form of the superalgebra. We will then discuss the exact form of the superalgebra for the examples exhibited in Sec. (1).

1 16+ Regular Hpp-waves

In [1] it was proven that if a Hpp-was has more than 16 preserved supersymmetries, then it necessarily must be regular homogeneous plane wave [2]. This means that, in stationary coordinates, such a solution can be written as

$$ds^2 = 2du\left(dv + H_{ij}y^iy^jdu + \tfrac{1}{2}y^iF_{ij}dy^j\right) - d\vec{y}^{\,2}_{(9)} , \qquad (1)$$

$$G_{(4)} = \tfrac{1}{3!}\Theta_{ijk}\, du \wedge dy^{ijk} , \qquad (2)$$

where H, F and Θ are constant. These metrics are geodesically complete and the underlying spacetime is symmetric, *i.e.* is a Cahen-Wallach spacetime, iff $[F, H] = 0$. If this is the case, then we can always eliminate the F from the metric by a change of coordinates. Making use of the characterization of Lorentzian homogeneous spaces in [3], one can see that in general these regular homogeneous plane waves are naturally reductive.[1]

[1] In short, this boils down to the fact that we can add a totally anti-symmetric torsion, S say, to the Levi-Cività connection such that the metric, the Riemann tensor and the torsion are parallel. For the regular Hpp-waves this torsion is $S = -du \wedge F$.

The equations of motion for the field-strength are trivially satisfied and the only non-trivial equation following from the Einstein equations, is

$$H_{ij}\eta^{ij} \ - \ \tfrac{1}{8}F^2 \ - \ \tfrac{1}{4!}\Theta_{ijk}\Theta^{ijk} \ = \ 0 \ , \tag{3}$$

which is however automatically satisfied when the solution is 16+ [1].

The M-theory supersymmetry variations can be rewritten to be

$$0 \ = \ \nabla_a\epsilon \ + \ \tfrac{i}{4!}\left[3\not{G}_{(4)}\Gamma_a \ - \ \Gamma_a\not{G}_{(4)}\right]\epsilon \ , \tag{4}$$

where in our conventions for the slash, we have $\not{G}_{(4)} = \frac{1}{4!}G_{abcd}\Gamma^{abcd}$. Making then use of the fact that for the wave we have $\not{G}_{(4)} = \Gamma^+\not{\Theta}$, the above variation can be rewritten as

$$0 \ = \ \nabla_a\epsilon \ + \ \tfrac{i}{4!}\left[3\Gamma^+\not{\Theta}\Gamma_a \ + \ \Gamma_a\not{\Theta}\Gamma^+\right]\epsilon \ , \tag{5}$$

Making use of the above equations, one sees that ϵ doesn't depend on v, as is usual, and the rest of the equations are reduced to

$$0 \ = \ \partial_i\epsilon \ - \ \Gamma^+\Omega_i\epsilon \quad : \quad \Omega_i \ = \ \tfrac{1}{4}F_{ij}\Gamma^j \ - \ \tfrac{i}{4!}\left[3\not{\Theta}\Gamma_i + \Gamma_i\not{\Theta}\right] \ ; , \tag{6}$$

$$0 \ = \ \partial_u\epsilon + \tfrac{1}{2}\partial_i H \ \Gamma^{+i}\epsilon - \tfrac{1}{4}\not{F}\epsilon \ - \ \tfrac{2i}{4!}\not{\Theta}\left[\Gamma^+\Gamma^- + 1\right]\epsilon \ , \tag{7}$$

Eq. (6) can of course always be integrated to give

$$\epsilon \ = \ \epsilon^+(u) \ + \ \left(1 \ + \ \Gamma^+y^i\Omega_i\right) \ \epsilon^-(u) \ . \tag{8}$$

where $\Gamma^\pm\epsilon^\pm = 0$. so we only need to have a look at Eq. (7). By bluntly plugging the above solution into Eq. (7), however leads to a piece which doesn't depend on the transverse coordinates and one that does. This then means that we must have

$$\partial_u\epsilon^\pm \ = \ \tfrac{1}{4}\not{F}\epsilon^\pm \ + \ \tfrac{2i}{4!}\not{\Theta}\left[\Gamma^+\Gamma^- + 1\right]\epsilon^\pm \ , \tag{9}$$

which after substitution into the y-dependent part of Eq. (7) leads to the constraint

$$y^i\Gamma^+\left[12\cdot 4!\left(H_{ij} - \tfrac{1}{8}F^2_{ij}\right)\Gamma^j + 9\not{\Theta}^2\Gamma_i + 6\not{\Theta}\Gamma_i\not{\Theta} + \Gamma_i\not{\Theta}^2 + 9i\left[\not{F},\not{\Theta}\right]\Gamma_i + 3i\Gamma_i\left[\not{F},\not{\Theta}\right]\right]\epsilon^- \ = \ 0 \ . \tag{10}$$

This equation is solved by putting $\hat{\epsilon}(u) = \mathcal{P}\chi(u)$, or rather $\mathcal{P}$ projects onto the trivial subspace. This imposition of the projection operator has to be done consistently with the u-evolution, and this implies the constraint

$$\left[\ \mathcal{P} \ , \ \not{\Theta} - 3i\not{F} \ \right] \ = \ 0 \ . \tag{11}$$

It is conventient to rewrite the above equations in terms of $Cl(0,9)$, generated by γ^i which are symmetric and purely imaginary, through the rule $\Gamma^+\Gamma^i = \Gamma^+\gamma^i$, for which we then find the representation

$$\Gamma^+ \;=\; \sqrt{2}i\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \,,\; \Gamma^+ \;=\; -\sqrt{2}i\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \,,\; \Gamma^i \;=\; \begin{pmatrix} -\gamma^i & 0 \\ 0 & \gamma^i \end{pmatrix} \,. \tag{12}$$

The projection operator for the extra supersymmetries will always be written in terms of the γ matrices, and the change between the 2 notations is almost trivial.

At the end of the day, the solution for the Killing spinor is

$$\epsilon \;=\; \exp\left(\tfrac{ui}{4}\left[\not{\Theta} - i\not{F}\right]\right)\epsilon_0^+ \;+\; \left(1 \;+\; \Gamma^+ y^i\Omega_i\right)\exp\left(\tfrac{2ui}{4!}\left[\not{\Theta} - 3i\not{F}\right]\right)\epsilon_0^- \;, \tag{13}$$

where we of course have that $\mathcal{P}\epsilon_0^- = \epsilon_0^-$.

1.1 Some examples of 16+ non-Symmetric Hpp-waves

The most famous of these 16+ Hpp-waves is the maximally supersymmetric solution found by Kowalski-Glikman [4], and a great bunch of them are to be found in the works of Gauntlett and Hull [5], Pope et al. [???], where they realize 24,22,20 and 18 preserved supercharges, and in the work of Michelson [6] who finds a solution that preserves 26 supersymmetries. All of these solution are Cahen-Wallach spaces, *i.e.* they are symmetric spaces.

It is also possible to find 16+ Hpp-wave solutions that are not Cahen-Wallach: The first example of these solutions was constructed in [1] by applying a Penrose limit on the M-theory Gödel solution found in [7]. Explicitly, this solution can be represented as

$$H \;=\; diag\left(0, 2\beta^2\left(1-p^2\right), \left[\tfrac{\beta^2}{2}\right]^2, 0^5\right) \;, \tag{14}$$

$$\not{F} \;=\; -2\beta p\, \Gamma^{12} \;, \tag{15}$$

$$\not{\Theta} \;=\; -2\beta\left[\Gamma^{125} \;-\; p\Gamma^{345} \;-\; \sqrt{1-p^2}\,\Gamma^{267} \;-\; \sqrt{1-p^2}\,\Gamma^{289}\right] \,. \tag{16}$$

This familiy of solutions interpolates between 2 Cahen-Wallach spaces: One at $p = 0$, which is nothing more than the uplift of the KG5 solution [9], which preserves 20 supersymmetries, and another one at $p = 1$, which preserves 24 supersymmetries. For generic values of p, the solution preserves 20 supersymmetries and the projector is given by

$$\mathcal{P} \;=\; \tfrac{1}{4}\left(1 \;-\; \sqrt{1-p^2}\gamma^{1589} \;-\; p\gamma^{1234}\right)\,\left(1 \;-\; \gamma^{6789}\right) \tag{17}$$

As a sidenote let us mention that the Killing spinor does not depend on the coordinates $y^5 \ldots y^9$ and that we can therefore obtain solutions of type IIB that have the same kind of behaviour.[2]

The authors of Ref. [11], thought that these kind of solutions were probably inexistent in IIB, a thing that clearly doesn't stand up. The only published NR type IIB solution is to be found in Ref. [15].

Using a notation similar to the one above but adapted to the IIB case, we can find 2 time-dependent 16+ waves. By reducing over x^9 and T-dualising over x^6 one ends up with

$$\begin{aligned} A &= diag\left(0, \tfrac{1}{2}(1-p^2), [\tfrac{1}{8}]^2, [0]^4\right) \ , \\ \not{F} &= p\Gamma^{12} \ , \\ \not{H} &= -\sqrt{1-p^2}\Gamma^+\Gamma^{28} \ , \\ \not{G}_{(3)} &= \sqrt{1-p^2}\Gamma^+\Gamma^{27} \ , \\ \not{G}_{(5)} &= -\Gamma^+\left[\Gamma^{1256} \pm \Gamma^{3478} \ - \ p\,\Gamma^{3456} \mp p\Gamma^{1278}\right] \ . \end{aligned} \tag{18}$$

Another one can be obtained by dimensionally reducing over x^5 and T-dualizing over x^9, which results in

$$\begin{aligned} A &= diag\left(0, \tfrac{1}{2}(1-p^2), [\tfrac{1}{8}]^2, [0]^4\right) \ , \\ \not{F} &= p\Gamma^{12} \ , \\ \not{H} &= \Gamma^+\left[\Gamma^{12} - p\Gamma^{34}\right] \ , \\ \not{G}_{(3)} &= \sqrt{1-p^2}\Gamma^+\Gamma^{27} \ , \\ \not{G}_{(5)} &= -\sqrt{1-p^2}\Gamma^+\left[\Gamma^{2568} \pm \Gamma^{1347}\right] \ . \end{aligned} \tag{19}$$

The reasoning of [1] can however also be applied to another Gödel-like solution in M-theory, namely the $n = 4$ case in [8], resulting in [16]

$$\begin{aligned} H &= diag(0, 2\beta^2(1-p^2), [\tfrac{\beta^2}{2}]^6, 0) \ , \\ \not{F} &= -2\beta p\Gamma^{12} \ , \\ \not{\Phi} &= -2\beta\left[\Gamma^{129} \ + \ 2\sqrt{1-p^2}\ \Gamma^{278} \ + \ p\Gamma^{349} \ + \ p\Gamma^{569} \ + \ p\Gamma^{789}\right] \end{aligned} \tag{20}$$

As in the foregoing example, there are 2 points where the above family becomes a Cahen-Wallach space: At $p = 0$ we have 24 supersymmetry solution and at $p = 1$ we have a 22 supersymmetry solution. For generic values of p, one can see that the projection operator is

$$\mathcal{P} \ = \ \tfrac{1}{8}\left(1 \ + \sqrt{1-p^2}\gamma^{1789} \ + pi\gamma^9\right)\left(3 \ + \gamma^{3456} \ + \gamma^{3478} \ + \gamma^{5678}\right) \ , \tag{21}$$

from which it is paramount that the solution preserves 22 supersymmetries.

The isometry algebra also jumps in this family. Generically the rotational part of the isometry algebra is $u(2) \oplus u(1)$, but at the Cahen-Wallach

[2] Apart from the 22-dimensional Heisenberg algebra, the isometry algebra also has rotational isometries: a $u(1) \oplus u(2)$ when $p \neq 0, 1$ and at $p = 0$ this is enhanced to $u(3) \oplus u(1)$, whereas it is enhanced to $u(1) \oplus su(2)^3$ at $p = 1$.

points it gets enhanced. At $p = 0$ one has a $u(2)^2$ whereas at $p = 1$ one has a $u(4)$. A point worth observing however, is that the point where the rotational isometries increase most is not the point where the supersymmetry gets enhanced.

The only supersymmetry preserving dualization of the above solution, for generic p, to a type IIB solution we have been able to find, although rather suggestive, is by dimensionally reducing over y^9 and T-dualizing over y^1. The resulting type IIB solution reads

$$
\begin{aligned}
ds^2 &= 2du\left(dv + \tfrac{\beta^2}{2}\vec{x}^2_{(8)}du\right) - dx^2_{(8)} , \\
H_{(3)} &= -2\beta\, pdu \wedge \left[dx^{12} + dx^{34} + dx^{56} + dx^{78}\right] , \\
G_{(5)} &= 4\beta\sqrt{1-p^2}du \wedge \left[dx^{1278} + dx^{3456}\right] ,
\end{aligned} \tag{22}
$$

which after a small coordinate relabeling is just the 28 susy solution Eq. (4.23) of [11].[3]

A special case is $p = 1$. In that case one can introduce an F on all the 2-planes, thus getting rid of H; furthermore in these coordinates the Killing spinor only depends on u. This means that we can safely T-dualize it to a type IIB solution, the result of which is Michelson's 28 supersymmetry solution [6]. Also observe that this 28 IIB solution can be described as a WZW model on a Heisenberg group [10]. This is a bit overkill, since the basic fact is that although the Killing spinor doesn't depend on the transverse coordinates, you still have to take coordinates such that they are from different 2-planes. As such the result written above is completely okay, and even the limit $p = 1$ is okay. On the other extreme the point $p = 0$ corresponds to the maximally supersymmetric KGB solution. An interesting question then is: What is the *Penrent* of this solution?? *I.e.* what is the configuration that gives rise, after a PL, to this interpolating wave?

2 The Generic Superalgebra

The bosonic part of the of the superalgebra was already discussed in [2], and we'll follow their line of reasoning in this section. Two Killing vectors are obvious in the metric (1), namely $V = \partial_v$ and $U = \partial_u$. The killing vectors would be the translations and the null-rotations in the flat-space limit are of the form $\xi = C^i(u)\partial_i + x_i B^i(u)\partial_v$. Imposing that they are Killing then leads to the set of equations The condition for Killingness then read

$$
\begin{aligned}
0 &= B^i + \dot{C}^i - \tfrac{1}{2}F^i{}_jC^j , && (23)\\
0 &= \dot{B}^i + 2H^i{}_jC^j + \tfrac{1}{2}F^i{}_j\dot{C}^j . && (24)
\end{aligned}
$$

[3] For completeness, let us state that $f = -2\beta p$ and $g = 4\beta\sqrt{1+p^2}$.

Taking the first to define B, we can derive

$$\ddot{C}^i \;-\; 2H^i{}_jC^j \;+\; F^i{}_j\dot{C}^j \;=\; 0\;. \tag{25}$$

We can choose a base for the solutions to the above equations by defining the boundary conditions

$$\begin{aligned}
X_i &= C^j_{(i)}\partial_i \;+\; x_jB^j_{(i)}\partial_v \quad\rightarrow\quad \left\{\begin{array}{llll} C^j_{(i)}\Big|_{u=0} = \eta_i{}^j & , & \dot{C}^j_{(i)}\Big|_{u=0} = -F_i{}^j \\ B^j_{(i)}\Big|_{u=0} = \frac{1}{2}F_i{}^j & , & \dot{B}^j_{(i)}\Big|_{u=0} = -2\left[H-\frac{1}{4}F^2\right]^j{}_i \end{array}\right. \\
\overline{X}_i &= \overline{C}^j_{(i)}\partial_i \;+\; x_j\overline{B}^j_{(i)}\partial_v \quad\rightarrow\quad \left\{\begin{array}{llll} \overline{C}^j_{(i)}\Big| = 0 & , & \dot{\overline{C}}^j_{(i)}\Big|_{u=0} = \eta_i{}^j \\ \overline{B}^j_{(i)}\Big|_{u=0} = -\eta_i{}^j & , & \dot{\overline{B}}^j_{(i)}\Big|_{u=0} = \frac{1}{2}F_i{}^j \end{array}\right.
\end{aligned} \tag{26}$$

With the above boundary conditions we can then calculate the algebra, whose explicit form is

$$\begin{array}{lllllll}
[X_i,X_j] &=& -F_{ij}V & , & [U,X_i] &=& 2H_i{}^j\overline{X}_j \;-\; F_i{}^jX_j\;, \\
\left[\overline{X}_i,X_j\right] &=& \eta_{ij}V & , & \left[\overline{X}_i,U\right] &=& -X_i\;.
\end{array} \tag{27}$$

from which the fact that these spaces are reductive is paramount. There can, and will, also be rotational symmetries, but since the only ones that will interest us arrise from the fermions, we will not discuss them here, let it suffice to say that they must be an automorphism of the above algebra, which implies that they must commute with F and H.

In order to derive the 'supertranslation' part of the algebra, let us write the Killing spinors as Let us write the generic solution as

$$\epsilon \;=\; e^{uN^+_-}\epsilon^+_{(0)} \;+\; \left(1\;+\;\Gamma^+x^i\Omega_i\right)e^{uN^-_-}\epsilon^-_{(0)}\;, \tag{28}$$

$$\overline{\epsilon} \;=\; \overline{\epsilon^+_{(0)}}e^{uN^+_-} \;+\; \overline{\epsilon^-_{(0)}}e^{uN^-_+}\left(1-\Gamma^+x^i\overline{\Omega}_i\right)\;, \tag{29}$$

where $\Gamma^\pm\epsilon^\pm_{(0)} \;=\; 0$ (and therfore also$\overline{\epsilon^\pm_{(0)}}\Gamma^\pm \;=\; 0$) and we introduced the abbreviations

$$N^+_\pm \;=\; \frac{i}{4}\left\{\not{H} \;\pm\; i\not{F}\right\}\;, \tag{30}$$

$$N^-_\pm \;=\; \frac{2i}{4!}\left\{\not{H} \;\pm\; 3i\not{F}\right\}\;, \tag{31}$$

$$\overline{\Omega}_i \;=\; \frac{1}{8}\left[\Gamma_i,\not{F}\right] \;+\; \frac{i}{4!}\left[3\Gamma_i\not{H} \;+\; \not{H}\Gamma_i\right]\;. \tag{32}$$

With this very enlightening knowledge we can then determine $\overline{\epsilon}_1\Gamma^A\epsilon_2\theta_A$ (remember that this is the equivalent of the $\{\mathcal{Q},\mathcal{Q}\}$)

$$++ \;:\; \overline{\epsilon^+_1}\Gamma^-\epsilon^+_2\;\partial_v\;, \tag{33}$$

$$
\begin{aligned}
+- \quad &: \quad \overline{\epsilon_1^+} e^{uN_+^+} \left[\Gamma^i \partial_i \ + \ 2x^i \ X_i \partial_v \ \right] e^{uN_-^-} \epsilon_2^- \ , && (34) \\
-- \quad &: \quad \overline{\epsilon_1^-} \Gamma^+ \epsilon_2^- \ \partial_u \ - \ \overline{\epsilon_1^-} \Gamma^+ e^{-uN_-^-} \left\{ \overline{\Omega}_j \Gamma^i \ + \ \Gamma^i \Omega_j \right\} e^{uN_-^-} \epsilon_2^- \ x^j \partial_i \\
& \quad -x^i x^j \overline{\epsilon_1^-} \Gamma^+ e^{-uN_-^-} \left[H_{ij} \ - \ \tfrac{1}{8} F_{ij}^2 \ + \ 2U^T_{(i} U_{j)} \right] e^{uN_-^-} \epsilon_2^- \ \partial_v \ , && (35)
\end{aligned}
$$

where $U_i \ = \ \frac{-i}{4!} \left[3\not{\Theta}\Gamma_i + \gamma_i \not{\Theta} \right] \ = \ \Omega_i - \frac{1}{8} \left[\gamma_i, \not{F} \right]$ (I should probably mention that the transposition in the last equation is the one from the 9d Clifford algebra). anyway, since the above thing has to be expressable in terms of killing vectors and they can be at most linear in the x's, we must have that, using the fact that the projection operator commutes with N_-^-,

$$
0 \ = \ P \left[H_{ij} \ - \ \tfrac{1}{8} F_{ij}^2 \ + \ 2U^T_{(i} U_{j)} \right] P \tag{36}
$$

which is a rather surprising identity (which holds for the 2 examples we have. See the file `qqconstr.map`).

Anyway, we can always evaluate the above rules at the point $u = 0$ and the general result is, here we have that $\Gamma^{\pm} \mathcal{Q}^{\mp} = 0$ and $(1 - P) \ \mathcal{Q}^+ = 0$,

$$
\begin{aligned}
\left\{ \mathcal{Q}^-, \mathcal{Q}^- \right\} \ &= \ -\sqrt{2} i \ V && (37) \\
\left\{ \mathcal{Q}^-, \mathcal{Q}^+ \right\} \ &= \ -\gamma^i \mathcal{P} \ X_i \ + \ 2\omega_i \mathcal{P} \ \overline{X}^i \ , && (38) \\
\left\{ \mathcal{Q}^+, \mathcal{Q}^+ \right\} \ &= \ -\sqrt{2} i \mathcal{P} \ U \ + \ \sqrt{2} i \mathcal{P} \left[\omega_i^t \gamma_j + \gamma_j \omega_i \right] \mathcal{P} \ y^i \partial^j \ , && (39)
\end{aligned}
$$

where we use the $Cl(0,9)$-version of Ω_i, *e.g.*

$$
\omega_i \ = \ \tfrac{1}{8} \left[\gamma_i, \not{f} \right] \ - \ \tfrac{i}{4!} \left(3\not{\theta} \gamma_i \ + \ \gamma_i \not{\theta} \right) \ , \tag{40}
$$

and ^{t} means transposition of the 16×16-matrices. Using the expression for the Killing vectors one can derive

$$
\pounds_{\xi} \epsilon \ = \ C^i \partial_i \epsilon \ + \ \tfrac{1}{2} \dot{C}_i \Gamma^{+i} \epsilon \ = \ -C^j \Gamma^+ \Omega_i \hat{\epsilon}^- \ - \ \tfrac{1}{2} \dot{C}_i \Gamma^{+i} \hat{\epsilon}^- \ , \tag{41}
$$

which after making use of the boundary conditions (26)

$$
\begin{aligned}
\left[U, \mathcal{Q}^+ \right] \ &= \ -\tfrac{2i}{4!} \left(\not{\theta} - 3i \not{f} \right) \mathcal{Q}^+ \ , && (42) \\
\left[U, \mathcal{Q}^- \right] \ &= \ +\tfrac{i}{4} \left(\not{\theta} + i \not{f} \right) \mathcal{Q}^- \ , && (43) \\
\left[\overline{X}_i, \mathcal{Q}^+ \right] \ &= \ \tfrac{i}{\sqrt{2}} \mathcal{P} \gamma_i \ \mathcal{Q}^- \ , && (44) \\
\left[X_i, \mathcal{Q}^+ \right] \ &= \ \sqrt{2} i \mathcal{P} \left[\omega_i^t - \tfrac{1}{2} F_{ij} \gamma^j \right] \ \mathcal{Q}^- \ . && (45)
\end{aligned}
$$

At this point we have fixed the complete Lie superalegbra corresponding to 16+ Hpp-waves, and the only remaining thing that is to be done, is to have a look at some examples!

2.1 The examples

Since we now have the general formulae at our disposal, we can derive the superalgebras for our examples in Sec. (1.1) with great ease. The explicit details are not very enlightening, so that we shall only talk about what possible rotational groups can appear as supervectors. As was mentioned in Sec. (1.1) the generic rotational symmetry of the solution (14) is just $u(2) \oplus u(1)$, and this is enhanced to $u(3) \oplus u(1)$ at the point $p = 0$ and to $u(2) \oplus so(4)$ when $p = 1$. A small calculation then shows that in this case, for all values of p, only a $u(1)$ appears. For $p \neq 1$ this is the rotation on the 34-plane, whereas when $p = 1$ it is the $u(1)$ in the $u(2)$ that acts on the directions 1234. In light of the isomorphism, we are quite lucky in this case: We deform the $p = 0$ solution in the 12-plane and this breaks the rotational invariance of the solution. Supersymmetry is however not affected by this since the only the rotation in the 34 plane is sitting there.

At the point $p = 1$ the same calculations show that there is a $u(1) \oplus u(1)$ appearing, basically rotations on the 12- and 34-plane. This is however not possible since the full automorphism-group is $u(2) \oplus so(4)$, with the $u(2)$ on 1234, and the aforementioned $u(1) \oplus u(1)$ is not an ideal of this algebra, as it ought be. Even worse, we can undo F in the metric and calculate the superalgebra only to find that in that case the super translation only contains the $u(1)$ in the $u(2)$.

What is confusing in the above paragraph is nothing more than natural, and at the end rather stunning! The solutions we are dealing with allow, algebraically speaking, a host of different descriptions, or said more mathematically, a host of reductive splits. Different reductive splits, however, need not admit the same automorphism groups nor need it admit the maximal automorphism algebra. It is clear that if we look at the automorphism algebra at $p = 1$ then since we have an F on 12 but not on 34, that in the algebra the automorphism is not $u(2)$ but rather $u(1) \oplus u(1)$. Once we rotate, by a coordinate transformation, F into H, we find the maxilimal automorphism algebra $u(2)$, and susy tells us that we only pick up the $u(1)$. The stunning fact is that the supercharges understand the subtleties involved!

In the case of Sol. (20), things are a bit more involved. At $p = 0$, the rotational isometries of the bosonic solution forms a $u(2) \oplus u(2)$, and once we start turning on p, this is broken to a $u(1) \oplus u(2)$ and then enhanced to $u(4)$ when p becomes 1. As far as the rotational contribution to the superalgebra is concerned, we start of with a $u(1) \oplus so(4)$ at $p = 0$, which then is broken, together with the supersymmetry, to $u(1) \oplus u(2)$. The point $p = 1$ is delicate: my calculations indicate that it should be a u(1) from a $u(3)$ and not one from a $u(4)$. However when we dualize this case to IIB, then we find Michelson's wave and in his superalgebra one only finds the $u(1)$ in the $u(4)$. Of course, the explication is the same as before.

So we have seen that there is a subtlety involved in talking about superalgebras associated to 16+ Hpp-waves: a straightforward calculation might lead to non-sensical answers when comparing the information to the isometry algebra (assuming that one didn't make a mistake, and found the maximal isometry algebra). The conclusion then is that in order to be on the safe side you either have to point this which reductive split you are using or always talk about the reductive split admitting the maximal automorphism algebra.

3 Heisenberg algebras and Bogolyubov transformations: From anti-Mach to Cahen-Wallach.

Let us consider the 4-dimensional anti-Mach metric, and investigate the Heisenberg part of the isometry algebra.[4] In fact consider the data

$$H \;=\; diag\left(0, 2\beta^2\left(1-p^2\right)\right) \quad ; \quad F_{12} \;=\; -2\beta p \;. \tag{46}$$

An explicit representation of the Heisenberg generators can then be seen to be $U = \partial_u$, $V = \partial_v$ and

$$\begin{aligned}
\xi_1 &= \partial_1 \;-\; \beta p x_2 \partial_v \;, \\
\xi_2 &= \cos(2\beta u)\,\partial_2 \;+\; p\sin(2\beta u)\,\partial_1 \;+\; \left[\beta\left(2-p^2\right)\sin(2\beta u)\,x_2 \;-\; \beta p\cos(2\beta u)\,x_1\right]\partial_v \;, \\
\overline{\xi}_1 &= 2\beta\left(1-p^2\right)\, u\xi_1 \;-\; p\partial_2 \;-\; \beta\left[2-p^2\right]x_1\partial_v \;, \\
\overline{\xi}_2 &= \sin(2\beta u)\,\partial_2 \;-\; p\cos(2\beta u)\,\partial_1 \;-\; \left[\beta\left(2-p^2\right)\cos(2\beta u)\,x_2 \;+\; \beta p\sin(2\beta u)\,x_1\right]\partial_v \;.
\end{aligned} \tag{47}$$

In the above base, the algebra reads

$$\begin{array}{lllllll}
\left[\xi^i, \overline{\xi}^j\right] = -\eta^{ij}\, V & , & \left[U, \xi^1\right] = 0 & , & \left[U, \overline{\xi}^1\right] = 2\beta\left(1-p^2\right)\,\xi^1 \;, \\
 & & \left[U, \xi^2\right] = -2\beta\, \overline{\xi}^2 & , & \left[U, \overline{\xi}^2\right] = 2\beta\,\xi^2 \;.
\end{array} \tag{48}$$

This form of the algebra can be mapped to the generic form (27) with data (46), by the invertible map[5]

$$\begin{array}{lllll}
Y^1 = \left(1-\frac{p^2}{2}\right)\,\xi^1 \;-\; \frac{p}{2}\,\overline{\xi}^2 & , & \overline{Y}^1 = \frac{1}{2\beta}\left[\overline{\xi}^1 \;+\; p\xi^2\right] \;, \\
Y^2 = \left(1-\frac{p^2}{2}\right)\,\xi^2 \;-\; \frac{p}{2}\,\overline{\xi}^1 & , & \overline{Y}^2 = \frac{1}{2\beta}\left[\overline{\xi}^2 \;+\; p\xi^1\right] \;,
\end{array} \tag{49}$$

and more importantly to the case $p = 0$ by redefining $Z^1 = \sqrt{1-p^2}\xi^1$, $\sqrt{1-p^2}\overline{Z}^i = \overline{\xi}^1$, $Z^2 = \xi^2$ and $\overline{Z}^2 = \overline{\xi}^2$. Clearly this map breaks down

[4] As we saw in Sec. (2) these are automatically isometries of the field strength so that we can safely ignore it. Furthermore, we also saw that the general Heisenberg algebra for our examples splits into a anti-Mach part and a Cahen-Wallach part, so that the discussion in this appendix can readily be applied to other cases.

[5] Of course this needs to be adapted to the new base $2X_i = 2Y_i - F_{ij}\overline{Y}^j$ and $\overline{X}_i = \overline{Y}_i$, but you should get the idea.

when $p = 1$. Even though we cannot map the Heisenberg algebra at $p = 1$ to the one at $p = 0$, there is nothing wrong with the $p = 1$ limit. In fact everything is regular and the only thing that changes is that the algebra at $p = 1$ allows for a bigger automorphism group, as one can see by observing that at the point $p = 1$ we are dealing with a Cahen-Wallach space, which after a coordinate transformation has $H = diag([\beta^2/2]^4, 0^5)$ and $F = 0$, singaling that the automorohism group now is $so(4) \oplus so(5)$.

An important implication of the fact that the isometry algebra for $p \neq 1$ is isomorphic to the one at $p = 0$, is to be found in the Abelian part of the Matrix models. It will be readily acknowledged that the positions are related to $\overline{Y}$'s, the momenta are related to the Y's and the Hamiltonian is proportional to U. The above mapping between the Heisenberg algebras then becomes a canonical transformation on the phase space, enabling us to map the system at $p \neq 1$ to the system at $p = 0$. As a result of this, such things as the vacuum energy of the bosonic sector must be p independent. As far as the fermions are concerned, a similar statement can be made: Seeing the isomorphism, it must always be possible, and in fact in the examples one can show this explicitly, to redefine the fermions in such a way that its coupling to the background, becomes the one at $p = 0$. Since we are dealing with a one dimensional theory, this redefinition cannot influence or break the necessary boundary conditions, so that also the fermionic vacuum energy should be unchanged. In fact an explicit calculation shows just that [14].

In fact, the coupling of the fermions to the background always is

$$\Psi^T \left[\not{\Theta} + i\not{F} \right] \Psi \ , \tag{50}$$

and by means of a $so(16)$ rotation this can always be brought to to a form where $\not{\Theta}$ and $\not{F}$ are expanded in mututally commuting gamma matrices. The redefinition of the spinors however should not change the fact that the model enjoys extra supersymmetry or not. Hence even after changing the base, it should be possible to find a new set of data that describes a Cahen-Wallach space. Please note that the above expansion is possible due to the fact that we are dealing with $so(16)$. Also, since we are expanding on the Cartan subalgebra of $so(16)$ and this is unique upto homomorphisms, the two bases that are being used in the literature can be mapped into each other.

4 Conclusions

We have discussed in some detail the structure of 16+ M-theory Hpp-waves, albeit quite algebraically. As far as the author is concerned, the really interesting parts are Eq. (11) and Sec. (3). Perhaps one day, someone will care enough to have a look at the classification problem of 16+ Hpp-waves and gain an understanding of the above equations and put them to good use or trivialize them.

References

[1] M. Blau, P. Meessen and M. O'Loughlin, "*Gödel, Penrose, anti-Mach: Extra supersymmetries of time-dependent plane waves*", JHEP **0309** (2003) 072 (`hep-th/0306161`).

[2] M. Blau and M. O'Loughlin, "*Homogeneous plane waves*", Nucl. Phys. B **654** (2003) 135 [arXiv:hep-th/0212135].

[3] P.M. Gadea, J.A. Oubiña: "*Reductive homogeneous pseudo-Riemannian manifolds*¨, Mh. Math. **124**(1997), 17-34.

[4] J. Kowalski-Glikman, "*Vacuum States In Supersymmetric Kaluza-Klein Theory*", Phys. Lett. B **134** (1984) 194.

[5] J. P. Gauntlett and C. M. Hull, "*waves in 11-dimensions with extra supersymmetry*", JHEP **0206** (2002) 013 [arXiv:hep-th/0203255].

[6] J. Michelson, "*A pp-wave with 26 supercharges*", Class. Quant. Grav. **19** (2002) 5935 [arXiv:hep-th/0206204].

[7] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis and H. S. Reall, "*All supersymmetric solutions of minimal supergravity in five dimensions*", arXiv:hep-th/0209114.

[8] T. Harmark and T. Takayanagi, "*Supersymmetric Goedel universes in string theory*", Nucl. Phys. B **662** (2003) 3 [arXiv:hep-th/0301206].

[9] P. Meessen, "*Small note on PP-wave vacua in 6 and 5 dimensions*", Phys. Rev. D **65** (2002) 087501 [arXiv:hep-th/0111031].

[10] A.A. Kehagias and P. Meessen, "*Exact string background from a WZW model based on the Heisenberg group*", Phys. Lett. **B331**(1994), 77 [arXiv:hep-th/9403041].

[11] I. Bena and R. Roiban, "*Supergravity pp-wave solutions with 28 and 24 supercharges*", Phys. Rev. D **67** (2003) 125014 [arXiv:hep-th/0206195].

[12] W. Ambrose, I.M. Singer: "*On homogeneous Riemannian manifolds*¨, Duke Math. J. **25**(1958), 647-669.

[13] F. Tricerri, L. Vanhecke: "*Homogeneous structures on Riemannian manifolds*¨, London Math. Soc. Lect. Notes **83**. C.U.P. 1983.

[14] M. Sakaguchi and K. Yoshida, "*M-theory on a time-dependent plane-wave*", JHEP **0311** (2003) 030 [arXiv:hep-th/0309025].

[15] T. Mateos, "*Marginal deformation of $N = 4$ SYM and Penrose limits with continuum spectrum*", JHEP **0508** (2005) 026 hep-th0505243.

[16] J. Figueroa-O'Farrill, P. Meessen and S. Philip, "*Supersymmetry and homogeneity of M-theory backgrounds*", Class. Quant. Grav. **22**(2005), 207 (`hep-th/0409170`).