Universidad Autónoma de Madrid
Facultad de Ciencias
Departamento de Física Teórica

# Type II String Duality and Massive Supergravity

Memoria de Tesis presentada por:

**Patrick A.A. Meessen**

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≫Vielleicht ist es aber auch unerlaubt≪, warf Walters Mutter ein, ≫immer gleich an die großen Gestalten wie Mozart oder Einstein zu denken. Der Einzelne hat meist nicht die Möglichkeit, an einer entscheidenen Stelle mitzuwirken. Er nimmt mehr im stillen, im kleinen Kreise teil, und da muß man sich doch eben überlegen, ob es nicht schöner ist, das D-Dur-Trio von Schubert zu spielen, als Apparate zu bauen oder mathematische Formeln zu schreiben.≪

Ich bestätigte, daß mir gerade an dieser Stelle viele Skrupel gekommen wären, und ich berichtete auch über mein Gespräch mit Sommerfeld und darüber, daß mein zukünftiger Lehrer das Schillerwort zitiert hatte: ≫Wenn die Könige bauen, haben die Kärner zu tun.≪

Rolf meinte dazu: ≫Darin geht es natürlich uns allen gleich. Als Musiker muß man zunächst unendlich viel Arbeit allein für die technische Beherrschung des Instruments aufwenden, und selbst dann kann man nur immer wieder Stücke spielen, die schon von hundert anderen Musikern noch besser interpretiert worden sind. Und du wirst, wenn du Physik studierst, zunächst in langer mühevoller Arbeit Apparate bauen müssen, die schon von anderen besser gebaut, oder wirst mathematischen Überlegungen nachgehen, die schon von anderen in aller Schärfe vorgedacht worden sind. Wenn dies alles geleistet ist, bleibt bei uns, sofern man eben zu den Kärnern gehört, immerhin der ständige Umgang mit herrlicher Musik und gelegentlich die Freude daran, daß eine Interpretation besonders gut geraten ist. Bei euch wird es dann und wann gelingen, einen Zusammenhang noch etwas besser zu verstehen, als es vorher möglich war, oder einen Sachverhalt noch etwas genauer zu vermessen, als die Vorgänger es gekonnt haben. Darauf, daß man an noch wichtigerem mitwirkt, daß man an entscheidener Stelle weiterkommen könnte, darf man nicht alzu bestimmt rechnen. Selbst dann nicht, wenn man an einem Gebiet mitarbeitet, in dem es noch viel Neuland zu erkunden gibt.≪[58]

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# Introducción

Aunque la formulación de la teoría de las (super)cuerdas aun se desconoce, lo que sí se sabe promete mucho [4, 53, 83, 95].

Una cuerda es la idealización de un cordón, donde se ignora su gordura: la cuerda es un espacio unidimensional cuando lo miramos a un tiempo fijo. Para los espacios unidimensionales sólo existen dos topologías diferentes: el círculo y el intervalo. Así que antes de empezar tenemos que distinguir entre las cuerdas cerradas, el círculo, y las cuerdas abiertas, el intervalo. La idea básica es la de introducir las interacciones: dos cuerdas cerradas pueden juntarse así formando una cuerda cerrada y a la inversa. En caso de la cuerda abierta, las interacciones se hacen juntando los extremos de las cuerdas abiertas. Esto también quiere decir que una cuerda abierta se puede cerrar: una teoría de cuerdas abiertas necesita las cuerdas cerradas para su consistencia.

Empezando en el primer capítulo con la acción de Nambu-Goto, que describe clásicamente cómo una cuerda se mueve en el espacio de Minkowski, se linealiza ésta, lo cual resulta en la acción de Polyakov. Utilizando las invariancias de la acción de Polyakov, ésta es equivalente a una teoría dos dimensional de $D$ bosones libres, donde $D$ es la dimensión en la que la cuerda se mueve. Esta teoría, cuando se cuantiza sólo es consistente, es decir que no haya estados de norma negativa en el espectro físico, si $D = 26$.

Al igual que para una cuerda de un violín, las vibraciones de la cuerda se clasifican en modos (tonos). La cuerda cerrada tiene dos tipos de modos: las vibraciones que se mueven a la derecha y las que se mueven a la izquierda.[1] La cuerda abierta sólo tiene, gracias a las condiciones de contorno, un tipo de modos. Analizando estos modos en términos del grupo de Poincaré en $D = 26$, las cuerdas siempre tienen un escalar de masa imaginario, un *taquión*, en el espectro (de hecho, el taquión es el vacío del espacio de Fock.). Las partículas sin masa son un gravitón, $G_{\mu\nu}$, un escalar, $\Phi$, llamado el *dilatón* y un campo antisimétrico, $B_{\mu\nu}$, llamado el Kalb-Ramond para la cuerda cerrada[2] y un campo vectorial para la cuerda abierta. Un hecho decepcionante de las cuerdas, aparte de vivir en 26 dimensiones y ser, posiblemente, inestables gracias al taquión, es la ausencia de fermiones en el espacio-tiempo. La *supercuerda* soluciona este problema introduciéndolos en la sábana por medio de *supersimetría.* Supersimetría es una simetría que intercambia bosones y fermiones. Sabiendo esto, las cargas que generan las transformaciones de supersimetría tienen que ser spinores, lo cual implica que dos transformaciones de supersimetría tienen que generar, entre otras cosas, traslaciones.

En la supercuerda, pues, aparte de los $D$ bosones de la cuerda bosónica, se introducen también $D$ spinores de Majorana-Weyl en la sábana, que se transforman como un vector bajo el grupo de Poincaré en $D$ dimensiones. Las condiciones de contorno para los spinores admiten dos posibilidades: o bien se transforman con un signo menos o no; dependiendo del signo se les llama

[1] La teoría pone la pega que el número de oscilaciones que van a la derecha tiene que ser el mismo que el de las que van a la izquierda.

[2] Estos tres estados aparecen en la mayoría de las teorías de cuerdas, y se llama el sector común de la cuerda.

spinores Neveu-Schwarz (NS) o spinores Ramond (R). La diferencia se nota cuando cuantizamos la cuerda fermiónica: los spinores NS se comportan como bosones en el espacio-tiempo, mientras que los spinores R son auténticos spinores en el espacio-tiempo. Además, la dimensión del espacio-tiempo para la supercuerda es 10.

Aplicando estas ideas a las cuerdas, vemos que todavía aparece el taquión en el espectro. Existe una projección sobre el espectro que es consistente y hace que el taquión desaparezca: la proyección de Gliozzi, Scherk y Olive (GSO). La proyección consiste, más o menos, en tirar todos los estados que se generan a partir del taquión por medio de la aplicación de cualquier conjunto de operadores de creación que no sea fermiónico. La proyección de GSO quiere decir que, ya que los estados de la cuerda bosónica se crean a partir del taquión utilizando sólo operadores bosónicos, todos los modos correspondientes a la cuerda bosónica son eliminados por la proyección de GSO.

Analizando el espectro de la supercuerda abierta, los modos sin masa son un vector y un spinor de tipo Majorana-Weyl. Un vector sin masa *on shell* tiene 8 grados de libertad y un spinor de tipo Majorana-Weyl también, lo cual es justamente un multiplete *on shell* de $N = 1$ supersimetría.

En el caso de la supercuerda cerrada hay mas que decir: existe la libertad de elegir la quiralidad de los spinores R de los modos que van a la derecha, $\mathrm{R}_d$, y los que van a la izquierda, $\mathrm{R}_i$. Distinguimos pues, la supercuerda tipo IIA, donde los modos $\mathrm{R}_d$ tienen la quiralidad opuesta de los modos $\mathrm{R}_i$, y la supercuerda tipo IIB, donde los modos $\mathrm{R}_i$ y $\mathrm{R}_d$ tienen la misma quiralidad. El espectro de los estados sin masa en los dos casos se parecen: el sector común de la cuerda, dos estados de tipo Rarita-Schwinger, también llamados *gravitinos*, y dos spinores, los *dilatinos*. La diferencia se nota en los estados RR: en la tipo IIA hay un vector, una tres forma, un tensor antisimétrico en sus tres índices, y una constante. En la tipo IIB hay un escalar, una dos forma y una cuatro forma, cuya *field strength* es auto-dual. En cada caso, el conjunto de estados llena los supermultipletes de $N = 2$ $D = 10$, indicando que las teorías de tipo II son invariantes bajo $N = 2$ supersimetría en el espacio-tiempo.

Debido al hecho de que la supersimetría incluye las traslaciones y que las cuerdas siempre contienen el sector común, que contiene el gravitón, las teorías de cuerdas a energías muy bajas, o equivalente a largas distancias, se describe por medio de teorías de supergravedad en diez dimensiones.

Del estudio de las supergravedades, que se originó más o menos al mismo tiempo que el estudio de las cuerdas, se sabe que hay una supergravedad máxima. Esta teoría vive en once dimensiones, es único y no se le puede añadir una constante cosmológica. Hoy en dìa, esta teoría esta considerado una teoría efectiva, de una teoría que se llama *la teoría M*.

A partir de $N = 2$ supersimetría existe la posibilidad de introducir *cargas centrales* en el álgebra. Gracias a esas cargas centrales, que dada una representación del álgebra no son más que números, se puede dar un valor mínimo a la energía de esa representación. Ésta es la llamada cota de Bogomoln'yi y simbólicamente es $M \geq |Z|$, donde $M$ es la energía y $Z$ es el valor de la carga central. La gracia de las cotas de Bogomoln'yi reside en el hecho de que cuando se satura, $M = |Z|$, las representaciones son mucho mas pequeñas que lo normal, lo cual quiere decir que, en una teoría no-anómala, siempre se seguirá cumpliendo $M = |Z|$.

Cuando se considera una cuerda cerrada que se mueve en un espacio que es el producto de un círculo y un espacio con topología trivial, la cuerda se puede enrollar sobre la dimensión compacta. El número de veces que la cuerda se enrolla sobre el círculo es un entero, llamado el número de *winding*, y crea estados con masa proporcional al radio del círculo sobre el cual la cuerda se enrolla. La mecánica cuántica también impone que el momento en la dirección del círculo está cuantizado en términos del inverso del radio del círculo. El resultado del todo esto en la fórmula de masas es que aparecen más términos que en el caso no compacto (ésta se recupera tomando el radio del círculo infinito), y además es invariante bajo el intercambio de los números winding y los números cuánticos asociados al momento en la dirección del círculo, cuando al mismo tiempo se intercambia el radio del círculo por su inverso. Esta simetría de la fórmula de masas se extiende a la teoría de las cuerdas cerradas y se le llama '*dualidad T*'. Aplicando dualidad T dos veces en la misma dirección, casi por definición, se recupera la teoría de partida, lo cual quiere decir que la dualidad T es el grupo $\mathbb{Z}_2$.

Aplicar la misma idea a la cuerda abierta no parece tener mucho sentido, ya que la cuerda abierta no se puede enrollar sobre un círculo. Aún así, si aplicamos dualidad T a las cuerdas abiertas, las condiciones de contorno cambian de Neumann a *Dirichlet*, que quiere decir nada más que los extremos de la cuerda abierta en la dirección donde hemos aplicado dualidad T están fijados. Aplicando la dualidad T en $D-p-1$ direcciones, se ve que los extremos de la cuerda abierta sólo se pueden mover sobre una subsuperficie del espacio-tiempo de dimensión $p+1$. A estas superficies se las llama *Dp-branas*.

El gran logro de Polchinski en el año 1995 [93] consistía en demostrar que, cuando se mezclan las cuerdas de tipo II con cuerdas abiertas, la teoría es consistente si al mismo tiempo se introducen las Dp-branas ($p$ par en caso de la tipo IIA e impar para la tipo IIB). En las teorías de tipo II, los campos RR, vistos como *formas*, se acoplan de forma natural a las Dp-branas. Polchinski también demostró que las Dp-branas son los objetos que, al igual que la cuerda cerrada genera el campo Kalb-Ramond, generan los campos RR. La consecuencia de introducir Dp-branas, o recíprocamente cuerdas abiertas, es que la supersimetría se rompe a la mitad,[3] y consecuentemente es un estado BPS en las teorías de tipo II.

Compactificando la supergravedades de tipo II sobre un círculo, es decir aplicando la reducción dimensional, se ve [15] que las acciones en $D=9$ están relacionadas por medio de la redefinición de algunos campos. Las relaciones generalizan las reglas de dualidad T encontradas por Buscher [30] en el sector común de la cuerda, y son la representación de la dualidad T entra las supercuerdas de tipo II al nivel de sus acciones efectivas.

Compactificando una supergravedad en diez dimensiones a cuatro dimensiones, la acción reducida es invariante bajo un grupo que contiene la dualidad T. Si, por ejemplo, compactificamos la acción efectiva del sector común de la cuerda sobre un toro de dimensión 6, la acción en cuatro dimensiones contiene el gravitón, el Kalb-Ramond, el dilatón, 12 vectores sin masa y una barbaridad de escalares, que automáticamente se agrupan de tal forma que la acción es invariante bajo $O(6,6)$. Además, en cuatro dimensiones el Kalb-Ramond es dual, en el sentido de la dualidad de Hodge entre formas, a un escalar, llamado el *axión*. El dilatón y el axión se agrupan de tal forma que la acción, despues de dualizar, sea invariante bajo el grupo $Sl(2,\mathbb{R})$. Este grupo incluye la inversión del dilatón, $\phi \to -\phi$, y por eso suele llamarse dualidad S, aunque su origen es diferente a la de la dualidad S en la tipo IIB.

Los agujeros negros son muy atractivos: aunque clásicamente no suele ser más que una aspiradora de gran tamaño, en cuanto se les aplica métodos cuantícos aparece la radiación de Hawking. Esa radiación es térmica y uno puede considerar el agujero negro como un sistema termodinámico. En ese sistema, la temperatura del sistema es proporcional a la fuerza de atracción sobre el

[3] La tipo II tiene $N=2$ supersimetría en el espacio-tiempo y la cuerda abierta sólo $N=1$, así que es intuible que al introducir Dp-branas, se rompa la mitad de la supersimetría.

horizonte y la entropía es propocional al área del horizonte.

Considerando las soluciones de tipo agujero negro en las teorías de supergravedad en cuatro dimensiones con, por lo menos $N = 2$ supersimetría, se puede derivar la cota de Bogomoln'yi en términos del *pelo* (el pelo son constantes que caracterizan asintóticamente el agujero negro. Ejemplos del pelo son la masa, el momento angular y la carga eléctrica.). La sorpresa es que para un agujero negro sin momento angular, el agujero se vuelve supersimétrico justamente cuando la temperatura de Hawking es cero y la singularidad todavía está protegida por un horizonte. Esto quiere decir que la supersimetría actúa como el *censor cósmico*: la supersimetría no permite la creación de singularidades desnudas. Para agujeros negros con momento angular, la supersimetría no actúa como el censor: antes de hacerse supersimétrico, el agujero negro se convierte en una singularidad desnuda. La razón para esto es que el momento angular no aparece en la cota de Bogomoln'yi pero sí aparece en las ecuaciones que determinan si hay un horizonte o no. La cuestión fundamental que se investiga en el capítulo (2), es si, de alguna forma, las dualidades mejoran las cosas.

La clase de soluciones de tipo agujero negro, con o sin momento angular, se caracteriza por el hecho de admitir dos vectores de Killing. Eso quiere decir que se puede reducir dimensionalmente a dos dimensiones, las ecuaciones de movimiento y aplicar la dualidad T para generar nuevas soluciones. Si entonces miramos a una teoría $N = 4$, tambien tenemos la dualidad S a nuestra disposición.

En el capítulo (2) pues, consideramos la clase más general de agujeros negros, llamado *TNbh*, definido en términos del pelo, embebido en una teoría que es invariante bajo dualidad S, y contiene dos campos vectoriales de tipo $U(1)$. A nuestra disposicion tenemos entonces la dualidad S y un grupo $O(2,4)$ para generar nuevas soluciones y para estudiar el comportamiento del pelo bajo esas dualidades.

El estudio de las dualidades indica que sólo un subgrupo, llamado el *ADS*, transforma el TNbh en el TNbh y que el pelo se descompone, de forma natural, en multipletes bajo el ADS. La forma de los multipletes es tal que el momento angular no se mezcla con el pelo que entra en la cota de Bogomoln'yi. La cota de Bogomoln'yi es expresada en términos de los multipletes y es automáticamente invariante bajo el ADS. En las conlusiones del capítulo (2), se discute la importancia de la *superradiance*, parecido a la emisión espontánea, en la cuestión de estabilidad de agujeros negros.

Hasta el año 1995, el hecho de que en la supercuerda de tipo IIA hubiese un modo de la cuerda que se manifesta como una constante, era una cosa de muy poca importancia. En este año Polchinski [93] se da cuenta, que existe una teoría de supergravedad masiva, que contiene una constante arbitraria y que se reduce a la supergravedad IIA de todo la vida cuando esta constante es cero. Teniendo en quenta que una constante no es más que una funcción que satiface una identidad de Bianchi, uno puede considerar la constante como la *field strength* de lo que sea y dualizarla [19]. Esta operación introduce una diez forma, que no es nada más que la field strength del campo asociado a la D8-brana, que deberìa estar en la tipo IIA por dualidad T desde el principio. Es decir, la supergravedad IIA masiva, llamado la teoría de Romans [98], es la verdadera acción efectiva de las supercuerdas de tipo IIA.

¿Cuál es entonces la influencia del D8 en la acción efectiva? El D8 se manifiesta como un parámetro de masa, resultando en una constante cosmológica en la supergravedad, y cambia los acoplos entre algunos campos. En el caso de la RR uno forma, el acoplo es tal que ésta es comida por el campo Kalb-Ramond, que se hace masivo. Dualizando todos los *field strengths*, véase la sección (3.1.1), que solo se puede hacer *on shell*, se puede ver que las mismas cosas le ocurre a la RR siete forma [69].

La supercuerdas de tipo IIA y de tipo IIB son T duales, lo cual se manifiesta en las respectivas supergravedades, no masivas, que al reducirlas dimensionalmente a $D = 9$, son idénticas [15]. Si compactificamos la teoría de Romans dimensionalmente, no se obtiene la supergravedad en $D = 9$ de antes, claro está. ¿Quiere decir eso que la dualidad T no esta representada en las acciones efectivas, mientras que sí es así en las teorías de supercuerdas?

Como fue demostrado por Scherk y Schwarz [101], cuando una teoría es invariante bajo una simetría global, se puede utilizar una dependencia específica en las coordenadas, sobre las cuales reducimos, de tal forma que la acción en dimensiones más bajas, no depende de ellas. La supergravedad IIB es invariante bajo el grupo global $Sl(2, \mathbb{R})$, y utilizando sólo un subgrupo $U(1)$ para aplicar la reducción de Scherk-Schwarz se ve que la dualidad T funciona entre las supergravedades de tipo II [19].

La reducción de Scherk-Schwarz es más que un simple truco para conseguir masas en la reducción dimensional: en la sección (3.2.1) se estudia, en un modelo muy fácil, la relación entre la reducción de Scherk-Schwarz y la dualidad de Hodge aplicada a escalares. La conclusión de esta sección es que la reducción de Scherk-Schwarz es la manera correcta de reducir dimensionalmente, si tenemos en cuenta los posibles duales de los escalares. Además se introduce un algoritmo para hacer la reducción de Scherk-Schwarz sin la necesidad de construir las transformaciones explícitamente [86].

La supergravedad IIB es invariante bajo el grupo $Sl(2, \mathbb{R})$, que es un grupo de tres dimensiones, mientras que sólo uno de esos sirve para establecer la dualidad T entre la tipo IIA y la tipo IIB. Utilizando todo el grupo para aplicar la reducción de Scherk-Schwarz, véase [78, 86] o la sección (3.2), resulta en una teoría con tres parametros de masas, donde todavía hay covariancia bajo del grupo $Sl(2, \mathbb{R})$: encontramos un multiplete de $N = 2$, $D = 9$ supergravedades masivas.

Para poder obtener la teoría de Romans a partir de $D = 11$, o de la teoría M, uno necesitaría por lo menos una constante cosmológica, lo cual es imposible: como fue demostrado en [11, 36, 37] no se puede extender la supergravedad en $D = 11$ con una constante cosmológica. En [25] las condiciones por las cuales no se puede añadir un constante cosmologico a $D = 11$ supergravedad son evitados, introduciendo explícitamente un vector de Killing en la teoría. Reduciendo esa teoría en la dirección del Killing se recupera la teoría de Romans. En la sección (3.3) seguiremos a [25] para construir una teoría en $D = 11$, que al reducir sobre un toro de dimensión dos resulta ser la teoría encontrada en la sección (3.2).

Las *intersecciones* son soluciones a las ecuaciones de movimiento de una supergravedad, que tienen la interpretación de un conjunto de varios objetos de las supercuerdas. De gran interés son las intersecciones que rompen tres cuartos de la supersimetría, ya que eso quiere decir que los objetos no se atraen el uno al otro, es decir que la solución es estable. En la literatura (véase por ejemplo [21, 23, 68, 116, 111, 117, 79]) siempre se han estudiado estas intersecciones utilizando ecuaciones de movimiento genéricos, que no capturan la esencia de la teoría de Romans: algunos campos se vuelven masivos cuando hay un D8 en la intersección. La sección (3.4) empieza enumerando las soluciones clásicas asociadas a las cuerdas y las Dp-branas. Después se habla de las intersecciones estudiadas en la literatura y termina discutiendo las dos intersecciones que hasta [69] no se conocía: la intersección del D8 con la cuerda y la intersección del D8 con el D6.

En la sección (3.6) identificamos los campos que dan lugar a las masas en $D = 9$. Utilizando las observaciones de la sección (3.2), se ve que el objeto que resulta en la masa de la teoría de Romans es la D7-brana. Pero dualizando los dos escalares en la supergravedad IIB, que también incluye el dilatón, solo es posible *on shell*. El resultado de la dualizacion son tres ocho formas, una para cada masa que se introduce por medio de la reducción de Scherk-Schwarz, lo cual quiere decir que hay tres objetos, relacionados por dualidad S, de tipo 7-brana en la tipo IIB. El capítulo termina verificando explícitamente las dualidades entre esos objetos y los demás objetos que existen en la teoría de las supercuerdas y M.

En el apéndice (B.1) se resumen todas las reglas de dualidad T para las supergravedades de tipo II. Las dualidades entre las soluciones, que representan un objeto fundamental de las cuerdas, está resumido en la figura (B.1).

# Chapter 1

# String Theory: A Short Introduction

Although we have already introduced some of the results on string theory in the introduction, here we will visualize them by means of formulae.

First we will introduce the basic notion of a (bosonic) string, and set up a possible action governing its propagation in normal Minkowski space. After canonical quantization, which introduces quanta of vibration in the string, the spectrum, as seen by a space-time observer is introduced and it will be shown that the theory contains a tachyon, rendering the theory, possibly, instable. Compactifying this theory then over a circle of radius $R$, one can see that in the case of the closed string, the physical spectrum is invariant under a mapping of type $\mathbb{Z}_2$, which is called T-duality.

Since the bosonic string does not lead to space-time fermions, we explore the formulation of the fermionic string, and see that it does lead to space-time fermions, but still contains a tachyon. After a consistent truncation of the theory, known as the GSO projection, the tachyon is thrown out of the spectrum, and the resulting theory leads automatically to space-time supersymmetry.

In section (1.3), we will deal with the propagation of a test-string in a given background, and will show that this propagation does not lead to anomalies iff the background satisfies some criteria, which contain the Einstein equations. Applying these ideas to the superstring, it is implied that the consistency criteria can be derived from the various ten dimensional supergravities.

## 1.1. Bosonic Strings

Everybody knows what a string is, if not just look at your shoes. Idealizing the concept of a string to something with only a length, we have something one can mathematically work with, which is called, very original, a *string*. Embedding this string into spacetime we see that during its evolution in spacetime it sweeps out a 2 dimensional surface in spacetime. Denoting this surface, called the strings *worldsheet*, by $\Sigma$, we see that a string in the mathematical sense is nothing but a mapping from the worldsheet to the spacetime. If we denote the coordinates on the worldsheet by $(\tau, \sigma)$ we see that the coordinates describing the position of the string in spacetime $X^\mu$ are: $X^\mu(\tau,\sigma) : \Sigma \to \mathcal{M}$, where $\mathcal{M}$ is some manifold representing spacetime.

What kind of action will govern the dynamics of our strings? Well, let us look at the particle for the moment. In General Relativity, a particle, coupled to gravity, is defined to move along geodesics, which is nothing but the shortest path as specified by the Riemannian geometry. Now, the string-analogue of the particle's path, rather the length of the trajectory a particle sweeps out in spacetime, is the worldsheet. Seeing this, Nambu and Goto proposed to govern the string dymanics by minimizing the *area* the string sweeps out when moving.

The Nambu Goto action reads

$$\mathcal{S}_{NG} \;=\; -T\int_\Sigma d^2\xi\,\sqrt{\det\left(\partial_a X^\mu \partial_b X_\mu\right)} \tag{1.1}$$

Note that this is just the minimal action we can write down: We could couple the string to some 2-form, but for simplicity we choose not to.

One can linearize the Nambu-Goto action, as was first done by Polyakov, by introducing a metric, $h_{ab}$, on the worldsheet:

$$S_{Polyakov}\left(X\right) \;=\; -\tfrac{T}{2}\int d^2\xi\sqrt{-h}\; h^{ab}\partial_a X^\mu\,\partial_b X^\nu\,\eta_{\mu\nu}\;, \tag{1.2}$$

By eliminating the worldsheet metric from the action, one can see that the Polyakov and the Nambu-Goto actions are, classically, equivalent.

Having a small look at the symmetries of this action, one sees that this action is invariant under worldsheet diffeomorphisms, *i.e.* under 2-dimensional coordinate transformations, under spacetime Lorentz transformations and under rescalings of the 2-dimensional metric, *i.e.* $h_{ab} \to \Omega(\xi)h_{ab}$, called *Weyl invariance.* As is well-known, we can use the 2d diffeomorphisms in order to bring the metric to the form[1]

$$h_{ab} \;=\; e^{-\varphi}\eta_{ab}\;, \tag{1.3}$$

where $\eta_{ab}$ is the 2D Minkowski metric. Due to the Weyl invariance, the $\varphi$ dependence drops out of the Polyakov action, which then reads

$$S_P\left(X\right) \;=\; -\tfrac{1}{2\alpha'}\int d^2\xi\;\eta^{ab}\,\partial_a X^\mu\partial_b X^\nu\,\eta_{\mu\nu}\;. \tag{1.4}$$

Note the beauty of the story: We started with a nonlinear theory, which we then linearized. Then we saw that the linearized action had the appropriate symmetries for taking a suitable gauge in which the theory is nothing but a 2-dimensional theory of $D$ free bosons.

The explicit variation of the Polyakov action Eq. (1.4), leads to

$$\delta S_P\left(X\right) \;=\; \int d^2\xi\;\delta X_\mu\left[-\partial_a\partial^a X^\mu\right] + \int_{\partial\Sigma}\delta X_\mu\;* dX^\mu\;. \tag{1.5}$$

From this one can see that the equation of motion is nothing but

$$\partial_a\partial^a X^\mu \;=\; 0\;. \tag{1.6}$$

Introducing the the coordinates $x_\pm = \tau \pm \sigma$, we see that the above equation read $\partial_+\partial_- X^\mu = 0$ so that the most general solution, without imposing boundary conditions, is $X^\mu = X^\mu(x_+) + \bar{X}^\mu(x_-)$.

Since we have fixed the gauge freedom in Eq. (1.4), we necessarily will encounter a constraint: This is obvious if we calculate the equations of motion from the pre-gauge-fixed action, since we also need to calculate the variation under $h_{ab}$. The restriction thus found is, in the conformal gauge,

$$0 \;=\; T_{ab} \;\equiv\; -\partial_a X^\mu\partial_b X_\mu \;+\; \tfrac{1}{2}\eta_{ab}\partial_c X^\mu\partial^c X_\mu\;. \tag{1.7}$$

In the lightcone coordinates the constraint (1.7) becomes

$$0 \;=\; T_{++} \;=\; -\partial_+ X^\mu\partial_+ X_\mu\;, \tag{1.8}$$
$$0 \;=\; T_{--} \;=\; -\partial_- X^\mu\partial_- X_\mu\;. \tag{1.9}$$

[1]This is called the *conformal gauge.*

These constraints will become important ingredients in our search for consistent, anomaly free string theories.

Here we ought to discuss the difference between open and closed strings: A String frozen in time is just a one-dimensional object, and as such admits to two different topologies: A segment and a circle. According to this we have to divide our attention to the two different classes called closed (open) when the topology is that of a circle (*resp.* segment). As we will see later on, the observable spectrum of the string also depends on the topology of spacetime, so that for the moment we will focus our attention to usual Minkowski space.

Anyway, looking at Eq. (1.5) and considering $\tau \in (-\infty, \infty)$, we see that in the case of the closed string the boundary term vanishes identically, *i.e.* an infinitely long cylinder has no boundary. For the open string however, we see that, if we want to maintain Poincaré invariance, we have to impose

$$*dX^{\mu}|_{\partial\Sigma} = 0 \quad \rightarrow \quad \partial_{\sigma} X^{\mu}|_{\sigma=0,\pi} = 0 , \tag{1.10}$$

which is called a *Neumann boundary condition*.[2]

### 1.1.1. Closed Strings in Minkowski Space

A circle is nothing but a segment where we identify the endpoints. Applying this idea to the case at hand, and taking the parameter space for the spacelike worldsheet coordinate to be $\sigma \in [0, 2\pi]$, we see that we have to impose

$$X^{\mu}(\tau, \sigma + 2\pi) = X^{\mu}(\tau, \sigma) . \tag{1.11}$$

Note that due to these boundary conditions, the surface terms in Eq. (1.5) drops out completely. Using this boundary condition we can write down the solutions to (1.6)

$$X^{\mu}(x_-) = \tfrac{1}{2} X_0^{\mu} + \tfrac{1}{4\pi T} p^{\mu} x_- + \tfrac{i}{\sqrt{4\pi T}} \textstyle\sum_{n\neq 0} \tfrac{1}{n} \alpha_n^{\mu} e^{-inx_-} \tag{1.12}$$

$$X^{\mu}(x_+) = \tfrac{1}{2} X_0^{\mu} + \tfrac{1}{4\pi T} p^{\mu} x_+ + \tfrac{i}{\sqrt{4\pi T}} \textstyle\sum_{n\neq 0} \tfrac{1}{n} \bar{\alpha}_n^{\mu} e^{-inx_+} \tag{1.13}$$

where the $a^{\mu}$'s and $\bar{a}^{\mu}$'s are arbitrary constants and from now on we fix $T = (4\pi)^{-1}$.

Canonical quantization is then introduced by promoting the $a^{\mu}$'s and $\tilde{a}^{\mu}$'s to operators and imposing the *Equal Time Commutation Relations*[3]

$$\left[\hat{P}_{\mu}(\tau,\sigma), \hat{X}^{\nu}(\tau,\sigma')\right] = -2\pi i \delta(\sigma - \sigma') \eta_{\mu}{}^{\nu} , \tag{1.14}$$

$$\left[\hat{P}_{\mu}(\tau,\sigma), \hat{P}_{\nu}(\tau,\sigma')\right] = 0 , \tag{1.15}$$

$$\left[\hat{X}^{\mu}(\tau,\sigma), \hat{X}^{\nu}(\tau,\sigma')\right] = 0 , \tag{1.16}$$

we find that

$$[a_m^{\mu}, a_n^{\nu}] = n \eta^{\mu\nu} \delta_{m+n,0} , \tag{1.17}$$

$$[\bar{a}_m^{\mu}, \bar{a}_n^{\nu}] = n \eta^{\mu\nu} \delta_{m+n,0} , \tag{1.18}$$

$$[a_m^{\mu}, \bar{a}_n^{\nu}] = 0 , \tag{1.19}$$

$$[X_0^{\mu}, P^{\nu}] = i \eta^{\mu\nu} , \tag{1.20}$$

[2] We could also have imposed the so-called *Dirichlet* boundary condition $X^{\mu}\ |_{\sigma=0,\pi} = C^{\mu}_{0,\pi}$. This condition breaks global Poincaré invariance, however, so that we will ignore. In a few pages it will reappear mysteriously!

[3] Remember that we are quantizing a 2d theory of free bosons.

where we have defined $a_0^\mu = \bar{a}_0^\mu = P^\mu$.

Since we have promoted the fields to operators, one is deemed to encounter normal ordering ambiguities: Actually, the only possibility to encounter such ambiguities is in the constraints, since they involve products of operators in the same place.[4] Decomposing the constraint $T_{--} = \sum L_n e^{-2inx_-}$ one can see that

$$L_n \;=\; -\tfrac{1}{2}\sum \,:\alpha^\mu_{m-n}\alpha_{n|\mu} \;-\; a\delta_{n,0} \;, \tag{1.21}$$

where we have included an, unknown, normal ordering constant.

If one then calculates the algebra generated by the $L_n$'s, one finds the '*Virasoro algebra*', *i.e.*

$$[L_m, L_n] \;=\; (m-n)L_{m+n} \;+\; \frac{c}{12}m(m^2-1)\delta_{m+n} \;, \tag{1.22}$$

where $c$ is known as the *conformal anomaly*, and in this case equals the dimension of the target space, *i.e.* $c = D$.

Up to now, we have introduced two arbitrary constants in our algebra, $c$ and $a$, which can however be fixed by imposing the Fock space, based on a vacuum satisfying

$$\alpha^\mu_m \mid 0\rangle \;=\; 0 \quad \text{for all } n > 0 \;, \tag{1.23}$$

to have no negative norm states: This fixes $D = 26$ and $a = 1$ (See *e.g.* [53]). Physical states, denoted $\mid phys\rangle$, are then required to satisfy the weak constraints

$$L_n \mid phys\rangle \;=\; 0 \quad \text{for } n > 0 \tag{1.24}$$

$$(L_0 - 1) \mid phys\rangle \;=\; 0 \;. \tag{1.25}$$

If one then analyzes the last constraint one can write, at least symbolically,

$$1 \;=\; L_0 \;=\; -\tfrac{1}{2}\sum :\alpha^\mu_{-n}\alpha_{n|\mu}: \;=\; -\tfrac{1}{2}P^\mu P_\mu \;+\; N_- \;, \tag{1.26}$$

where we have defined

$$N_- \;=\; -\tfrac{1}{2}\textstyle\sum_{n\neq 0} :\alpha_{-n}\alpha_n: \;=\; -\textstyle\sum_{n>0}\alpha_{-n}\alpha_n \,, \tag{1.27}$$

which acts on a state $\mid m\rangle = \alpha_{-m} \mid 0\rangle$, as $N_- \mid m\rangle = m \mid m\rangle$, so that it defines a counting operator for the level of string oscillations. Using then the identity $P^2 = m^2$ we can write

$$m^2 \;=\; 2\left(N_- - 1\right) \;. \tag{1.28}$$

Another constraint, known as *level matching*, is found by also considering the + branch: One finds, once again, $m^2 = 2(N_+ - 1)$, which then implies that

$$N_+ \;=\; N_- \tag{1.29}$$

As we have seen, and as was to be expected, a closed string can vibrate according to modes. How are we, being spacetime observers, to interpret these modes? Special relativity and Quantum field theory teaches us that all particles, as described by fields, are classified according to their mass and their transformation under the Lorentz group, *i.e.* the spin. This then means that in order to compare the modes to something we know, we should classify them according to the Poincaré group.

[4]Seeing the great similarity between the two independent branches we will treat only one of them.

The first state that satisfies the level matching constraint is the state $\mid 0\,k\rangle$, which is annihilated by the annihilation operators in the $+$ and the $-$ branch, and satisfies $P^\mu \mid 0\,k\rangle = k^\mu \mid 0\,k\rangle$, so that it bears a certain momentum. This then means that it has level zero resulting in

$$m^2 \mid 0\,k\rangle \;=\; -2 \mid 0\,k\rangle \;, \tag{1.30}$$

which is nothing but a tachyon, thus endangering the consistency of the theory.

The next level is massless and is generated by

$$a^\mu_{-1}\bar{a}^\nu_{-1} \mid 0\rangle \;. \tag{1.31}$$

Since this has no apparent symmetry and the Poincaré group does not transform symmetric matrices into scalars nor in antisymmetric matrices, we should split this into a scalar, $a^\mu_{-1}\bar{a}_{-1\nu} \mid 0\rangle$, a symmetric part and an anti-symmetric part. The symmetric part can be shown to be a spin 2 field and the anti-symmetric part can be shown to be a spin 1 field[5]. Higher levels can be analyzed in the same manner, but since in this thesis we are mainly interested in the low energy approximation, or equivalently the large distance behaviour, the interesting modes are massless.

### 1.1.2. Open Strings in Minkowski Space

The solution for the open string, we use that parameterization $\sigma \in [0,\pi]$, reads

$$X^\mu \;=\; X^\mu_0 \;+\; \frac{1}{\pi T}P^\mu\tau + \tfrac{i}{\sqrt{\pi T}}\textstyle\sum_{n\neq 0}\tfrac{1}{n}\alpha^\mu_n\, e^{-in\tau}\cos\left(n\sigma\right) \;, \tag{1.32}$$

and by defining $P^\mu = \sqrt{\pi T}\alpha^\mu_0$ one can write

$$\partial_\pm X^\mu \;=\; \frac{1}{2\sqrt{\pi T}}\sum_n \alpha^\mu_n \tag{1.33}$$

The canonical quantization for the open string is actually the same as for the $-$, or the $+$ branch in the closed string case [53]: The only difference being that we have no level matching. Having said this, we can set up the Fock space as in the foregoing section, and see immediately that the vacuum is once again a tachyon. The real difference lies however in the first level: It is massless and reads

$$a^\mu_{-1} \mid 0\rangle \;. \tag{1.34}$$

This then means that the massless sector of the open string is a vector in $D=26$, and therefore must be a gauge field.

### 1.1.3. Compactification and T-duality

As we said we are not forced to look at the Polyakov string moving on the usual Minkowski space: We could take it to be compactified, *i.e.* $\mathcal{M}_D = \mathcal{M}_{D-1}\oplus S^1$, or even stranger things. It is the aim of this section, to have a look at some of the phenomena one encounters when looking at string propagation on other manifolds.

[5] Had we been working in 4 dimensions, we would have said that since the $a^\mu$'s transform as a vector under $SO(1,3)$, the above decomposition is nothing but the Clebsch-Gordon series $(\frac{1}{2},\frac{1}{2})\otimes(\frac{1}{2},\frac{1}{2}) = (1,1)\oplus(1,0)\oplus(0,1)\oplus(0,0)$, which speaks for itself. Note that in higher dimensons the same reasoning works, although the identification is harder.

### Circle Compactifications of Closed Strings and T-Duality

Let us take a $D$ dimensional target space, which is divided in a $D-1$ dimensional Minkowski part and a circle of radius $R$. Since the analysis of the $D-1$ dimensional part is the same as in Sec. (1.1.1), we will focus on the circle.

Since we have a map of circle onto a circle, classified topologically by $\pi_1\left(S^1\right)=\mathbb{Z}$, we have the possibility of taking that the string winds several times around the circle, *i.e.*

$$X(\tau,\sigma+2\pi) \;=\; X(\tau,\sigma) \;+\; 2\pi R\, n\;, \tag{1.35}$$

where $n\in\mathbb{Z}$ is the winding number of the map.

The solution to Eq. (1.6) given the above boundary condition reads

$$\begin{aligned} X_- &= \tfrac{1}{2}X_0 + \left(\tfrac{m}{R}+\tfrac{nR}{2}\right)x_- + i\textstyle\sum_{n\neq 0} n^{-1}\alpha_n e^{-inx_-}\;, && (1.36)\\ X_+ &= \tfrac{1}{2}X_0 + \left(\tfrac{m}{2}-\tfrac{nR}{2}\right)x_+ + i\textstyle\sum_{m\neq 0} n^{-1}\bar{\alpha}_n e^{-inx_+}\;, && (1.37)\end{aligned}$$

The mass of the states can then be written as

$$\begin{aligned} m_-^2 &= \frac{1}{2}\left(\frac{n}{R}+\frac{mR}{2}\right)^2 + N_- - 1\;,\\ m_+^2 &= \frac{1}{2}\left(\frac{n}{R}-\frac{mR}{2}\right)^2 + N_+ - 1\;. \end{aligned} \tag{1.38}$$

Level matching, $m_-=m_+$, implies that there is a relationship between momentum and winding numbers on the one hand, and the oscillator excess on the other

$$N_+ - N_- \;=\; nm\;. \tag{1.39}$$

From this formula we see that it is invariant under $R\rightarrow 2/R$, $m\leftrightarrow n$ and $\bar{a}^\mu\rightarrow -\bar{a}^\mu$. In fact the whole theory is invariant under this mapping. This mapping is equivalent to $X(x_+)\rightarrow X(x_+)$ and $X(x_-)\rightarrow -X(x_-)$, making clear that it is a purely stringy symmetry. It is baptized to the name *T-duality*.

As can be seen from the above mass formula, at the point $R=\sqrt{2}$ there are a few new massless modes (See *e.g.* [83]), which in its turn can be shown to generate a $SU(2)\otimes SU(2)$ gauge group. This then shows that in general compactifications, where one calls the parameterspace of the distinct compactifications the *moduli space*, there are points in the moduli space where one has an enhanced symmetry. On the other hand one can say that one has a broken symmetry in the rest of the moduli space.

### Circle Compactifications of Open Strings, T-Duality and D-Branes

Let us now ask ourselves the question what happens when we compactify the open string on a circle. The naive answer is that nothing special will happen, because an open string wound around a circle can always be unwound. This is however too naive: Looking at the boundary condition for the open string, and using the result for the closed string one sees that iff we apply T-duality on the boundary that

$$\partial_\sigma X^\mu \;=\; \partial_+X^\mu - \partial_-X^\mu \;=\; 0 \;\xleftrightarrow{\text{T-duality}}\; \partial_+X^\mu + \partial_-X^\mu \;=\; \partial_\tau X^\mu \;=\; 0\;, \tag{1.40}$$

This then means that the Neumann boundary condition for the open string is transformed into a Dirichlet boundary condition, *i.e.* the open string is restricted to move on some $d-1$

Figure 1.1: An artists impression of an open string ending on a D-brane. On the D-brane we have Neumann conditions, whereas transverse to it we have Dirichlet conditions.

dimensional hypersurface. Likewise, performing T-duality in $n$ different directions one reaches to the conclusion that the open string is restricted to move on some $d-n$ dimensional hypersurface.

Does this then mean that there is no T-duality in open string theory, although it must have closed strings in its spectrum? It might, but one may also extend open string theory by saying that there exist some extended objects called *D-p-branes*,[6] being defined as the surfaces on which an open string can end, such that T-duality is also a symmetry of the, extended, open string theory. Due to the fact that an open string with Dirichlet boundary condition leads to a leakage of momentum out of the string, one is forced to conclude that this momentum must be absorbed by the D-brane, implying that these D-branes are dynamical.

It was shown by Polchinsky [93], that when one considers type II strings together with open strings with mixed boundary conditions, and thus admitting D-branes, one can arrive at a consistent theory. Furthermore, in this setting the D-branes then carry charges with respect to the RR fields occurring, and in fact can be identified as the sources of the RR-fields.

Note that although the original type II string had $N = 2$ targetspace supersymmetry, once one allows for the D-branes, or equivalently an open string, supersymmetry is broken to $N = 1$: This is a result from the identification of left- and right-modes of the closed string leading to the open string.

## 1.2. Fermionic Strings

A clear drawback of the bosonic string is, apart from living in $D = 26$ and possibly being inconsistent due to the tachyon, is that it does not lead to spacetime fermions. In this section we will see how to introduce fermions by means of *supersymmetry*.

Supersymmetry, as is well-known, relates bosons to fermions, and *vise versa*. Since we are looking at a string, there are two places where we can introduce supersymmetry: On the worldsheet and/or on the targetspace. The first possibility will lead to the *Neveu-Schwarz-Ramond* string and the other will lead to the so-called *Green-Schwarz* superstring. Since no one has been able to quantize the GS superstring however, we will focus on the NSR string.

[6] $p$ in this setting is $d - n - 1$. It is customary only to mention the spacelike directions.

The two-dimensional locally supersymmetric action generalizing the one used above for the bosonic string reads (once the auxiliary fields have been eliminated)

$$\begin{aligned} S \ &= -\tfrac{1}{8\pi}\int d^2\xi\sqrt{h}\ \ \Big[h^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + 2i\bar{\psi}^\mu\gamma^a\partial_a\psi^\nu\eta_{\mu\nu} \\ &\qquad\qquad -i\ \bar{\chi}_a\gamma^b\gamma^a\psi^\mu(\partial_b X_\mu - \tfrac{i}{4}\bar{\chi}_b\psi_\mu)\Big]\ . \end{aligned} \tag{1.41}$$

This action includes a scalar supermultiplet $(X^\mu, \psi^\mu, F^\mu)$, where $F^\mu$ are auxiliary fields, and the two-dimensional gravity supermultiplet $(e^a, \chi_a, A)$, where again $A$ is an auxiliary field.

The gravitino $\chi_a$ is a world-sheet vector-spinor. Using all the gauge symmetries of the action (reparametrizations, local supersymmetry and Weyl transformations) it is formally possible to reach the *superconformal gauge* where $h_{ab} = \delta_{ab}$ and $\chi_a = 0$. Off the critical dimension, however, there are obstructions similar, although technically more involved, to those present already in the bosonic string.

In this gauge, and using again lightcone coordinates on the worldsheet, the action reads

$$\mathcal{S} \ = \ \frac{1}{2\pi}\int d^2\xi \left\{\partial_+ X^\mu \partial_- X_\mu \ + \ i\left(\psi^\mu_+\partial_-\psi^{\mu|+} \ + \ \psi^\mu_-\partial_+\psi_{\mu|-}\right)\right\}\ , \tag{1.42}$$

which leads to the following equations of motion

$$\begin{aligned} 0 \ &= \ \partial_+\partial_-\, X^\mu\ , \\ 0 \ &= \ \partial_+\psi_- \ = \ \partial_-\psi_+\ , \end{aligned} \tag{1.43}$$

and boundary term resulting from the spinors is

$$0 \ = \ (\psi_+\delta\psi_+ - \psi_-\delta\psi_-|^{\sigma=2\pi}_{\sigma=0}\ \ , \tag{1.44}$$

for a closed string, and

$$0 \ = \ (\psi_+\delta\psi_+ - \psi_-\delta\psi_-|_{\sigma=0,\pi}\ \ , \tag{1.45}$$

for an open string. The energy-momentum tensor reads

$$T(-) \equiv T_{--} \ = \ -\frac{1}{2}\partial_- X^\mu\partial_- X_\mu - \frac{i}{2}\psi^\mu_-\partial_-\psi_{-\mu}\ , \tag{1.46}$$

and depends on $x_-$ only, due to its conservation, $\partial_+ T_{--} = 0$.

The supercurrent (associated to supersymmetry) reads

$$T^F_- \ = \ -\frac{1}{2}\psi^\mu_-\partial_- X_\mu\ , \tag{1.47}$$

and satisfies $\partial_+ T^F_- = 0$.

For open strings, in order to satisfy Eq. (1.45), we fix arbitrarily at one end

$$\psi_+(0,\tau) = \psi_-(0,\tau)\ , \tag{1.48}$$

and the equations of motion then allows for two possibilities at the other end

$$\psi_+(\pi,\tau) = \pm\psi_-(\pi,\tau)\ . \tag{1.49}$$

The two sectors are called Ramond (for the + sign) and Neveu-Schwarz (for the − sign). The solution for the bosons $X^\mu$ is the same as in the case for the bosonic string and the fermions are

$$\psi^\mu_- \ \ = \ \ \tfrac{1}{\sqrt{2}}\textstyle\sum_{r=\mathbb{Z},\mathbb{Z}+\frac{1}{2}}\ b^\mu_r e^{-inx_-}\ , \tag{1.50}$$

$$\psi^\mu_+ \ \ = \ \ \tfrac{1}{\sqrt{2}}\textstyle\sum_{r=\mathbb{Z},\mathbb{Z}+\ \frac{1}{2}}\ \bar{b}^\mu_r e^{-inx_+}\ , \tag{1.51}$$

where $r = \mathbb{Z}, \mathbb{Z} + \frac{1}{2}$ means that in case of the R sector $r$ is an integer and that in case of the NS sector $r$ is an integer plus $\frac{1}{2}$.

In the closed string case, Eq. (1.44) means that fermionic fields need only be periodic up to a sign.

$$\psi_\mu(e^{2\pi i} z) \;=\; \pm\psi_\mu(z) \; . \tag{1.52}$$

Periodic fields are, once again, said to obey the R(amond) boundary conditions; Antiperiodic ones are said to obey N(eveu)S(chwarz) ones. The solution for the spinors[7] reads

$$\begin{aligned} \psi^\mu_- &= \sum_{r=\mathbb{Z},\mathbb{Z}+\frac{1}{2}} b^\mu_r \, e^{irx_-} \; , & (1.53) \\ \psi^\mu_+ &= \sum_{r=\mathbb{Z},\mathbb{Z}+\frac{1}{2}} \bar{b}^\mu_r \, e^{irx_+} \; , & (1.54) \end{aligned}$$

where one should note that the $+$ and $-$ branch are completely independent. This leads, in the closed string sector, to four possible combinations (for left as well as right movers), namely: (R,R), (NS,NS), (NS,R), (R,NS).

Fourier decomposing the generators of the Virasoro algebra as before, one finds

$$L_m \;=\; -\tfrac{1}{2}\left[\textstyle\sum_n : \alpha^\mu_{-n}\alpha^\mu_{m+n} : + \sum_{r\in\mathbb{Z},\mathbb{Z}+\frac{1}{2}} (r + \frac{m}{2}) : b^\mu_{-r} b^\mu_{m+r} :\right] \; , \tag{1.55}$$

and the modes of the supercurrent can be similarly shown to be equal to

$$G_r \;=\; -\sum_n \alpha^\mu_{-n} b^\mu_{r+n} \; . \tag{1.56}$$

The reality conditions then imply as usual

$$L^\dagger_n \;=\; L_{-n} \quad , \quad G^\dagger_r \;=\; G_{-r} \; . \tag{1.57}$$

The unitary operator $U_\delta \equiv e^{i\delta(L_0 - \bar{L}_0)}$ implements spatial translations in $\sigma$, *i.e.*

$$U^\dagger_\delta X^\mu(\tau,\sigma) U_\delta \;=\; X^\mu(\tau, \sigma + \delta) \; . \tag{1.58}$$

This transformation should be immaterial for closed strings, which means that in that case we have the further constraint

$$L_0 \;=\; \bar{L}_0 \; , \tag{1.59}$$

which is nothing but the level matching constraint.

Canonical quantization can then be shown to lead to the canonical commutators

$$\begin{aligned} [x^\mu, p^\nu] &= i\eta^{\mu\nu} \; , \\ [\alpha^\mu_m, \alpha^\nu_n] &= -m\delta_{m+n}\eta^{\mu\nu} \; , \\ [b^\mu_r, b^\nu_s] &= \eta^{\mu\nu}\delta_{r+s} \; , \end{aligned} \tag{1.60}$$

(all other commutators vanishing) and similar relations for the commutator of the $\bar{\alpha}$'s, if considering the closed string.

This means that we can divide all modes in two sets, positive and negative, and identify one of them (for example, the positive subset) as annihilation operators for harmonic oscillators. On

[7]The solution for the bosonic fields, stays as in the purely bosonic case.

the cylinder, the modding for the NS fermions is half-integer and integer for the R fermions. We can now set up a convenient Fock vacuum (in a sector with a given center of mass momentum, $p^\mu$.) by

$$\left\{\begin{array}{lcll} \alpha_m^\mu \mid 0, p^\mu\rangle & = & 0 & (m>0)\ , \\ b_r^\mu \mid 0, p^\mu\rangle & = & 0 & (r>0)\ , \\ P^\mu \mid 0, p^\mu\rangle & = & p^\mu \mid 0, p^\mu\rangle & . \end{array}\right. \tag{1.61}$$

There are a few things to be noted here: The first one is that $\alpha_{-m}^0 \mid 0\rangle$ $(m>0)$ are negative-norm, ghostly, states, *i.e.* $\langle 0 \mid \alpha_m^0 \alpha_{-m}^0 \mid 0\rangle = -m\langle 0 \mid 0\rangle < 0$. The second thing to note is that, in the case of the R sector, the zero mode operators span a Clifford algebra, $\{b_0^\mu, b_0^\nu\} = \eta^{\mu\nu}$, so that they can be represented in terms of Dirac $\gamma$-matrices.

Recalling again that the Virasoro algebra reads

$$[L_m, L_n] \;=\; (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}\ , \tag{1.62}$$

where the central charge is equal to the dimension of the external spacetime, $c = d$, it is plain that we cannot impose the vanishing of the $L_n$'s as a strong constraint. Instead we can impose them as a weak constraint. Therefore we impose

$$NS\ :\ \left\{\begin{array}{lcll} L_m \mid Phys\rangle & = & 0 & m>0 \\ (L_0 - a) \mid Phys\rangle & = & 0 & \\ G_r \mid Phys\rangle & = & 0 & r \geq 1/2 \\ \big(L_0 - \bar{L}_0\big) \mid Phys\rangle & = & 0 & \end{array}\right. \tag{1.63}$$

$$R\ :\ \left\{\begin{array}{lcll} L_m \mid Phys\rangle & = & 0 & m \geq 0 \\ G_r \mid Phys\rangle & = & 0 & r \geq 0 \end{array}\right.$$

Imposing the absence of negative norm states, then once again states that $a = 1$, but that $D = 10$, *i.e.* the superstring lives in a ten dimensional space-time.

**Open String Spectrum and GSO Projection**

It is now necessary to discriminate between the different sectors.

**NS sector**: The ground state, *i.e.* the oscillator vacuum, satisfies $M^2 \mid 0, p^i\rangle = -2 \mid 0, p^i\rangle$. The first excited state $b_{-1/2}^i \mid 0, p^i\rangle$ is a $(d-2)$ vector and is massless.

**R sector**: Let $\mid a\rangle$ be a state such that $b_0^\mu \mid a\rangle = \frac{1}{\sqrt{2}}(\gamma)^{\mu a}{}_b \mid b\rangle$, meaning that it defines an $SO(1,9)$ spinor with a priori $2^5 = 32$ complex components, which after imposing the Majorana-Weyl condition are reduced to 16 real components (8 on shell). This number is exactly the number that can be created with the oscillators $b_0^i$. The root of this fact is the famous triality symmetry of $SO(8)$ between the vector and the two spinor representations, the three of having dimension 8.

There are then two possible chiralities: $\mid a\rangle$ or $\mid \bar{a}\rangle$, and $M^2 = 0$, because oscillators do not contribute.

We are free to attribute arbitrarily a given fermion number to the vacuum.

$$(-)^F \mid 0\rangle_{NS} \;=\; - \mid 0\rangle_{NS} \tag{1.64}$$

This gives $(-)^F = -1$ for states created out of the NS vacuum by an even number of fermion operators. Gliozzi, Sherk and Olive (GSO) [52] proposed to truncate the theory, by eliminating all states with $(-)^F = -1$. It is highly nontrivial to show that this leads to a consistent theory, but actually it does, moreover, it leads to spacetime supersymmetry.[8] We demand then that all states obey $(-)^F_{NS} = 1$, thus eliminating the tachyon. This is called the GSO projection. On the Ramond sector, we define a generalized chirality operator, such that it counts ordinary fermion numbers and on the R vacuum,

$$(-)^F \mid a\rangle = \mid a\rangle \ , \ (-)^F \mid \bar{a}\rangle \ = \ - \mid \bar{a}\rangle \ , \tag{1.65}$$

There is now some freedom: To be specific, on the R sector we can demand either $(-)^F_R = 1$ or $(-)^F_R = -1$.

There is a rationale for all this: Since the tachyon is a bosonic state and is highly undesirable, we want to get rid of it. This is done by projection out all the states, created out of the tachyon state by applying a bosonic set of creation operators on it. Note that this also means that the ever present tower of states belonging to the bosonic string is also projected out. To put it differently: If we accept as physical the vector boson state, GSO amounts to projecting away all states related to it through an *odd* number of fermionic $\psi$-oscillators.

**Closed Superstring Spectrum**

The difference with the above case is that one has to consider as independent sectors the left and right movers.

**(NS,NS) sector**: The composite ground state is the tensor product of the NS vacuum for the right movers and the NS vacuum for the left-movers, and as such it drops out after the GSO projection. The first states surviving the GSO projection, that is $(-1)^F = (1,1)$, are

$$\bar{b}^i_{-1/2} \mid 0\rangle_L \otimes b^j_{-1/2} \mid 0\rangle_R \ . \tag{1.66}$$

Decomposing this in irreducible representations of the little group $SO(8)$ yields $\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}$ showing that it is equivalent to a scalar $\phi$, the singlet, an antisymmetric 2-form field $B_{\mu\nu}$, the $\mathbf{28}$, and a symmetric 2-tensor field $g_{\mu\nu}$, the $\mathbf{35}$.

**(R,R) sector, type IIA**: The massless states are of the form $(-1)^F = (-1,1)$

$$\mid \bar{a}\rangle_L \otimes \mid b\rangle_R \ , \tag{1.67}$$

and decompose as $\mathbf{8}_v \oplus \mathbf{56}_v$, corresponding to a vector field, a one-form $A_1$, and a 3-form field, $A_3$.

This is however but part of the story: The state created is a bispinor, which, by making use of the ten dimensional Clifford algebra, can be expanded in term of some form fields [95], to wit: a function $F_{(0)}$, a two form $F_{(2)}$ and a four form $F_{(4)}$. One can also introduce a six and an eight form, which are however Hodge dual, due to the Clifford algebra, to the two and the four form fields. Analysis of the superstring constraints, then reveals the fact that the form fields need satisfy the Bianchi identity $dF_{(2n)} = 0$. In other words, the fundamental fields are a one form field and a three form field, and $F_{(0)}$ has to be constant. Now since this constant transforms is not a field, it does not occur in the above mentioned Clebsch-Gordan series, and has been overlooked until Polchinski noted [93], that there is a supergravity of type IIA, where there is an arbitrary constant. This massive type IIA goes under the name of Romans' theory [98].

**(R,R) sector, type IIB**: The massless states, with $(-1)^F = (1,1)$ are

$$\mid a\rangle_L \otimes \mid b\rangle_R \ , \tag{1.68}$$

[8] Note that in this case, the massless fields consist in 8 boson- and 8 spinor degrees of freedom, which is enough to fill an $N = 1$ on shell supermultiplet.

and they decompose as $\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_s$ corresponding to a pseudo scalar, $\chi$, a 2-form field, $A_2$, and a selfdual 4-form field, $A_4$.

**(R,NS) sector, Type IIA**: The first GSO surviving states, with $(-1)^F = (-1,1)$, are

$$\mid \bar{a}\rangle_L \otimes b^i_{-1/2} \mid 0\rangle_R \;, \tag{1.69}$$

and they decompose as $\mathbf{8}_s \oplus \mathbf{56}_s$.

**(R,NS) sector, Type IIB**: The first GSO surviving states, with $(-1)^F = (1,1)$ are

$$\mid a\rangle_L \otimes b^i_{-1/2} \mid 0\rangle_R \;, \tag{1.70}$$

and they decompose as $\mathbf{8}_c \oplus \mathbf{56}_c$.

**(NS,R) sector, Type IIA**: The first GSO surviving states, with $(-1)^F = (1,-1)$ are

$$\bar{b}^i_{-1/2} \mid 0\rangle_L \otimes \mid \bar{a}\rangle_R \;, \tag{1.71}$$

and decompose as $\mathbf{8}_s \oplus \mathbf{56}_s$.

**(NS,R) sector, Type IIB**: The first GSO surviving states, with $(-1)^F = (1,1)$, are

$$\bar{b}^i_{-1/2} \mid 0\rangle_L \otimes \mid \; a\rangle_R \;, \tag{1.72}$$

and decompose as $\mathbf{8}_c \oplus \mathbf{56}_c$. The $\mathbf{56}_c$ corresponds to two gravitinos.

Although it lies outside the scope of this meager introduction to string theory, it can be shown that due to the GSO projection, the closed superstring leads to $N = 2$ space-time supersymmetry [53].

## 1.3. Strings on Curved Manifolds

Up to now we have had a look at (super)strings moving on manifolds where we had something like Lorentz invariance, where we saw that the targetspace spectrum always included a massless spin 2 field. This of course looks like a *graviton*, but does it lead to gravitation? And how does it lead to gravitation?

One would be inclined to think that the string, or rather a set of strings would determine a possible targetspace curvature dynamically. Although this might be true, it is still out of reach. One can however be less ambitious, and couple the string to some background fields, however this may arise, and look for the conditions of consistent string evolution. Consistency is then obviously defined by the non-breaking at the quantum level of the classical symmetries. This will lead to the *$\beta$-equations*, which in its turn will lead to the effective actions.

Let us have a look at the bosonic closed string: The massless modes are a symmetric, an anti-symmetric fields and a scalar. Coupling this to some test string, in the only possible way, leads to the so-called *non-linear sigma model*:

$$\mathcal{S}_\sigma \;=\; \frac{T}{2}\int d^2\xi\; \eta^{ab}G(X)_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \;+\; \epsilon^{ab}B(X)_{\mu\nu}\partial_a X^\mu\partial_b X^\nu \;+\; \frac{1}{T}\sqrt{-h}\Phi(X)R^{(2)} \;, \tag{1.73}$$

Note that in this case the $G$, $B$ and $\Phi$ are given on the forehand, and should therefore be regarded as couplings: They are the long-range fields generated by the strings, and the only interaction between the test string and the rest of the strings is mediated by these fields.

An interesting thing to note is that shifting the dilaton as

$$\Phi \;=\; \langle\Phi\rangle \;+\; \hat{\Phi} \;, \tag{1.74}$$

every 'diagram' in the perturbative expansion of $e^{-\mathbf{S}_\sigma}$ bears a factor $e^{(2r-2)\langle\Phi\rangle}$, where $r$ is the genus of the Riemann surface. Comparing this to a normal perturbative expansion, one is forced to identify $g_s = e^{\langle\Phi\rangle}$, where $g_s$ denotes the *string coupling constant*. This physically means that the vacuum expectation value of the dilaton gives the coupling constant, which is then promoted to a dynamical field.

If we then impose conformal invariance of the string up to first loop, we end up with the $\beta$-functions of the couplings

$$
\begin{aligned}
0 &= R_{\mu\nu} - 2\nabla_\mu\partial_\nu\Phi + \tfrac{1}{4}H_{\mu\kappa\rho}H_\nu{}^{\kappa\rho} \,, \\
0 &= 4\Box\Phi - 4\,(\partial\Phi)^2 - R - \tfrac{1}{2\cdot 3!}H^2 \,, \\
0 &= \nabla_\mu\left(e^{-2\Phi}H^{\mu\nu\rho}\right) \,, \qquad (1.75)
\end{aligned}
$$

where $H \equiv dB$ and all the curvatures are calculated using $G_{\mu\nu}$ as the metric. As it so happens, the above equations can be derived from

$$
\mathcal{S}_\sigma = \int d^{10}x\sqrt{-G}e^{-2\Phi}\left[R - 4\,(\partial\Phi)^2 + \tfrac{1}{2\cdot 3!}H^2\right] \,. \qquad (1.76)
$$

This action is not the 'standard' form of Einstein's action, and is therefore said to be written in the '*Sigma*' frame. By redefining the metric as

$$
G_{\mu\nu} \equiv e^{-\Phi/2}g_{\mu\nu} \,, \qquad (1.77)
$$

action (1.76) is transformed to the, so called, '*Einstein frame*', *i.e.*

$$
\mathcal{S}_E = \int d^{10}x\sqrt{-g}\left\{R + \tfrac{1}{2}\,(\partial\Phi)^2 + \tfrac{1}{2\cdot 3!}e^{-\Phi}H^2\right\} \,. \qquad (1.78)
$$

What does this tell us about the effective actions for the various superstrings? As we saw in section (1.2), every string contains the bosonic string, for which the above arguments hold, and leads to targetspace supersymmetry. These two things together imply that the low-energy effective actions for the superstring should be some *10-dimensional supergravity*!!

Since the aim of this thesis is type II supergravity, we will discuss only the two type II supergravities.

### Type IIA Supergravity

Apart from the SCS, the type IIA sugra consists of a one-form field, $C^{(1)}$, and a 3-form field, $C^{(3)}$. This field content represents the $D0$- and $D2$-brane occurring as the massless RR sector in type IIA strings. By Hodge duality, one can introduce the form-fields representing the $D4$- and $D6$-branes. Note that the $D8$-brane is not included: It will be dealt with in chapter (3).

The, bosonic part of the, action reads

$$
\begin{aligned}
\mathcal{S}_{IIA} &= \int d^{10}x\sqrt{\mid g\mid}\left[e^{-2\phi}\left\{R - 4\,(\partial\phi)^2 + \tfrac{1}{2\cdot 3!}H^2\right\} - \tfrac{1}{2}G_{(2)}^2 - \tfrac{1}{2\cdot 4!}G_{(4)}^2\right] \\
&\quad -\tfrac{1}{144}\int d^{10}x\,\epsilon\,\partial C^{(3)}\partial C^{(3)}B \,, \qquad (1.79)
\end{aligned}
$$

where $G_{(2)} = dC^{(1)}$ and $G_{(4)} = dC^{(3)} - H\wedge C^{(1)}$.

This action is, apart from coordinate invariance and the invariance of the SCS, invariant under

$$
\begin{aligned}
\delta\,C^{(1)} &= d\Lambda^{(0)} \,, \\
\delta\,C^{(3)} &= d\Lambda^{(2)} - H\,\Lambda^{(0)} \,. \qquad (1.80)
\end{aligned}
$$

The supersymmetry variations, in the sigma frame, are given by

$$\begin{aligned}
\delta\psi_\mu &= \nabla_\mu\epsilon - \tfrac{1}{8}\,\not{H}_\mu\gamma_{11}\epsilon + \tfrac{i}{16}e^{\phi}\,\not{G}_{(2)}\gamma_{11}\gamma_\mu\epsilon + \tfrac{i}{8\cdot 4!}e^{\phi}\,\not{G}_{(4)}\gamma_\mu\epsilon\,, \\
\delta\lambda &= \not{\partial}\phi + \tfrac{1}{2\cdot 3!}\gamma_{11}\,\not{H} - \tfrac{3i}{8}e^{\phi}\,\not{G}_{(2)}\gamma_{11}\epsilon + \tfrac{i}{4\cdot 4!}e^{\phi}\,\not{G}_{(4)}\epsilon\,.
\end{aligned} \tag{1.81}$$

**Type IIB Supergravity**

It is well-known [102] that it is not possible to write a covariant action whose minimization gives the equations of motion 10-dimensional type IIB supergravity. The problematic equation of motion is the self-duality of the 5-form field strength. However, we can use it to find an alternative equation of motion just by replacing the 5-form field strength by its Hodge dual in the Bianchi identity. This alternative equation of motion has the conventional form of the equation of motion of a 4-form potential and it is possible to find an action from which to derive this and the other equations of motion *but not self-duality.* This NSD action, supplemented by the self-duality constraint gives all the equations of motion of the type IIB theory.

The (bosonic sector of the) string-frame NSD action is[9]

$$\begin{aligned}
S_{\rm NSD} &= \int d^{10}x\sqrt{|\jmath|}\,\Big\{e^{-2\varphi}\Big[R(\jmath) - 4\left(\partial\varphi\right)^2 + \tfrac{1}{2\cdot 5!}\mathcal{H}^{\in}\Big] \\
&\qquad +\tfrac{1}{2}\left(G_{(1)}\right)^2 + \tfrac{1}{2\cdot 3!}\left(G_{(3)}\right)^2 + \tfrac{1}{4\cdot 3!}\left(G_{(5)}\right)^2 \\
&\qquad -\tfrac{1}{192}\frac{1}{\sqrt{|\jmath|}}\,\epsilon\,\partial C^{(4)}\partial C^{(2)}\mathcal{B}\Big\}\,,
\end{aligned} \tag{1.82}$$

where $\{\jmath_{\mu\nu}, \mathcal{B}_{\mu\nu}, \varphi\}$ are the NS-NS fields: The type IIB string metric, the type IIB NS-NS 2-form and the type IIB dilaton respectively.

$$\mathcal{H}_{\mu\mu\rho} = 3\partial_{[\mu}\mathcal{B}_{\nu\rho]}\,, \qquad (\mathcal{H} = 3\partial\mathcal{B})\,, \tag{1.83}$$

is the NS-NS 2-form field strength. $\{C^{(0)}, C^{(2)}{}_{\mu\nu}, C^{(4)}{}_{\mu\nu\rho\sigma}\}$ are the RR potentials. Their field strengths and gauge transformations are

$$\left\{\begin{aligned}
G_{(1)} &= \partial C^{(0)}\,, \\
G_{(3)} &= 3\left(\partial C^{(2)} - \partial\mathcal{B}C^{(0)}\right)\,, \\
G_{(5)} &= 5\left(\partial C^{(4)} - 6\partial\mathcal{B}C^{(2)}\right)\,.
\end{aligned}\right. \tag{1.84}$$

and

$$\left\{\begin{aligned}
\delta C^{(0)} &= 0\,, \\
\delta C^{(2)} &= 2\partial\Lambda^{(1)}\,, \\
\delta C^{(4)} &= 4\partial\Lambda^{(3)} + 6\mathcal{B}\partial\Lambda^{(1)}\,,
\end{aligned}\right. \tag{1.85}$$

[9] Our conventions are essentially those of Ref. [25] but we change the symbols denoting NS-NS fields in the type IIB theory to distinguish them from those of the We use the index-free notation of Ref. [25]: When indices are not explicitly shown, they are assumed to be completely antisymmetrized with weight one. The definition of field strengths and gauge transformations are inspired by those of Refs. [54, 22].

respectively.

The equations of motion derived from the above action have to be supplemented by the self-duality condition

$$G_{(5)} = + {}^{\star}G_{(5)}\,. \tag{1.86}$$

In the original version of the 10-dimensional, chiral $N = 2$ supergravity [102] the theory has a classical $SU(1,1)$ global symmetry. The two scalars parametrize the coset $SU(1,1)/U(1)$, $U(1)$ being the maximal compact subgroup of $SU(1,1)$, and transform under a combination of a global $SU(1,1)$ transformation and a local $U(1)$ transformation which depends on the global $SU(1,1)$ transformation. They are combinations of the dilaton and the RR scalar. The group $SU(1,1)$ is isomorphic to $SL(2,\mathbb{R})$, the conjectured classical S duality symmetry group for the type IIB string theory [63]. A simple field redefinition [15] is enough to rewrite the action in terms of two real scalars parametrizing the coset $SL(2,\mathbb{R})/SO(2)$ which can now be identified with the dilaton $\varphi$ and the RR scalar $C^{(0)}$.

In order to make the S duality symmetry manifest, we first have to rescale the metric as to go to the Einstein frame:

$$\jmath_{E\,\mu\nu} = e^{-\varphi/2}\jmath_{\mu\nu}\,. \tag{1.87}$$

We now have to make some further field redefinitions. For instance, while the NS-NS and RR 2-forms we are using form an $SL(2,\mathbb{R})$ doublet, their field strengths do not. Furthermore, our self-dual RR 4-form potential $C^{(4)}$ is not $SL(2,\mathbb{R})$-invariant. Thus, for the purpose of exhibiting the $SL(2,\mathbb{R})$ symmetry it is convenient to perform the following field redefinitions[10]:

$$\left\{\begin{array}{rcl} \vec{\mathcal{B}} & = & \begin{pmatrix} C^{(2)} \\ \mathcal{B} \end{pmatrix}, \\ \\ D & = & C^{(4)} - 3\mathcal{B}C^{(2)}\,, \end{array}\right. \tag{1.88}$$

These new fields undergo the following gauge transformations:

$$\left\{\begin{array}{rcl} \delta\vec{\mathcal{B}} & = & 2\vec{\Sigma}\,, \\ \\ \delta D & = & 4\partial\Delta + 2\vec{\Sigma}^{\,T}\eta\,\vec{\mathcal{H}}\,, \end{array}\right. \tag{1.89}$$

and have field strengths

$$\left\{\begin{array}{rcl} \vec{\mathcal{H}} & = & 3\partial\vec{\mathcal{B}}\,, \\ \\ F & = & G_{(5)} = + {}^{\star}F \\ \\ & = & 5\left(\partial D - \vec{\mathcal{B}}^{\,T}\eta\,\vec{\mathcal{H}}\right)\,, \end{array}\right. \tag{1.90}$$

where $\eta$ is the $2\times 2$ matrix

$$\eta = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\eta^{-1} = -\eta^T\,, \tag{1.91}$$

Given the isomorphism $SL(2,\mathbb{R}) \sim Sp(2,\mathbb{R})$, it can be identified with an invariant metric:

[10]Our conventions are such that all fields are either invariant or transform *covariantly* as opposed to *contravariantly*.

$$\Lambda\eta\Lambda^T = \eta\,, \;\Rightarrow\; \eta\Lambda\eta^T = (\Lambda^{-1})^T\,, \qquad \Lambda \in SL(2,\mathbb{R})\,. \tag{1.92}$$

Finally, it is convenient to define the $2\times 2$ matrix $\mathcal{M}_{ij}$

$$\mathcal{M} = e^{\varphi}\begin{pmatrix} |\lambda|^2 & C^{(0)} \\ C^{(0)} & 1 \end{pmatrix}\,, \qquad \mathcal{M}^{-1} = e^{\varphi}\begin{pmatrix} 1 & -C^{(0)} \\ -C^{(0)} & |\lambda|^2 \end{pmatrix}\,, \tag{1.93}$$

where $\lambda$ is the complex scalar

$$\lambda = C^{(0)} + ie^{-\varphi}\,. \tag{1.94}$$

Observe that $\mathcal{M}$ is a symmetric $SL(2,\mathbb{R})$ matrix and therefore, as a consequence of Eq. (1.92) it has the property

$$\mathcal{M}^{-1} = \eta\mathcal{M}\eta^T\,, \tag{1.95}$$

To see that $\lambda$ parametrizes the $SL(2,\mathbb{R})/SO(2)$ coset, it is convenient to consider how one arrives at $\mathcal{M}$. First one considers the non-symmetric $SL(2,\mathbb{R})$ matrix $V$

$$V = \begin{pmatrix} e^{-\varphi/2} & e^{\varphi/2}C^{(0)} \\ 0 & e^{\varphi/2} \end{pmatrix}\,. \tag{1.96}$$

This $SL(2,\mathbb{R})$ matrix is generated by only two of the three $SL(2,\mathbb{R})$ generators and it should cover the $SL(2,\mathbb{R})/SO(2)$ coset. The choice for the form of $V$ can be understood as a choice of gauge or as a choice of coset representatives. However, an arbitrary $SL(2,\mathbb{R})$ transformation $\Lambda$ will transform $V$ into a non-upper-triangular matrix $\Lambda V$ (which is not a coset representative). A further $\Lambda$-dependent $SO(2)$-transformation $h$ will, by using the definition of a coset, take us to another coset representative $V' = \Lambda V h$. The transformation $h$ will be *local* but not arbitrary. It can be thought of as a compensating gauge transformation. The condition that $V'$ is upper-triangular fully determines $h(\Lambda, V)$ and the transformations of $C^{(0)}$ and $\varphi$:

$$\begin{aligned} V' &= \begin{pmatrix} e^{-\varphi'/2} & e^{\varphi'/2}C^{(0)\,\prime} \\ 0 & e^{\varphi'/2} \end{pmatrix} = \Lambda V h = \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e^{-\varphi/2} & e^{\varphi/2}C^{(0)} \\ 0 & e^{\varphi/2} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\,, \end{aligned} \tag{1.97}$$

where $ad - bc = 1$. The result is that the parameter $\theta$ of the compensating transformation $h$ is given by

$$\tan\theta = \frac{c}{e^{\varphi}\left(cC^{(0)} + d\right)}\,, \tag{1.98}$$

and the transformation of the scalars can be written in the compact form

$$\lambda' = \frac{a\lambda + b}{c\lambda + d}\,. \tag{1.99}$$

The symmetric matrix $\mathcal{M}$ is now $\mathcal{M} = V\,V^T$ and transforms under $\Lambda \in SL(2,\mathbb{R})$ according to

$$\mathcal{M}' = \Lambda\mathcal{M}\Lambda^T\,, \tag{1.100}$$

which is completely equivalent to the above transformation of $\lambda$. Observe that it is not necessary to worry about the $h$-transformations anymore.

It is also worth stressing that the only $SL(2,\mathbb{R})$ transformations that leave invariant $\lambda$ or, equivalently, $\mathcal{M}$ or $V$ are $\pm\mathbb{I}_{2\times 2}$. This is an important point: $SO(2)$ is sometimes referred to as the "stability subgroup". Had we defined the coset by the equivalence relation $V \sim h|\,V\,,\ h\in SO(2)$, then, by definition, $V$ would have been invariant under any $\Lambda \in SO(2)$. Then, $SO(2)$ would have been the subgroup of $SL(2,\mathbb{R})$ leaving invariant the coset scalars. This is, however, *not* the way in which this coset is constructed and (as it can be explicitly checked) there is no stability subgroup of $SL(2,\mathbb{R})$ in that sense apart from this almost trivial $\mathbb{Z}_2$.

Under this $\Lambda$, the doublet of 2-forms transforms

$$\vec{\mathcal{B}}' = \Lambda\vec{\mathcal{B}}\,, \tag{1.101}$$

and the 4-form $D$ and the Einstein metric are inert.

Now, it is a simple exercise to rewrite the NSD type IIB action in the following manifestly S duality invariant form

$$\begin{aligned}\mathcal{S}_{\rm NSD} \;=\;& \tfrac{1}{16\pi G_N^{(10)}}\int d^{10}x\,\sqrt{|\jmath_E|}\,\Big\{R(\jmath_E)+\tfrac{1}{4}{\rm Tr}\left(\partial\mathcal{M}\mathcal{M}^{-1}\right)^2\\ & +\tfrac{1}{2\cdot 3!}\vec{\mathcal{H}}^{\,T}\mathcal{M}^{-1}\vec{\mathcal{H}}+\tfrac{1}{4\cdot 3!}F^2-\tfrac{1}{2^7\cdot 3^3}\tfrac{1}{\sqrt{|\jmath_E|}}\,\epsilon\,D\,\vec{\mathcal{H}}^{\,T}\eta\,\vec{\mathcal{H}}\Big\}\,,\end{aligned} \tag{1.102}$$

It is easy to find how the fields $\mathcal{H}, G_{(3)}, C^{(4)}$ in the action Eq. (1.82) transform under $SL(2,\mathbb{R})$:

$$\left\{\begin{array}{rcl}\mathcal{H}' &=& \left(d+cC^{(0)}\right)\mathcal{H}+cG_{(3)}\,,\\[2mm] G_{(3)}\,' &=& \frac{1}{|c\lambda+d|^2}\left[\left(d+cC^{(0)}\right)G_{(3)}-ce^{-2\varphi}\mathcal{H}\right]\,,\\[2mm] C^{(4)\,\prime} &=& C^{(4)}-3\left(\begin{array}{cc}C^{(2)} & \mathcal{B}\end{array}\right)\left(\begin{array}{cc}ac & bc\\ bc & db\end{array}\right)\left(\begin{array}{c}C^{(2)}\\ \mathcal{B}\end{array}\right)\,.\end{array}\right. \tag{1.103}$$

$\lambda$ transforms as above and we stress that the string metric does transform under $SL(2,\mathbb{R})$:

$$\jmath' = |c\lambda+d|\jmath\,. \tag{1.104}$$

The string-frame supersymmetry variations[11] are

$$\begin{aligned}\delta\psi_\mu &= \nabla_\mu\epsilon-\tfrac{1}{8}\,\not{\mathcal{H}}_\mu\sigma^3\epsilon+\tfrac{1}{16}e^{\varphi}\textstyle\sum_{n=1}^{5}\tfrac{1}{(2n-1)!}\;\not{G}^{(2n-1)}\Gamma_\mu\mathcal{P}_n\epsilon\\ \delta\lambda &= \not{\partial}\varphi\epsilon-\tfrac{1}{2\cdot 3!}\,\not{\mathcal{H}}\,\sigma^3\epsilon+\tfrac{1}{4}e^{\varphi}\textstyle\sum_{n=1}^{5}\tfrac{n-3}{(2n-1)!}\;\not{G}^{(2n-1)}\,\mathcal{P}_n\epsilon\end{aligned} \tag{1.105}$$

where

$$\mathcal{P}_n \;=\; \left\{\begin{array}{lcl}\sigma^1 & : & n\ even\\ i\sigma^2 & : & n\ odd\end{array}\right. \tag{1.106}$$

[11]The 2 chiral spinors are grouped into a 2 vector.

### 1.3.1. T-duality and Background Fields: Buscher's Map

As we have seen in Sec. (1.1.3), a string moving on a manifold compactified on a circle, enjoys T-duality. We now ask ourselves whether there is something analogous to this in supergravity theories. The answer was first found by Buscher [30] but we will use ideas taken from [2, 97]. The basic idea in T-duality is that one of the target space directions was compactified. In the sigma model, this translates to the requirement that the background fields admit to an isometry, *i.e.* the background fields are independent, up to possible gauge transformations, of this direction. Taking the isometry direction to be $Y$, only the combination $\partial_a Y$ occurs in the sigma model. This then means that we can Hodge dualize the field strength $\partial_a Y$: Substitute $\Theta_a = \partial_a Y$, introduce a Lagrange multiplier $\chi$ imposing $d\Theta = 0$[12] and making use of the equation of motion of $\Theta_a$ to integrate it out of the action, *et voilá* the result is another sigma model, once we interpret $\chi$ as a new coordinate. The explicit mapping, known as Buscher's transformation [30], is given by

$$\begin{aligned}
\tilde{G}_{\chi\chi} &= \tfrac{1}{G_{YY}} , & \tilde{G}_{\chi m} &= \tfrac{B_{mY}}{G_{YY}} , \\
\tilde{B}_{m\chi} &= \tfrac{G_{Ym}}{G_{YY}} , & \tilde{G}_{mn} &= G_{mn} - \tfrac{G_{mY}G_{nY} - B_{mY}B_{nY}}{G_{YY}} , \\
\tilde{B}_{mn} &= B_{mn} + 2\tfrac{B_{Y[m}G_{n]Y}}{G_{YY}} .
\end{aligned} \tag{1.107}$$

As one can see by doing the above transformation twice, it defines a $\mathbb{Z}_2$ mapping between sigma models.

The thing is that, although the above formulae relate sigma models, starting with a model with vanishing $\beta$-functions, one does not end up with a background that satisfies them. The situation is however ameliorated by taking the dilaton to transform as well [30], *i.e.*

$$\tilde{\Phi} = \Phi - \tfrac{1}{2}\log\left(\mid G_{YY} \mid\right) . \tag{1.108}$$

Although there are various ways of demonstrating the above, in this case it best shown using the action (1.76): Imposing the existence of an isometry in our theory, we can apply Kaluza-Klein reduction. Using the standard techniques we decompose the Zehnbein as

$$E_{\hat{\mu}}{}^{\hat{a}} = \begin{pmatrix} e_m{}^a & kA_m \\ 0 & k \end{pmatrix} \longrightarrow \left\{ \begin{array}{lcl} k &=& \sqrt{-G_{YY}} , \\ A_m &=& G_{Ym}G_{YY}^{-1} , \\ g_{mn} &=& G_{mn} - G_{mY}G_{nY}G_{YY}^{-1} , \end{array} \right. , \tag{1.109}$$

and decompose the Kalb-Ramond field, in form notation, as

$$\hat{B} = B - \tfrac{1}{2}A \wedge C + C \wedge dy , \tag{1.110}$$

and the dilaton, $\hat{\Phi}$, is taken to be $y$ independent. Thus reducing (1.76) to 9 dimensions, one obtains

$$\int d^9x\sqrt{g}e^{-2\hat{\Phi}}k\left[R(g) + 4\partial\log(k)\partial\hat{\Phi} - 4\left(\partial\hat{\Phi}\right)^2 + \tfrac{1}{2\cdot 3!}H^2 - \tfrac{1}{2\cdot 3!}k^2F(A)^2 - \tfrac{1}{2\cdot 3!}k^{-2}F(C)^2\right] , \tag{1.111}$$

where $F(A) = dA$, $F(C) = dC$ and

$$H = dB - \tfrac{1}{2}A \wedge F(C) - \tfrac{1}{2}C \wedge F(A) . \tag{1.112}$$

[12] This imposes that, locally, $\Theta$ is indeed a field strength, so that one does not introduce extra degrees of freedom.

By defining then the field $\phi \equiv \hat{\Phi} - \frac{1}{2}\log(k)$, the above action takes the form

$$\int d^9x\sqrt{g}e^{-2\phi}\left[R(g) + (\partial\log(k))^2 - 4\,(\partial\phi)^2 + \tfrac{1}{2\cdot 3!}H^2 - \tfrac{1}{2\cdot 3!}k^2F(A)^2 - \tfrac{1}{2\cdot 3!}k^{-2}F(C)^2\right] . \tag{1.113}$$

It is obvious that the above, 9-dimensional, action is invariant under the $\mathbb{Z}_2$ mapping

$$k \;\rightarrow\; k^{-1} \quad , \quad A_\mu \;\leftrightarrow\; C_\mu \; , \tag{1.114}$$

and the rest of the fields invariant. It will not come as a surprise that upon re-expressing this 9-dimensional mapping in terms of 10-dimensional fields, one obtains the Buscher transformation (1.107,1.108).

## 1.4. $D = 11$ Sugra and M Theory

It was known for quite some time that the maximal admissible sugra was $N = 1$ $D = 11$ supergravity.[13] The gravitational $N = 1$ multiplet consists of a spin 2 field, the metric, $g_{\mu\nu}$, or, if you like, the Elfbein, a Majorana Rarita-Schwinger field, $\Psi_\mu$, and a 3-form field, $C^{(3)}$. The bosonic part of the sugra action reads

$$\mathcal{S} \;=\; \int d^{11}x\sqrt{-g}\left[R(g) \;+\; \tfrac{1}{2\cdot 4!}G^2_{(4)}\right] \;-\; \tfrac{1}{6}\int_{11} C^{(3)}\wedge dC^{(3)}\wedge dC^{(3)} \; , \tag{1.115}$$

where $G_{(4)} \;=\; dC^{(3)}$. Apart from invariance under general coordinate transformations, the action is invariant under $\delta C^{(3)} \;=\; d\Lambda^{(2)}$ which then is about it regarding symmetries of the action. There is a scaling symmetry that scales the action, though.

The supersymmetry variation reads, all spinors are Majorana,

$$\delta\Psi_\mu \;=\; \nabla_\mu\epsilon \;+\; \tfrac{i}{2^5\cdot 3^2}\left[\Gamma_{\alpha\beta\gamma\delta\mu} - 8\Gamma_{\beta\gamma\delta}\; g_{\alpha\mu}\right] G^{\alpha\beta\gamma\delta}_{(4)}\epsilon \; . \tag{1.116}$$

Using dimensional reduction, *i.e.* using the decomposition as displayed in appendix (B.1), one can see that the above action becomes the type IIA action given above.

It has been argued [6], that the correct and complete supertranslation algebra which is valid for this case is

$$\{\mathcal{Q},\mathcal{Q}\} \;=\; i\Gamma^\mu\mathcal{C}^{-1}\,P_\mu + \tfrac{1}{2!}\Gamma^{\mu\nu}\mathcal{C}^{-1}Z^{(2)}_{\mu\nu} + \tfrac{i}{5!}\Gamma^{\mu_1\cdots\mu_5}\mathcal{C}^{-1}Z^{(5)}_{\mu_1\cdots\mu_5} \; , \tag{1.117}$$

where $\mathcal{Q}$ is a Majorana spinor operator, generating the susy transformations, and the $Z$'s are central form-charges, *i.e.* they transform as forms under the Lorentz group but are central with respect to the rest of the Susy algebra.

Since the $D = 11$ supergravity contains a three form, and this couples naturally to a three surface in space-time, one might be inclined to think that there is an extended object, a membrane, also called the '*M2*' brane, in a theory having as a low energy limit the $D = 11$ supergravity.[14] Dually, the theory can also be formulated with a 6 form field [25], which then signals the existence of a *M5* brane.

Taking the existence of these M-branes seriously, one can compactify the membrane over a circle, thus naively leading to a string when we restrict ourselves to the zero modes. If the

[13]Going beyond this limit, would mean that the gravitational multiplets necessarily incorporate particles with spin 5/2 and up, which cannot be done consistently.

[14]One can write down Nambu-Goto type actions for the M2 brane, moving in a given background [13], and then consistency implies that the background satisfies the equations of motion for $D = 11$ supergravity.

radius of the eleventh dimension is R, and we denote the M-2-brane tension by $T_3 \equiv l_{11}^{-3}$, the string tension (traditionally denoted by $\alpha'$) will be given by $T_2 = \frac{R}{l_{11}^3} \equiv \frac{1}{l_s^2} \equiv \frac{1}{\alpha'}$. This gives the string length as

$$l_s = \frac{l_{11}^{3/2}}{R^{1/2}} \; . \tag{1.118}$$

The mass of the first Kaluza-Klein excitation with one unit of momentum in the eleventh direction is $M(KK) \equiv R^{-1}$. As we shall see later on, this state is interpreted, from the 10-dimensional point of view, as a D0-brane, and its mass could serve as a *definition* of the *string coupling constant*, $M(D0) \equiv \frac{1}{g_s l_s}$. Equating the two expressions gives

$$g_s = \frac{R}{l_s} = \left(\frac{R}{l_{11}}\right)^{3/2} \; . \tag{1.119}$$

This formula is very intriguing, because it clearly suggests that the string will only live in 10 dimensions as long as the coupling is small. The historical way in which Witten [115] arrived to this result was exactly the opposite, by realizing that the mass of a D0 brane (in 10 dimensions) goes to zero at strong coupling, and interpreting this fact as the opening of a new dimension. Although some partial evidence exists on how the full O(1,10) can be implemented in the theory (as opposed to the O(1,9) of ten-dimensional physics), there is no clear understanding about the rôle of conformal invariance (which is equivalent to BRST invariance, and selects the critical dimension) in eleven dimensional physics.

The radius could also be eliminated, yielding the beautiful formula

$$g_s = \left(\frac{l_{11}}{l_s}\right)^3 \; . \tag{1.120}$$

# Chapter 2

# Black holes and Duality

Black-hole physics is probably the only non perturbative problem in gravity in which non trivial progress has recently been made owing to the different perspective afforded by "string dualities" (for a recent review see e.g. [85]).

In the literature [29] a systematic analysis was made of the behaviour of asymptotic charges under T duality (see e.g. [51, 2] for four-dimensional non-rotating black holes[1]. The main goal of the present chapter is to extend their results, by essentially widening the class of metrics considered both by allowing a more general asymptotic behaviour and by including more non-trivial fields. This simultaneously widens the subgroup of the duality group that acts on that class preserving the asymptotic behaviour.

Therefore, we will define the asymptotic behaviour considered ("TNbh") and we will identify the subgroup that preserves it (the "ADS"). We will find that the charges naturally fit in multiplets under the action of this subgroup and that the Bogomol'nyi bound can be written as a natural invariant of this subgroup. This was to be expected since duality transformations in general respect unbroken supersymmetries, but, since duality transformations in general transform conserved charges that appear in the Bogomol'nyi bound into non-conserved charges (associated to primary scalar hair) that, in principle, do not, the consistency of the picture will require us to include those non-conserved charges into the generalized Bogomol'nyi bound. A by-product of our study will be the identification of the known supersymmetric massless black holes as the T duals in the time direction of the usual supersymmetric massive black holes. These are the main results of this chapter.

One of the motivations for this investigation was to try to constrain the angular momentum of black holes using duality and supersymmetry in such a way that the extreme limit would never be surpassed. As it is well known the striking difference between the black-hole extremality bound and the supersymmetry (or Bogomol'nyi) bound: although the angular momentum appears in the extremality bound, it does not enter the supersymmetry bound. This difference is even more surprising in view of the fact that in presence of only NUT charge (that is, for some stationary, non-static, cases) both bounds still coincide; the NUT charge squared must simply be added to the first member in the two bounds [72]. On the other hand, it is also known that some T duality transformations seem to break spacetime supersymmetry making it non-manifest [7]. These two facts could perhaps give rise to an scenario in which extremal Kerr-Newman black holes (which are not supersymmetric) could be dual to some supersymmetric configuration. At the level of the supersymmetry bounds one would see the angular momentum

[1]Since some of the objects studied are singular, as opposite to black holes with a regular horizon, the name black hole will be used in a generalized sense for (usually point-like) objects described by asymptotics such that a mass, angular momentum etc. can be assigned to them.

transforming under a non-supersymmetry-preserving duality transformation into a charge that does appear in the supersymmetry bound (like the NUT charge). In this way, the constraints imposed by supersymmetry on the charge would constraint equally the angular momentum.

Although this scenario has been disproved by the calculations that we are going to present[2] the transformation of black-hole charges and the corresponding Bogomol'nyi bounds under general string duality transformations remains an interesting subject in its own right and its study should help us gain more insight into the physical space-time meaning of duality.

Thus, in the sequel, the transformation of asymptotic observables (as multipoles of the metric or of other physical fields) of four-dimensional "black holes" under the T duality and S duality groups will be systematically analyzed, from the four-dimensional effective action point of view[3].

We are going to consider for simplicity a consistent (from the point of view of the equations of motion and of the supersymmetry transformations) truncation[4] of the four-dimensional heterotic string effective action including the metric, dilaton and two-form field plus two Abelian vector fields. This truncation is, however, rich enough to contain solutions with 1/4 of the supersymmetries unbroken [70, 88].

Since all the configurations we are going to consider are stationary and axially symmetric, the theory can be reduced to two dimensions were the T dualities due to the presence of isometries in four dimensions become evident. This we will do in Section 2.1 getting manifest $O(2,4)$ due to the presence of the two Abelian vector fields in four dimensions [84]. We will also find the S duality transformations in their four-dimensional form.

Then, in Section 2.2 we will define the asymptotic behaviour of those fields in the configurations we are interested in: (charged, rotating) black holes, Taub-NUT metrics etc. which are stationary and axially-symmetric. This class of asymptotic behaviour will be referred to as *TNbh asymptotics*. Any configuration in this class will be characterized by a set of parameters (*charges*) such as the electric and magnetic charges with respect to the gauge fields, the ADM mass, the angular momentum, the NUT charge and some other charges forced upon us by duality.

The rest of the chapter is organized as follows: In Section 2.3 we study the transformation of the charges under different elements of the T and S duality groups and show explicitly the transformations that preserve TNbh asymptotics including the effect of constant terms in the asymptotics of the vector fields (Section 2.3.4). In Section 2.4 we define the *Asymptotic Duality Subgroup* as the subgroup of the duality group that preserves TNbh asymptotics and study the transformation of the Bogomol'nyi bound under duality. We will find full agreement with the preservation of unbroken supersymmetry if we admit the presence of primary scalar hair in the generalized Bogomol'nyi bound. We illustrate this with several examples in Section 2.4.3 where we also find that the known massless supersymmetric black holes are the T duals of the common massive supersymmetric ones. Section 2.5 contains our conclusions.

## 2.1. The Derivation of the Duality Transformations

In this chapter we are going to consider a consistent truncation of the four-dimensional heterotic string effective action in the string frame including the metric, axion 2-form and two

[2] In fact, the angular momentum is part of a set of charges which transform amongst themselves under duality and never appear in the Bogomol'nyi bound.

[3] The transformation of some of the charges we are going to consider here was studied previously in Refs. [88, 71]. Here we will extend that study to other sets of charges.

[4] This truncation is also invariant under duality transformations in the compact six-dimensional space directions.

Abelian vector fields given by[5]:

$$S = \int d^4x \sqrt{|\hat{g}|}\; e^{-\hat{\phi}} \left[ R(\hat{g}) + \hat{g}^{\hat{\mu}\hat{\nu}} \partial_{\hat{\mu}}\hat{\phi} \partial_{\hat{\nu}}\hat{\phi} - \tfrac{1}{12} \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} \hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}} - \tfrac{1}{4} \hat{F}^{I}{}_{\hat{\mu}\hat{\nu}} \hat{F}^{I\hat{\mu}\hat{\nu}} \right] , \tag{2.1}$$

where $I = 1,2$ sums over the Abelian gauge fields $\hat{A}^{I}{}_{\hat{\mu}}$ with standard field strengths $\hat{F}^I = d\hat{A}^I$ and the two-form field strength is $\hat{H} = d\hat{B} - \frac{1}{2}\hat{F}^I\, \hat{A}^I$.

Before proceeding, an explanation of the origin of this action is in order. This action can be obtained from the ten-dimensional heterotic string effective action by first considering only the lowest order in $\alpha'$ terms (so the Yang-Mills fields and $R^2$ terms are consistently excluded), then compactifying the theory on $T^6$ to four dimensions following essentially Ref. [84] and afterwards setting to zero all the scalars and identifying the six Kaluza-Klein vector fields with the six vector fields that come from the ten-dimensional axion. This last truncation is done in the equations of motion and it is perfectly consistent with them and with the supersymmetry transformation rules. The result of this truncation is the action of $N = 4, d = 4$ supergravity [31] in the string frame and with the axion 2-form. Setting to zero four of the six vector fields one gets the above action.

The above action is invariant under Buscher's T duality transformations in the six compact directions because these interchange the vector fields whose origin is the ten-dimensional metric with the the vector fields whose origin is the ten-dimensional axion and we have identified these two sets of fields. There are still some trivial T duality transformations which correspond to rotations in the internal compact space. They correspond to global $O(2)$ rotations of the two vector fields ($O(6)$ rotations in the full $N = 4$ supergravity theory).

The reason why we consider two vector fields instead of six or just one is that the generating solution for black-hole solutions of the full $N = 4, d = 4$ theory only needs two non-trivial vector fields. Starting from this generating solution and performing T duality transformations in the compact space and S duality transformations (to be described later) which do not change the Einstein metric one can generate the most general black-hole solution (if the no-hair theorem holds). Also, the minimal number of vector fields required in this theory for allowing solutions with $1/4$ of the $N = 4$ supersymmetries unbroken, is two [70, 88].

As announced in the Introduction, it will be assumed that the metric has a timelike and a spacelike rotational isometry[6]. The former is physically associated to the stationary (but not static, in general) character of the spacetime and the other to the axial symmetry[7]. They commute with each other and, thus, one can find two coordinates, in this case the time $t$ and the angular variable $\varphi$, adapted to them, such that the background does not depend on them. This, then, implies that the theory can be dimensionally reduced. Using the standard technique [101] the resulting dimensionally reduced, Euclidean, action turns out to be

$$\begin{aligned} S \;=\; \int d^2x \sqrt{|g|}\; e^{-\phi} \big[ & R(g) + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \tfrac{1}{8}\mathrm{Tr}\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}^{-1} \\ & -\tfrac{1}{4} W^i_{\mu\nu} (\mathcal{M}^{-1})_{ij} W^{j\mu\nu} \big] \;, \end{aligned} \tag{2.2}$$

Now the spatial indices $\mu\nu = 2,3$ for simplicity and we also have internal indices $\alpha, \beta = 0,1$. The two-dimensional fields are the metric $g_{\mu\nu}$, six vector fields $\mathcal{K}^i{}_\mu = (K^{(1)\alpha}{}_\mu, K^{(2)}{}_{\alpha\mu}, K^{(3)I}{}_\mu)$ with

[5]The signature used in this chapter, and only in this chapter, is $(-,+,+,+)$. All hatted symbols are four-dimensional and so $\hat{\mu}, \hat{\nu} = 0,1,2,3$. The relation between the four-dimensional Einstein metric $\hat{g}_{E\hat{\mu}\hat{\nu}}$ and the string-frame metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ is $\hat{g}_{E\hat{\mu}\hat{\nu}} = e^{-\hat{\phi}}\hat{g}_{\hat{\mu}\hat{\nu}}$.

[6]The action of rotational isometries has fixed points, while translational isometries act with no fixed points.

[7]The axis corresponds obviously to the set of fixed points of the isometry.

the standard Abelian field strengths $W^i{}_{\mu\nu}$ $(i = 1, \ldots, 6)$ and a bunch of scalars $G_{\alpha\beta}, \hat{B}_{\alpha\beta}, \hat{A}^I{}_\alpha$ that appear combined in the $6 \times 6$ matrix $\mathcal{M}_{ij}$. They are given by

$$
\begin{array}{rclrcl}
G_{\alpha\beta} & = & \hat{g}_{\alpha\beta}\,, & \phi & = & \hat{\phi} - \frac{1}{2}\log|\det G_{\alpha\beta}|\,, \\[2mm]
K^{(1)\alpha}{}_\mu & = & \hat{g}_{\mu\beta}(G^{-1})^{\beta\alpha}\,, & C_{\alpha\beta} & = & \frac{1}{2}\hat{A}^I{}_\alpha \hat{A}^I{}_\beta + \hat{B}_{\alpha\beta}\,, \\[2mm]
g_{\mu\nu} & = & \hat{g}_{\mu\nu} - K^{(1)\alpha}{}_\mu K^{(1)\beta}{}_\nu G_{\alpha\beta}\,, & K^{(3)I}_\mu & = & \hat{A}^I{}_\mu - \hat{A}^I{}_\alpha K^{(1)\alpha}{}_\mu\,, \\[2mm]
K^{(2)}{}_{\alpha\ \mu} & = & \hat{B}_{\mu\alpha} + \hat{B}_{\alpha\beta}K^{(1)\beta}{}_\mu + \frac{1}{2}\hat{A}^I{}_\alpha K^{(3)I}{}_\mu\,, & & &
\end{array}
\tag{2.3}
$$

and

$$
(\mathcal{M}_{ij}) = \begin{pmatrix} G^{-1} & -G^{-1}C & -G^{-1}A^T \\ -C^T G^{-1} & G + C^T G^{-1} C + A^T A & C^T G^{-1} A^T + A^T \\ -AG^{-1} & AG^{-1}C + A & \mathbb{I}_2 + AG^{-1}A^T \end{pmatrix}, \tag{2.4}
$$

$A$ being the $2 \times 2$ matrix with entries $\hat{A}^I{}_\alpha$. If $B$ stands for the $2 \times 2$ scalar matrix $(\hat{B}_{\alpha\beta})$, then the $2 \times 2$ scalar matrix $C$ is given by

$$
C = \tfrac{1}{2}A^T A + B\,. \tag{2.5}
$$

Any explicit contribution from the three-form automatically vanishes in two dimensions, which explains why it does not occur in Eq. (2.2). On the other hand, the dynamics of a two-dimensional vector field is trivial[8] and this seems to suggest that we can safely ignore it. However, the correct procedure to eliminate the vector fields is to solve their equation of motion and then substitute the solution into the equations of motion of the other fields. The equations of motion for the vector fields in the action above tell us that the components of the fields $(\mathcal{M}^{-1})_{ij}W^j{}_{\mu\nu}$ are constant. In Ref. [106] the constants were chosen to be zero by setting the vector fields themselves to zero, which can be consistently done at the level of the action. This is obviously an additional restriction on the backgrounds considered[9]. This restriction was also made (in the purely gravitational sector) in the original article by Geroch [45] and it has been done in all the subsequent literature on this subject in the form proposed in Refs. [77] where it was expressed as the requirement that the background possess "orthogonal insensitivity", i.e. it is invariant under $(t, \varphi) \to (-t, -\varphi)$.

This restriction is crucial to obtain an infinite-dimensional algebra of invariances of the equations of motion of the two-dimensional system. As we are going to explain, though, in this chapter we are not interested in the infinite-dimensional algebra but only in its zero-mode subalgebra and so we will not impose this restriction. Nevertheless, all the configurations that we will explicitly consider will obey it.

### 2.1.1. T Duality Transformations

The matrix $\mathcal{M}$ satisfies $\mathcal{M}\mathcal{L}\mathcal{M}\mathcal{L} = \mathbb{I}_6$, with

[8] The equation of motion of a two-dimensional vector field implies that the single independent field-strength component is a constant.

[9] Other choices could lead to two-dimensional cosmological terms.

$$\mathcal{L} \equiv \begin{pmatrix} 0 & \mathbb{I}_2 & 0 \\ \mathbb{I}_2 & 0 & 0 \\ 0 & 0 & \mathbb{I}_2 \end{pmatrix} . \tag{2.6}$$

It can be immediately seen from Eq. (2.2) that the dimensionally reduced action, is invariant under the global transformations given by

$$\mathcal{M} \to \Omega\ \mathcal{M}\ \Omega^T , \qquad \mathcal{K}^i{}_\mu \to \Omega^i{}_j\ \mathcal{K}^j{}_\mu , \tag{2.7}$$

if the transformation matrices $\Omega$ satisfy the identity

$$\Omega\ \mathcal{L}\ \Omega^T = \mathcal{L} . \tag{2.8}$$

The matrix $\mathcal{L}$ given in Eq. (2.6) can be diagonalized and put into the form $\eta = \mathrm{diag}(-,-,+,+,+,+)$ by a change of basis associated to the orthogonal matrix $\rho$:

$$\rho\ \mathcal{L}\ \rho^T = \eta , \qquad \rho = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I}_2 & -\mathbb{I}_2 & 0 \\ \mathbb{I}_2 & \mathbb{I}_2 & 0 \\ 0 & 0 & \sqrt{2}\mathbb{I}_2 \end{pmatrix} , \qquad \rho\ \rho^T = \mathbb{I}_6 , \tag{2.9}$$

where now $\eta$ is the diagonal metric of $O(2,4;\mathbb{R})$ which implies that the $\Omega$'s are $O(2,4;\mathbb{R})$ transformations in a non-diagonal basis. The transformations in the diagonal ($\Omega_\eta$) and non-diagonal basis are related by

$$\Omega_\eta = \rho\ \Omega\ \rho^T , \qquad \Omega_\eta\ \eta\ \Omega_\eta^T = \eta . \tag{2.10}$$

This symmetry group corresponds to the classical T duality group. From the quantum-mechanical point of view, $O(2,4;\mathbb{R})$ is broken to $O(2,4;\mathbb{Z})$ and this group is an exact perturbative symmetry of string theory.

We must stress at this point that no S duality transformations are included in this group. S duality is a non-local symmetry while T duality consists only on local transformations[10]. So, where are the S duality transformations that were present in four dimensions?

It is well-known [8, 106] that this finite symmetry group can be extended to the infinite algebra $\widehat{o(2,4)}$. The zero-mode subalgebra corresponds to the algebra $o(2,4;\mathbb{R})$ of the symmetry we just described. The S duality transformations are included in this algebra as non-local transformations which are not in the zero-mode subalgebra.

Observe that we could have proceeded in a completely different way: we could have started by reducing the theory in the time direction to three dimensions and we could have dualized in three dimensions all vectors into scalars (as in Ref. [105]). In this way we would have gotten two scalars from each vector field: one would be the electrostatic potential $\hat{A}^I{}_t$ and the other would be the magnetostatic potential $\tilde{\hat{A}}^I{}_t$, non-locally related to the other three components of the vector. In this three-dimensional theory, S duality would be realized by local transformations rotating the electrostatic and magnetostatic potentials into each other. Further reduction to two dimensions would give us a different ("dual") version of the two-dimensional theory related to the one we have obtained and we are going to study by a non-local transformation. The dual theory has also a $\widehat{o(2,4)}_2$ invariance but now the S duality transformations are in the zero-mode subalgebra $o(2,4;\mathbb{R})_2$ [106].

Another possibility is to study the S duality transformations directly in four dimensions.

[10] At the level of the effective action, of course.

### 2.1.2. S Duality Transformations

The $N=4, d=4$ supergravity equations of motion [31] have another duality symmetry nowadays called S duality that consists of electric-magnetic duality rotations accompanied of the inversion of the dilaton (the string coupling constant) and constant shifts of pseudoscalar axion (see e.g. Ref. [104] for a review with references). This symmetry is only manifest in the Einstein frame and with the pseudoscalar axion. To study it we have to rewrite the action Eq. (2.1) in the Einstein frame and then trade the axion two-form $\hat{B}_{\hat{\mu}\hat{\nu}}$ by the pseudoscalar axion $\hat{a}$ by means of a Poincaré duality transformation. (One would get an inconsistent result if one replaced $\hat{H}$ by its dual field strength directly in the action). Thus, we consider first the above action as a functional of $\hat{H}$ which is now unrelated to $\hat{B}$. Then, we have to introduce a Lagrange multiplier ($\hat{a}$) to enforce the Bianchi identity of $\hat{H}$. Eliminating $\hat{H}$ in the action by using its equation of motion one finally gets the following action

$$S=\int d^4x\sqrt{|\hat{g}_E|}\left[\hat{R}(\hat{g}_E)-\tfrac{1}{2}(\partial\hat{\phi})^2-\tfrac{1}{2}e^{2\hat{\phi}}(\partial\hat{a})^2-\tfrac{1}{4}e^{-\hat{\phi}}\hat{F}^I\hat{F}^I+\tfrac{1}{4}\hat{a}\hat{F}^{I\star}\hat{F}^I\right]. \tag{2.11}$$

It is important for our purposes to have a very clear relation between the fields in both formulations since we have to identify the same charges in both and track them after their transformation. The (non-local) relation between $\hat{a}$ and $\hat{B}$ and the relation between the Einstein- and string-frame metric are given by

$$\left\{\begin{array}{rcl}\partial_{\hat{\mu}}\hat{a} & = & \frac{1}{3!\sqrt{|\hat{g}_E|}}e^{-2\hat{\phi}}\hat{\epsilon}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\hat{H}^{\hat{\nu}\hat{\rho}\hat{\sigma}}\,,\\ \hat{g}_{E\hat{\mu}\hat{\nu}} & = & e^{-\hat{\phi}}\,\hat{g}_{\hat{\mu}\hat{\nu}}\,.\end{array}\right. \tag{2.12}$$

Defining now the complex scalar $\hat{\lambda}$ and the S dual vector field strengths $\tilde{\hat{F}}^I$ [71]

$$\hat{\lambda}\equiv\hat{a}+ie^{-\hat{\phi}}\,,\qquad \tilde{\hat{F}}^I\equiv e^{-\hat{\phi}}\,{}^\star\hat{F}^I+\hat{a}\hat{F}^I=\hat{\lambda}\hat{F}^{I\,+}+\text{c.c.}\,, \tag{2.13}$$

where

$$\hat{F}^{I\,\pm}\equiv\tfrac{1}{2}\left(\hat{F}^I\mp i\,{}^\star\hat{F}^I\right),\qquad {}^\star\hat{F}^{I\,\pm}=\pm i\hat{F}^{I\,\pm}\,, \tag{2.14}$$

one gets the action

$$S=\int d^4x\sqrt{|g_E|}\left[\hat{R}(\hat{g}_E)-\tfrac{1}{2}\frac{\partial_{\hat{\mu}}\hat{\lambda}\partial^{\hat{\mu}}\bar{\hat{\lambda}}}{\left(\Im\text{m}\hat{\lambda}\right)^2}+\tfrac{1}{4}\hat{F}^I\,{}^\star\tilde{\hat{F}}^I\right]. \tag{2.15}$$

The equations of motion plus the Bianchi identities for the vector field strengths can be written in the following convenient form

$$\begin{array}{l}\hat{G}_{E\hat{\mu}\hat{\nu}}+\dfrac{2}{(\hat{\lambda}-\bar{\hat{\lambda}})}\left[\partial_{(\hat{\mu}}\hat{\lambda}\partial_{\hat{\nu})}\bar{\hat{\lambda}}-\tfrac{1}{2}\hat{g}_{E\hat{\mu}\hat{\nu}}\partial\hat{\lambda}\partial\bar{\hat{\lambda}}\right]\\ -\tfrac{1}{4}\left({}^\star\tilde{\hat{F}}^I{}_{\hat{\mu}}{}^{\hat{\rho}}\;{}^\star\hat{F}^I{}_{\hat{\mu}}{}^{\hat{\rho}}\right)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\tilde{\hat{F}}^I{}_{\hat{\nu}\hat{\rho}}\\ \hat{F}^I{}_{\hat{\nu}\hat{\rho}}\end{pmatrix}\quad=\quad 0\,,\end{array} \tag{2.16}$$

$$\nabla^2\hat{\lambda} - 2\frac{(\partial\hat{\lambda})^2}{(\hat{\lambda}-\bar{\hat{\lambda}})} + \tfrac{i}{8}(\hat{\lambda}-\bar{\hat{\lambda}})^2\left(\hat{F}^{I-}\right)^2 \;=\; 0\,, \tag{2.17}$$

$$\nabla_{\hat{\mu}}\left(\begin{array}{c} {}^\star\tilde{\hat{F}}^{I\;\hat{\mu}\hat{\nu}} \\ \\ {}^\star\hat{F}^{I\;\hat{\mu}\hat{\nu}} \end{array}\right) \;=\; 0\,. \tag{2.18}$$

In this way, it is easy to see that the last equation is covariant under linear combinations of the vector fields and the S dual vector fields

$$\left(\begin{array}{c} \tilde{\hat{F}}^{I\prime\;\hat{\mu}\hat{\nu}} \\ \\ \hat{F}^{I\prime\;\hat{\mu}\hat{\nu}} \end{array}\right) = \left(\begin{array}{cc} a & b \\ \\ c & d \end{array}\right)\left(\begin{array}{c} \tilde{\hat{F}}^{I\;\hat{\mu}\hat{\nu}} \\ \\ \hat{F}^{I\;\hat{\mu}\hat{\nu}} \end{array}\right), \tag{2.19}$$

with the only requirement that the transformation matrix is non-singular. However, these vector fields are not independent and consistency implies the following non-linear transformations for the complex scalar $\hat{\lambda}$

$$\hat{\lambda}' = \frac{a\hat{\lambda}+b}{c\hat{\lambda}+d}\,. \tag{2.20}$$

The Einstein equation and the scalar equations are invariant if the constants $a, b, c, d$ are the entries of an $SL(2,\mathbb{R})$ ($Sp(2,\mathbb{R})$) matrix i.e.

$$ad - bc = 1\,. \tag{2.21}$$

These transformations do not act on the Einstein metric. Observe that, although they are local transformations of the vector field strengths they are in fact non-local transformations in terms of the true variables; the vector fields themselves. Observe that the equations of motion of the vector fields are nothing but the Bianchi identities for the dual vector fields $\tilde{\hat{F}}^{I}{}_{\hat{\mu}\hat{\nu}}$ implying the local existence of the dual vector fields $\tilde{\hat{A}}^{I}{}_{\hat{\mu}}$ such that

$$\tilde{\hat{F}}^{I}{}_{\hat{\mu}\hat{\nu}} = 2\partial_{[\hat{\mu}}\tilde{\hat{A}}^{I}{}_{\hat{\nu}]}\,, \tag{2.22}$$

which justifies the definition of the $\tilde{\hat{F}}^{I}$'s. $\tilde{\hat{A}}^{I}$ depends non-locally on $\hat{A}^{I}$ and the pair $\tilde{\hat{A}}^{I}$, $\hat{A}^{I}$ transforms as an $SL(2,\mathbb{R})$ doublet.

$SL(2,\mathbb{R})$ is generated by three types of transformations[11]: rescalings of $\hat{\lambda}$

$$\left(\begin{array}{cc} a & 0 \\ 0 & 1/a \end{array}\right), \qquad \hat{\lambda}' = a^2\hat{\lambda}\,, \tag{2.23}$$

continuous shifts of the axion

$$\left(\begin{array}{cc} 1 & b \\ 0 & 1 \end{array}\right), \qquad \hat{\lambda}' = \hat{\lambda} + b\,, \tag{2.24}$$

and the discrete transformation

$$\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right), \qquad \hat{\lambda}' = -1/\hat{\lambda}\,. \tag{2.25}$$

[11] $SL(2,\mathbb{Z})$ can be generated by the discrete versions of the last two.

## 2.2. TNbh Asymptotics

In this section we will present the asymptotic behaviour that we will assume for the solutions of the equations of motion originating from the action (2.1).

As advertised in the Introduction we are going to consider generalizations of asymptotically flat Einstein metrics. The asymptotic behaviour of four-dimensional asymptotically flat metrics is completely characterized to first order in $1/r$ by only two charges: the ADM mass $M$ and the angular momentum $J$. However, duality transforms asymptotically flat metrics into non-asymptotically flat metrics which need different additional charges to be asymptotically characterized. One of them [29] is the NUT charge $N$ and closure under duality forces us to consider it. We will not need any more charges in the metric but, for completeness we define a possible new charge $u$ which we will simply ignore in what follows.

With these conditions on the asymptotics of the four-dimensional metric it is always possible to choose coordinates such that the Einstein metric in the $t - \varphi$ subspace has the following expansion in powers of $1/r$:

$$(\hat{g}_{E\alpha\beta}) = \begin{pmatrix} -1 + 2M/r & 2N\cos\theta + \left[2J\sin^2\theta - 4M(N+u)\cos\theta\right]/r \\ \\ 2N\cos\theta + \left[2J\sin^2\theta - 4M(N+u)\cos\theta\right]/r & (r^2 + 2Mr)\sin^2\theta \end{pmatrix} + \ldots \qquad (2.26)$$

We will assume the following behaviour for the dilaton

$$e^{-\hat{\phi}} = 1 - 2\mathcal{Q}_d/r + 2\mathcal{W}\cos\theta/r^2 - 2\mathcal{Z}/r^2 + \ldots \qquad (2.27)$$

where $\mathcal{Q}_d$ is the dilaton charge, $\mathcal{W}$ is a charge related to the angular momentum that will be forced upon us by S duality and $\mathcal{Z}$ is a charge which is not independent, but a function of the electric and magnetic charges (see below) and is also forced upon us by S duality . This implies for the two-dimensional scalar matrix $G$:

$$(G_{\alpha\beta}) = \begin{pmatrix} -1 + 2(M - \mathcal{Q}_d)/r & 2N\cos\theta + \left[2J\sin^2\theta - 4N(M - \mathcal{Q}_d)\cos\theta\right]/r \\ \\ 2N\cos\theta + \left[2J\sin^2\theta - 4N(M - \mathcal{Q}_d)\cos\theta\right]/r & \left[r^2 + (M + \mathcal{Q}_d)r\right]\sin^2\theta \end{pmatrix} \qquad (2.28)$$

where we have already set $u = 0$.

Observe that we have fixed its constant asymptotic value equal to zero using the same reasoning as Burgess *et.al.* [29], i.e. rescaling it away any time they arise. The time coordinate, when appropriate, will be rescaled as well, in order to bring the transformed Einstein metric to the above form (i.e. to preserve our coordinate (gauge) choice), but in a duality-consistent way.

Sometimes it will also be necessary to rescale the angular coordinate $\varphi$ in order to get a metric looking like (2.26). Conical singularities are then generically induced, and then the metric is not asymptotically TNbh in spite of looking like (2.26).

The objects we will consider will generically carry electric ($\mathcal{Q}^I_e$) and magnetic ($\mathcal{Q}^I_m$) charges with respect to the Abelian gauge fields $\hat{A}^I{}_{\hat{\mu}}$. Since we allow also for angular momentum, they will also have electric ($\mathcal{P}^I_e$) and magnetic ($\mathcal{P}^I_m$) dipole momenta. This implies for the two-dimensional scalar matrix $A$ the following asymptotic behaviour

$$\left(\hat{A}^I{}_\alpha\right) = -2\begin{pmatrix} \mathcal{Q}^1_e/r - \mathcal{P}^I_e\cos\theta/r^2 & \mathcal{Q}^1_m\cos\theta + \mathcal{P}^1_m\sin^2\theta/r \\ \mathcal{Q}^2_e/r - \mathcal{P}^I_e\cos\theta/r^2 & \mathcal{Q}^2_m\cos\theta + \mathcal{P}^2_m\sin^2\theta/r \end{pmatrix} + \ldots \qquad (2.29)$$

Electric dipole momenta appear at higher order in $1/r$ and it is not strictly necessary to consider them from the point of view of T duality, since it will not interchange them with any of the other charges we are considering and that appear at lower orders in $1/r$. However, S duality will interchange the electric and magnetic dipole momenta and we cannot in general ignore them.

The different behaviour of T and  duality is due to the fact that T duality acts on the potential's components and S duality acts on the field strengths. Thus, for the purpose of performing T duality transformations the electric charge and the magnetic momentum terms in the potentials are of the same order in $1/r$. From the point of view of S duality, the electric and magnetic charge terms are of the same order in $1/r$.

To the matrix $A$ in (2.29) we could have added a constant $2 \times 2$ matrix which would be the constant value of the $t, \varphi$ components of the vector fields at infinity. Usually these constants are not considered because they can be removed by a four-dimensional gauge transformation with gauge parameters depending linearly on $t$ and $\varphi$.

In [29] it was claimed those constants (in particular a constant term in the asymptotic expansion of $\hat{A}^I{}_t$), although pure gauge, do have an influence on physical characteristics of the dual solutions (actually this fact was interpreted there as evidence against the possibility of performing duality with respect to isometries with non-compact orbits).

However, a glance at the steps necessary to derive the $O(2,4)$ invariance of the dimensionally reduced theory [84] immediately reveals the necessity of not only staying in an adapted coordinate system, but also that the allowed four-dimensional gauge transformations are those which correspond to two-dimensional gauge transformations which are obviously independent of cyclic coordinates (in this case $t$ and $\varphi$) and keep the matrix $A$ invariant.

In other words: a constant shift in the matrix $A$, is not a symmetry of the two-dimensional theory but relates two inequivalent vacua[12]. The situation from the point of view of S duality is not different: the result of the same classical S duality transformation (i.e. $SL(2,\mathbb{R})$ transformation) depends on the asymptotic constant values of the dilaton and axion. These can always be absorbed by further classical S duality transformations, but they do not relate equivalent vacua in general.

Thus, at least from the point of view adopted here, constant terms are indeed physically meaningful. From the point of view of obtaining a closed class of solutions under duality they are necessary because they are generated by duality transformations. Setting the constant terms to zero is just a specific gauge choice (as much as the coordinate choice made for the metric is also a coordinate choice). Duality transformations do not respect these gauge choices. In the next section we will study the inclusion of these constant terms in a consistent way by performing gauge transformations and coordinate changes in all the fields. However, the transformations with constant terms become very clumsy and we will consider most transformations on the configurations we are describing in this section, with zero constant terms. Only in Section 2.3.4, we will briefly consider a discrete duality transformation on the most general configuration.

The two-index form will have the usual charge $\mathcal{Q}_a$. Closure under duality again demands the introduction of a new extra parameter ("charge") that we denote by $\mathcal{F}$ and which will play an important role in what follows. At the same order in $1/r$ it is possible to define another charge $\mathcal{H}$ which is not independent, but a function of the electric and magnetic charges, as we

[12]In any case, one should not be too dogmatic in this issue. After all, we are studying only the massless spectrum of four-dimensional string theory and performing dimensional reduction to two dimensions disregarding all the massive Kaluza-Klein modes which are associated to specific functional dependences on the coordinates $t$ and $\varphi$. A full answer on whether $t$- or $\varphi$-dependent gauge transformations are allowed and their effect on the two-dimensional theory can only be obtained from the study of the full theory and it is beyond the scope of the effective theory that describes the massless spectrum.

will show. Its presence, is required by closure of duality but it transforms as a dependent charge and it does not play a relevant role. The asymptotic expansion are, then

$$\left(\hat{B}_{\alpha\beta}\right) = 2\begin{pmatrix} 0 & \mathcal{Q}_a\cos\theta + \mathcal{F}\sin^2\theta/r + \mathcal{H}\cos\theta/r \\ -\mathcal{Q}_a\cos\theta - \mathcal{F}\sin^2\theta/r - \mathcal{H}\cos\theta/r & 0 \end{pmatrix} + \ldots \tag{2.30}$$

As we will show in Section 2.3.3, the necessity of the new charge $\mathcal{F}$ becomes clear when looking at discrete duality subgroups checking that the subgroup's multiplication table is satisfied. From the physical point of view it is clear that the presence of angular momentum should induce such a charge.

Observe that we could have added a constant term to $\hat{B}_{t\varphi}$ as well which could be reabsorbed by four-dimensional gauge transformations which are not allowed from the two-dimensional point of view. We choose them to be initially zero as well for simplicity[13].

Now we have to show that the asymptotics that have been assumed on the gauge fields $\hat{A}^I$, and on the two-index field $\hat{B}$ correspond to the gauge-invariant charges that one can define by looking directly in the asymptotic expansions of the field strengths $\hat{F}^I$, or $\hat{H}$. The field strengths corresponding to the above potentials are

$$\begin{aligned} \hat{F}^I &= 2\left(\mathcal{Q}_e^I - 2\frac{\mathcal{P}_e^I}{r}\cos\theta\right)\frac{1}{r^2}dr\wedge dt + 2\mathcal{P}_m^I\sin^2\theta\frac{1}{r^2}dr\wedge d\varphi \\ &\quad +2\left(\mathcal{Q}_m^I - 2\frac{\mathcal{P}_m^I}{r}\cos\theta\right)\sin\theta\, d\theta\wedge d\varphi - 2\mathcal{P}_e^I\sin\theta\frac{1}{r^2}d\theta\wedge dt + \ldots \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} \hat{H} &= -2\left\{\mathcal{Q}_a - \left[\left(\mathcal{Q}_e^I\mathcal{Q}_m^I + \mathcal{H}\right) - 2\mathcal{F}\cos\theta\right]\frac{1}{r}\right\}\sin\theta\, d\theta\wedge dt\wedge d\varphi \\ &\quad -2\left[\mathcal{F}\sin^2\theta - \left(\mathcal{Q}_e^I\mathcal{Q}_m^I - \mathcal{H}\right)\cos\theta\right]\frac{1}{r^2}\, dr\wedge dt\wedge d\varphi + \ldots \end{aligned} \tag{2.32}$$

Observe that the effect of taking the Hodge dual of $\hat{F}^I$ is equivalent to replacing $(\mathcal{Q}_e^I, \mathcal{P}_e^I)$ by $(\mathcal{Q}_m^I, \mathcal{P}_m^I)$ and $(\mathcal{Q}_m^I, \mathcal{P}_m^I)$ by $(-\mathcal{Q}_e^I, -\mathcal{P}_e^I)$.

Now we have to identify $\mathcal{H}$. A convenient way of doing this is to dualize the three-form field strength to find the asymptotics of the pseudoscalar axion $\hat{a}$ defined in Eq. (2.12). The partial-differential equation $\partial_{\hat{\mu}}\hat{a}$ for $\hat{a}$ the consistency condition $\partial_{[\hat{\mu}}\partial_{\hat{\nu}]}\hat{a} = 0$ (which is the Bianchi identity for $\hat{a}$ and, therefore, the equation of motion for $\hat{B}$) has to be satisfied and this implies that the combination $\mathcal{Q}_e^I\mathcal{Q}_m^I - \mathcal{H}$ vanishes, so

$$\mathcal{H} = \mathcal{Q}_e^I\mathcal{Q}_m^I\,, \tag{2.33}$$

and we find

$$\begin{aligned} \hat{H} &= -2\left\{\mathcal{Q}_a - 2\mathcal{F}\cos\theta\frac{1}{r} + 2\mathcal{Q}_e^I\mathcal{Q}_m^I\frac{1}{r}\right\}\sin\theta\, d\theta\wedge dt\wedge d\varphi \\ &\quad -2\mathcal{F}\sin^2\theta\frac{1}{r^2}dr\wedge dt\wedge d\varphi + \ldots \end{aligned} \tag{2.34}$$

[13] A constant term in $\hat{B}_{t\varphi}$ implies via duality a constant term in $G_{t\varphi}$ which we have also initially set to zero for the same reason. We will consider both kinds of constant terms in the next section

From this expression and (2.31) we see that all charges considered have a gauge-invariant meaning. The asymptotic expansion of the pseudoscalar axion $\hat{a}$ is (allowing for a constant value at infinity $\hat{a}_0$ that we will set to zero in the initial configuration)

$$\hat{a} = \hat{a}_0 + 2\mathcal{Q}_a/r - 2\mathcal{F}\cos\theta/r^2 + 2\mathcal{Q}_e^I\mathcal{Q}_m^I/r^2 + \ldots \tag{2.35}$$

which shows that $\mathcal{Q}_a$ is the standard axion charge defined, for instance, in Ref. [88]. With the pseudoscalar axion and the dilaton we find the asymptotic expansion of the complex scalar $\hat{\lambda}$ (allowing also for a non-vanishing asymptotic value for the dilaton $\hat{\phi}_0$)

$$\hat{\lambda} = \hat{\lambda}_0 + 2e^{-\hat{\phi}_0}\Upsilon/r - 2e^{-\hat{\phi}_0}\chi\cos\theta/r^2 + 2e^{-\hat{\phi}_0}\Theta/r^2 + \ldots \tag{2.36}$$

where

$$\begin{aligned} \hat{\lambda}_0 &= \hat{a}_0 + ie^{-\hat{\phi}_0}\,, \\ \Upsilon &= \mathcal{Q}_a - i\mathcal{Q}_d\,, \\ \chi &= \mathcal{F} - i\mathcal{W}\,, \\ \Theta &= \mathcal{Q}_e^I\mathcal{Q}_m^I - i\mathcal{Z}\,. \end{aligned} \tag{2.37}$$

### 2.2.1. Inclusion of Constant Terms

The inclusion of constant terms in the asymptotics of the matrices $G$, $A$ and $B$ in a consistent way is trickier than it seems at first sight. Let us start by discussing the modifications needed to include constant terms in $A$.

In the presence of constant terms in $A$ one has to be very careful when identifying the *right* axion charges. If we consider the presence of only constant terms $v^I$ in $\hat{A}^I{}_t$ for the moment

$$\left(\hat{A}^I{}_\alpha\right) = \begin{pmatrix} v^1 - 2\mathcal{Q}_e^1/r + 2\mathcal{P}_e^1\cos\theta/r^2 & -2\mathcal{Q}_m^1\cos\theta - 2\mathcal{P}_m^1\sin^2\theta/r \\ v^2 - 2\mathcal{Q}_e^2/r + 2\mathcal{P}_e^2\cos\theta/r^2 & -2\mathcal{Q}_m^2\cos\theta - 2\mathcal{P}_m^2\sin^2\theta/r \end{pmatrix} + \ldots \tag{2.38}$$

the above expression for the axion field strength changes due to the Chern-Simons terms to

$$\begin{aligned} \hat{H} &= -2\left\{\left(\mathcal{Q}_a - \tfrac{1}{2}v^I\mathcal{Q}_m^I\right) - 2\left(\mathcal{F} - \tfrac{1}{2}v^I\mathcal{P}_m^I\right)\cos\theta/r \;+\; 2\mathcal{Q}_e^I\mathcal{Q}_m^I\frac{1}{r}\right\}\sin\theta\; d\theta\wedge dt\wedge d\varphi \\ &\quad -2\left(\mathcal{F} - \tfrac{1}{2}v^I\mathcal{P}_m^I\right)\sin^2\theta/r^2 dr\wedge dt\wedge d\varphi + \ldots \end{aligned} \tag{2.39}$$

Now, the *right* charges are no longer $\mathcal{Q}_a$ and $\mathcal{F}$ but the combinations $\mathcal{Q}_a - \frac{1}{2}v^I\mathcal{Q}_m^I$ and $\mathcal{F} - \frac{1}{2}v^I\mathcal{P}_m^I$ that appear in $\hat{H}$. This really means that in presence of constant terms in $\hat{A}^I{}_t$ as above, the asymptotic expansion of $B$ that gives the right charges as in Eq. (2.34), and the one that on has to use is (setting $\mathcal{H} = 0$)

$$\begin{aligned} \hat{B}_{t\varphi} &= 2\left(\mathcal{Q}_a + \tfrac{1}{2}v^I\mathcal{Q}_m^I\right)\cos\theta + 2\left(\mathcal{F} + \tfrac{1}{2}v^I\mathcal{P}_m^I\right)\sin^2\theta/r \\ &\quad +2\mathcal{Q}_e^I\mathcal{Q}_m^I\cos\theta/r + \ldots \end{aligned} \tag{2.40}$$

Analogous results would have been obtained by creating the constant terms via a $t$-dependent gauge transformation of the gauge fields (which induces, due to the Chern-Simons term present in $\hat{H}$ a gauge transformation of the two-form field) of and looking for a gauge-independent definition of the axion charges. This way of thinking (i.e. that the terms arise because of gauge transformations that take us from the gauge in which we have written the asymptotic expansion of the potentials and metric in the previous section) is the most appropriate to study the inclusion of constant terms in $G$ and $B$. For instance, let us now perform a $\varphi$-dependent gauge transformation of the gauge fields with parameter $\Lambda^I = w^I\varphi$ that induces a constant term in $\hat{A}^I{}_\varphi$

$$\delta\hat{A}^I{}_\varphi = w^I\,. \tag{2.41}$$

This transformation induces on the two-form field Eq. (2.40) (taking into account the constant terms $v^I$) a gauge transformation in $B$. To make the story short, we will simply say that if we consider a general matrix $A$ with constant terms

$$\left(\hat{A}^I{}_\alpha\right) = \begin{pmatrix} v^1 - 2\mathcal{Q}_e^1/r + 2\mathcal{P}_e^1\cos\theta/r^2 & w^1 - 2\mathcal{Q}_m^1\cos\theta - 2\mathcal{P}_m^1\sin^2\theta/r \\ & \\ v^2 - 2\mathcal{Q}_e^2/r + 2\mathcal{P}_e^2\cos\theta/r^2 & w^2 - 2\mathcal{Q}_m^2\cos\theta - 2\mathcal{P}_m^2\sin^2\theta/r \end{pmatrix} + \ldots \tag{2.42}$$

we must consider a $B$ matrix of the form (we only write the $\hat{B}_{t\varphi}$ entry)

$$\begin{aligned} \hat{B}_{t\varphi} \;\;=\;\; & x + \tfrac{1}{2}v^I w^I 2\left(\mathcal{Q}_a + \tfrac{1}{2}v^I\mathcal{Q}_m^I\right)\cos\theta - w^I\mathcal{Q}_e^I/r \\ & +2\left(\mathcal{F} + \tfrac{1}{2}v^I\mathcal{P}_m^I\right)\sin^2\theta/r + 2\mathcal{Q}_e^I\mathcal{Q}_m^I\cos\theta/r + \ldots \end{aligned} \tag{2.43}$$

to get an axion field strength of the form (2.34) so the constants $\mathcal{Q}_a$ and $\mathcal{F}$ are still the axion charges. Now, a new $t$- or $\varphi$-dependent gauge transformation of the form $\Lambda = \delta v^I t + \delta w^I\varphi$ is reabsorbed in a redefinition of the constants $x, v^I, w^I$ and does not affect the charges, that keep their gauge-invariant meaning. The constant $x$ can also be generated or absorbed by a gauge transformation of the two-form field and it does not induce any other changes in the asymptotics of other fields.

Finally, we will see that duality sometimes creates a constant term in $\hat{g}_{Et\varphi}$. This term can be reabsorbed or induced by a reparametrization of the time coordinate $t \to t - q\varphi$. This transformation changes $G$ and $A$ to

$$(G_{\alpha\beta}) = \begin{pmatrix} -1 + 2(M - \mathcal{Q}_d)/r & (q + 2N\cos\theta)\left[1 - 2\left(M - \mathcal{Q}_d\right)/r\right] + 2J\sin^2\theta/r \\ & \\ (q + 2N\cos\theta)\left[1 - 2\left(M - \mathcal{Q}_d\right)/r\right] + 2J\sin^2\theta/r & \left[r^2 + (M + \mathcal{Q}_d)r\right]\sin^2\theta \end{pmatrix} \ldots \tag{2.44}$$

and

$$\left(\hat{A}^I{}_\alpha\right) = \begin{pmatrix} v^1 - 2\mathcal{Q}_e^1/r + 2\mathcal{P}_e^1\cos\theta/r^2 & (w^1 - qv^1) - 2\mathcal{Q}_m^1\cos\theta \\ & +2q\mathcal{Q}_e^1/r - 2\mathcal{P}_m^1\sin^2\theta/r \\ & \\ v^2 - 2\mathcal{Q}_e^2/r + 2\mathcal{P}_e^2\cos\theta/r^2 & (w^2 - qv^2) - 2\mathcal{Q}_m^2\cos\theta \\ & +2q\mathcal{Q}_e^2/r - 2\mathcal{P}_m^2\sin^2\theta/r \end{pmatrix} + \ldots \tag{2.45}$$

Observe that the electric dipole momenta do not appear in the right column because they are of higher order.

It is easy to see that there is no need to do further changes in $B$. Thus, the most general asymptotic expansions that we will consider are given by the matrices $G$ in Eq. (2.44) $A$ in

Eq. (2.45) and $B$ in Eq. (2.43) which define gauge-invariant charges in the sense that $t$ and $\varphi$-dependent gauge transformations $\Lambda^I = \delta v^I t + (\delta w^I - q\delta v^I)$ and reparametrizations of the form $t \to t + \delta q\varphi$ which are the ones that do not take us out of the Kaluza-Klein Ansatz become simple redefinitions of the constants $v^I \to v^I + \delta v^I$ etc. leaving the charges invariant (which justifies their name).

The class of asymptotic behaviour just described, determined by the twelve charges

$$\{M, J, N, \mathcal{Q}_a, \mathcal{F}, \mathcal{Q}_d, \mathcal{Q}_e^I, \mathcal{Q}_m^I, \mathcal{P}_m^I\} \tag{2.46}$$

(with or without constant terms in the matrices $G, B, A$) will be referred to henceforth as *TNbh asymptotics*.

## 2.3. Transformation of the Charges under Duality

In this Section we are going to study the transformation of the charges of asymptotically TNbh configurations under the T and S duality transformations found in Section 2.1.

### 2.3.1. Deriving the Form for the T Duality Transformation Matrices

The problem that now will be tackled is how to generate the explicit transformations of the full $O(2,4,\mathbb{R})$ classical T duality group to find which subgroup maps TNbh asymptotics into TNbh asymptotics. $O(2,4)$ is a non-compact, non-connected group and our first task is to elucidate its structure.

It is known that every element from a group $G$, can be written as a sequence of operators, which are always part of the connected component containing the identity $G_0$ (which is itself a subgroup of $G$), and elements from the coset $G/G_0$. The action of these elements on any element of $G$ is to take them from a connected part to a different connected part. This coset is called the *mapping-class group* $\pi_0(G)$.

$O(2,4)$ has four connected pieces: two correspond to matrices with determinant $+1$ and two to matrices with determinant $-1$. The former two connected pieces constitute the subgroup $SO(2,4)$ and are related to the other two by a discrete transformation that generates the group $O(2,4)/SO(2,4) = \mathbb{Z}_2^{(B)}$. The two connected components of the subgroup[14] $SO(2,4)$ differ by the sign of the $(1,1)$ component of the matrices of the defining representation. The component with positive sign contains the identity and is the subgroup $SO^{\uparrow}(2,4)$ and is related to the other connected component (which is not a subgroup and we denote by $SO^{\downarrow}(2,4)$) by a discrete transformation that generates another $\mathbb{Z}_2^{(S)} = SO(2,4)/SO^{\uparrow}(2,4)$ subgroup.

Thus, the mapping-class group of $O(2,4)$ is $O(2,4)/SO^{\uparrow}(2,4) = \mathbb{Z}_2^{(B)} \times \mathbb{Z}_2^{(S)}$.

We will study it in detail later. Now we are going to concentrate on describing the duality transformations in the component connected with the identity $SO^{\uparrow}(2,4)$.

Every element of the connected component of a group can be written as a sequence of its one-parameter-subgroups [50] and we are going to study these first.

In our case these are the exponentiated versions of the generators of the Lie algebra $so(2,4)$, which we write in the covariant form $M_{ij}$

$$\Omega_{ij}(\alpha_{ij}) = \exp\{-\alpha_{(ij)} M_{(ij)}\}\,, \tag{2.47}$$

and which satisfy the commutation relations

[14] All groups $SO(n,m)$ have two connected components [60].

$$[M_{ij}, M_{kl}] = \eta_{il} M_{jk} - \eta_{ik} M_{jl} + \eta_{jk} M_{il} - \eta_{jl} M_{ik} \ , \tag{2.48}$$

where, again, the indices $i, j, k, l = 1\ldots, 6$ and $\eta = \mathrm{diag}(-,-,+,+,+,+)$ is the diagonal metric of $O(2,4)$.

It should be noted that the action of $O(2,4)$ is 6-dimensional, which means that the group acts through its vector representation on the matrix $\mathcal{M}_{ij}$ and on the vectors $\mathcal{K}^i{}_\mu$. The generators of $so(2,4)$ in the vector representation, denoted by $\Gamma$, are given by

$$\Gamma\left(M_{ij}\right)^k{}_l = 2\eta_{l[i}\eta^k{}_{j]} \, . \tag{2.49}$$

Upon exponentiation of a single generator, one gets a one-parameter subgroup. In this way to get all the basic one-parameter subgroups of $SO^{\uparrow}(2,4)$ *in the diagonal basis* with metric $\eta$. Thus, we still need to transform the on-parameter subgroup transformations to the non-diagonal basis using Eq. (2.10) and finally we can study the effect of these transformations on the fields using Eq. (2.7).

### 2.3.2. The One-Parameter Subgroups of the T Duality Group

The one-parameter subgroups of $SO^{\uparrow}(2,4)$ are either boosts involving one of the indices $1,2$ and one of the indices $3,4,5,6$ or rotations involving the indices 1 and 2 or two of the indices $3,4,5,6$.

Boost matrices are taken to have the form

$$\Omega_\eta(\text{boost}) = \begin{pmatrix} \text{ch} & .. & \text{sh} & .. \\ .. & .. & .. & .. \\ \text{sh} & .. & \text{ch} & .. \\ .. & .. & .. & .. \end{pmatrix} \ , \tag{2.50}$$

and generate a non-compact $SO^{\uparrow}(1,1) = \mathbb{R}^+$ subgroup and every rotation will be taken to have the form

$$\Omega_\eta(\text{rotation}) = \begin{pmatrix} \cos & .. & \sin & .. \\ .. & .. & .. & .. \\ -\sin & .. & \cos & .. \\ .. & .. & .. & .. \end{pmatrix} \ , \tag{2.51}$$

and generates a compact $U(1)$ subgroup. Here the operators will be labelled by the Lie algebra generator that generates the operators. For instance, we have[15]

$$\Omega_{\eta 13} \equiv \exp\left\{-\alpha_{13} M_{\eta 13}\right\} = \begin{pmatrix} \text{ch} & 0 & \text{sh} & \\ 0 & 1 & 0 & \\ \text{sh} & 0 & \text{ch} & \\ & & & \mathbb{I}_3 \end{pmatrix} \ , \tag{2.52}$$

and in the non-diagonal basis

[15]Throughout we shall use the abbreviations $\mathrm{c} = \cos(\alpha_{ij})$, $\mathrm{s} = \sin(\alpha_{ij})$, $\mathrm{ch} = \cosh(\alpha_{ij})$ and $\mathrm{sh} = \sinh(\alpha_{ij})$, the $ij$ being the indices of the transformation.

$$
\Omega_{13} = \begin{pmatrix} \mathrm{ch}+\mathrm{sh} & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & \mathrm{ch}-\mathrm{sh} & \\ & & & \mathbb{I}_3 \end{pmatrix}, \tag{2.53}
$$

etc. Therefore it is not difficult to compute the action of these subgroups on the background fields. It turns out that only a few of them (seven, but only five with a non-trivial action and just three with different actions on the charges) preserve TNbh asymptotics.

When the transformations that leave TNbh asymptotics intact are known the exact change in the asymptotic charges can be computed. The charges transform linearly. Actually, the T duality transformations close on sets of four charges and their effect can be described by matrices acting on three four-component charge vectors:

$$
\vec{M} \equiv \begin{pmatrix} M \\ \mathcal{Q}_d \\ \mathcal{Q}_e^1 \\ \mathcal{Q}_e^2 \end{pmatrix}, \qquad \vec{N} \equiv \begin{pmatrix} N \\ \mathcal{Q}_a \\ \mathcal{Q}_m^1 \\ \mathcal{Q}_m^2 \end{pmatrix}, \qquad \vec{J} \equiv \begin{pmatrix} J \\ \mathcal{F} \\ \mathcal{P}_m^1 \\ \mathcal{P}_m^2 \end{pmatrix}, \tag{2.54}
$$

which will be referred to, respectively, as *electric, magnetic* and *dipole* charge vectors.

There is a fourth charge vector that contains the electric dipole momenta $\mathcal{P}_e^I$, the dilaton dipole-type charge $\mathcal{W}$ and an unidentified geometrical charge that we denote by $K$

$$
\vec{K} \equiv \begin{pmatrix} K \\ \mathcal{W} \\ \mathcal{P}_e^1 \\ \mathcal{P}_e^2 \end{pmatrix}. \tag{2.55}
$$

The presence of this fourth charge vector is required by S duality, as we will explain later.

For each TNbh duality transformation there is a unique matrix action on the three vectors. This, for the moment, can be considered merely a convenient representation of the duality transformations. It will be shown later in the chapter that the Bogomol'nyi bound can be rewritten in terms of our multiplets in an exceedingly convenient way.

Let us examine now examine the interesting duality transformations case by case.

$\Omega_{13}$

The action of this subgroup is simply equivalent to a rescaling of the time coordinate and obviously it preserves TNbh asymptotics. Using the inverse rescaling to rewrite the metric in the gauge (2.26) we find the the action of this duality transformation is trivial.

$\Omega_{15}$

This subgroup preserves TNbh asymptotics and our gauge choice for the metric (2.26) and for the matrices $A, B$. In particular it does not generate any constant term in the $A, B, G$ matrices. Thus, one can proceed to compute the transformation of the charges. This transformation is described by the $4 \times 4$ symmetric matrix

$$\Omega^{(4)}_{15} \equiv \begin{pmatrix} \frac{1+\mathrm{ch}}{2} & \frac{1-\mathrm{ch}}{2} & \frac{\mathrm{sh}}{\sqrt{2}} & 0 \\ \frac{1-\mathrm{ch}}{2} & \frac{1+\mathrm{ch}}{2} & -\frac{\mathrm{sh}}{\sqrt{2}} & 0 \\ \frac{\mathrm{sh}}{\sqrt{2}} & -\frac{\mathrm{sh}}{\sqrt{2}} & \mathrm{ch} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.56}$$

so

$$\tilde{\vec{M}} = \Omega^{(4)}_{15} \vec{M}\,, \qquad \tilde{\vec{N}} = \Omega^{(4)}_{15} \vec{N}\,, \qquad \tilde{\vec{J}} = \Omega^{(4)}_{15} \vec{J}. \tag{2.57}$$

$\Omega_{16}$

The effect of this transformation is identical to the previous one with the interchange of the labels $I = 1$ and $I = 2$. This, the matrix that describes it on the charges is

$$\Omega^{(4)}_{16} \equiv \begin{pmatrix} \frac{1+\mathrm{ch}}{2} & \frac{1-\mathrm{ch}}{2} & 0 & \frac{\mathrm{sh}}{\sqrt{2}} \\ \frac{1-\mathrm{ch}}{2} & \frac{1+\mathrm{ch}}{2} & 0 & -\frac{\mathrm{sh}}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ \frac{\mathrm{sh}}{\sqrt{2}} & -\frac{\mathrm{sh}}{\sqrt{2}} & 0 & \mathrm{ch} \end{pmatrix}. \tag{2.58}$$

$\Omega_{24}$

This transformation is analogous to the transformation $\Omega_{13}$: its effect is equivalent to a rescaling of the coordinate $\varphi$ that preserved it periodicity, which is initially fixed to be $2\pi$, i.e. all components of fields with indices $\varphi$ are rescaled, but the coordinate itself is not rescaled. Now, to go back to our coordinate choice (2.26) we have to rescale $\varphi$, changing its periodicity and, thus, introducing conical singularities. Therefore, this transformation does not preserve TNbh asymptotics.

$\Omega_{35}$

The result of this transformation is another asymptotically TNbh metric written in our gauge (2.26) up to a rescaling of the time coordinate and up to constant term in the matrix $A$:

$$v^1 = \sqrt{2} \sin \alpha_{35}\,, \tag{2.59}$$

and this has to be taken into account in the definitions of the axion charges that have to be identified in the transformed configurations using the expansion of $B$ in Eq. (2.40). The rescaling of the time coordinate can be performed combining $\omega_{35}$ with an $\omega_{13}$ transformation with the right parameter. The result of this composition is a one-parameter subgroup of transformations that do preserve TNbh asymptotics and our gauge choice except for the non-vanishing $v^1$. The effect of this composite transformation on the charges can be described by the $4 \times 4$ symmetric matrix

$$(\Omega_{13}\Omega_{35})^{(4)} \equiv \begin{pmatrix} \frac{c+1}{2c} & \frac{c-1}{2c} & \frac{1}{\sqrt{2}}\frac{s}{c} & 0 \\ \frac{c-1}{2c} & \frac{c+1}{2c} & -\frac{1}{\sqrt{2}}\frac{s}{c} & 0 \\ \frac{1}{\sqrt{2}}\frac{s}{c} & -\frac{1}{\sqrt{2}}\frac{s}{c} & \frac{1}{c} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.60}$$

Now, this matrix is exactly the same as $\Omega_{15}^{(4)}$ with the replacement

$$\cos\alpha_{35} = 1/\cosh\alpha_{15}\,, \tag{2.61}$$

and, so, these transformations are identical on the charges.

$\Omega_{36}$

This transformation is identical to the previous one with the interchange of the labels $I = 1$ and $I = 2$. Thus, it also generates a constant term in the matrix $A$ which has to be taken care of when identifying the axion charges of the transformed configurations:

$$v^2 = \sqrt{2}\sin\alpha_{36}\,, \tag{2.62}$$

Therefore, although it does not preserve TNbh asymptotics, it can be combined with an $\Omega_{13}$ transformation into a TNbh-preserving one-parameter subgroup of transformations that can be described by the action of the $4 \times 4$ symmetric matrix

$$(\Omega_{13}\Omega_{36})^{(4)} \equiv \begin{pmatrix} \frac{c+1}{2c} & \frac{c-1}{2c} & 0 & \frac{1}{\sqrt{2}}\frac{s}{c} \\ \frac{c-1}{2c} & \frac{c+1}{2c} & 0 & -\frac{1}{\sqrt{2}}\frac{s}{c} \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}}\frac{s}{c} & -\frac{1}{\sqrt{2}}\frac{s}{c} & 0 & \frac{1}{c} \end{pmatrix} . \tag{2.63}$$

on the three four-component charge vectors $\vec{M}, \vec{N}, \vec{J}$.

$\Omega_{56}$

This transformation is just the $SO(2)$ subgroup acting on the gauge fields only and rotates the electric and magnetic charges and the magnetic dipole momenta.Thus, it can be described by the $4 \times 4$ antisymmetric matrix

$$\Omega_{56}^{(4)} \equiv \begin{pmatrix} \mathbb{I}_2 & & \\ & c & s \\ & -s & c \end{pmatrix} . \tag{2.64}$$

### 2.3.3. The Mapping-Class Group T Duality Transformations $\mathbb{Z}_2^{(B)}$

The last "elementary" duality transformations of $O(2,4)$ that we have to study are those in the coset $O(2,4)/SO^{\uparrow}(2,4) = \mathbb{Z}_2^{(B)} \times \mathbb{Z}_2^{(S)}$. What do these transformations correspond to? In Ref. [16] the simpler duality groups $O(1,1)$ and $O(1,2)$ where analyzed in detail and it was found that the subgroup $\mathbb{Z}_2^{(S)}$ is essentially generated by a reflection in all directions in the scalar $\sigma$-model target space. Here we can do the same and take the generator of $\mathbb{Z}_2^{(S)}$ as the total reflection $-\mathbb{I}_6$. In the same reference it was also found that the subgroup $\mathbb{Z}_2^{(B)}$ corresponds essentially to Buscher's duality transformations [30].

The generator of $\mathbb{Z}_2^{(B)}$ is not unique (it is a representative element of a coset group). Two obvious choices correspond to the Buscher transformations in the directions $t$ and $\varphi$. The Buscher transformation in the direction $\varphi$ does not preserve TNbh asymptotics and so we will take as generator of $\mathbb{Z}_2^{(B)}$ the Buscher transformation in the direction $t$, that we denote by $\tau$, with matrix

$$\Omega_\eta(\tau) = \begin{pmatrix} +1 & \\ & -\mathbb{I}_5 \end{pmatrix} . \tag{2.65}$$

Observe that there is no inconsistency in taking one and not the other as inequivalent representatives because from the point of view of the TNbh-preserving duality subgroup they are no related: only an infinite boost ($\Omega_{14}$) will completely rotate $t$ into $\varphi$ and, in any case, this subgroup does not preserve TNbh asymptotics itself.

As was said in Section 2.2, the necessity of introducing additional "charges" like $\mathcal{F}$ becomes evident[16] when one studies discrete duality subgroups like $\mathbb{Z}_2^{(B)}$. If we analyze the $\tau$-transformation explicitly, we see that the $\tau$-transform of $\hat{g}_{t\varphi}$ is

$$\tilde{\hat{g}}_{t\varphi} = \frac{1}{2\Omega}\left\{\hat{g}_{tt}\left[\hat{A}^I{}_t\hat{A}^I{}_\varphi - 2\hat{B}_{t\varphi}\right] - \hat{g}_{t\varphi}\hat{A}^I{}_t\hat{A}^I{}_t\right\} , \tag{2.66}$$

where $\Omega$ goes asymptotically as

$$\Omega = 1 - \frac{4M}{r} + \mathcal{O}(r^{-2}) . \tag{2.67}$$

Looking at the asymptotic behaviour of the terms involved, it is easy to see that to get a contribution to $J$, the initial configuration has to have a $r^{-1}\sin^2\theta$ term in its asymptotic expansion of the Kalb-Ramond field. This also shows that $J$ transforms into the new charge $\mathcal{F}$ since $\tau^{-1} = \tau$.

The effect of $\tau$ on all the charges can be expressed in terms of the same symmetric $4\times 4$ matrix $\Omega_\tau^{(4)}$

$$\Omega_\tau^{(4)} = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & \mathbb{I}_2 \end{pmatrix} , \tag{2.68}$$

acting on the charge vectors $\vec{M}, \vec{N}, \vec{J}$. The involutive property, that on the charges $\tau^2 = id$ is immediately apparent.

A natural worry at this point is whether a combination of transformations that do not preserve TNbh asymptotics, can result in a TNbh asymptotics-preserving transformation.

[16] To see it in the continuous subgroups one has to study the invertibility of the transformations, which is much harder.

This is a complicated and time-consuming problem that can only be handled by computational methods for just products of two transformations. The result of our (CPU-limited) search is negative.

### 2.3.4. Constant Parts in the Gauge Fields and the Closed Set of Asymptotic "Charges" Under $\tau$

We have to find the way in which the transformations of the charges change when we include constant terms in the matrices $G, A, B$. To do a general study would take to much CPU time. Thus, we will only perform a full check of only the $\tau$ transformation, although the general picture should become quite clear from our results for $\tau$ and other general arguments.

First of all, the consistency in the way we have defined charges and constant terms (which will be referred to as *moduli*) implies that the moduli transform non-linearly amongst themselves and, thus, their transformations can be studied by setting to zero the charges. For the $\tau$ transformation this allows us to immediately get[17]

$$\left\{ \begin{array}{rclcrcl} \tilde{v}^I &=& v^I\,, & \qquad & \tilde{q} &=& \xi x\,, \\ \\ \tilde{w}^I &=& w^I\,, & \qquad & \tilde{x} &=& \xi^{-1} q\,. \end{array} \right. \tag{2.69}$$

where we have used $\xi = (1-\vec{v}^{\,2}/2)^{-1}$.

Next, we expect the multiplet structure of the duality transformations to remain valid in the presence of non-trivial moduli. (The multiplets contain multipole terms of the same order of different fields.) This can be checked explicitly, but it also allows us to set to zero all charges except for those in one multiplet and find their transformation more easily. The result is that we can describe in all of them the $\tau$ transformation with a unique moduli-dependent matrix $\Omega^{(4)}_\tau(x,q,v,w)$

$$\Omega^{(4)}_\tau(x,q,v,w) = \begin{pmatrix} -\frac{1}{2}\xi\vec{v}^{\,2} & \xi & \xi v^1 & \xi v^2 \\ \\ \xi & -\frac{1}{2}\xi\vec{v}^{\,2} & -\xi v^1 & -\xi v^2 \\ \\ -\xi v^1 & \xi v^1 & 1+\xi(v^1)^2 & \xi v^1 v^2 \\ \\ -\xi v^2 & \xi v^2 & \xi v^1 v^2 & 1+\xi(v^2)^2 \end{pmatrix}. \tag{2.70}$$

Observe that this matrix indeed squares to the identity.

What happens to the other transformations in presence of non-trivial moduli? The rule is that now the $4\times 4$ matrices $\Omega^{(4)}_{ij}$ will become moduli-dependent matrices $\Omega^{(4)}_{ij}(x,q,v,w)$ and the group multiplication table is satisfied in the following sense:

$$\Omega^{(4)}_{T_2}(\tilde{x},\tilde{q},\tilde{v},\tilde{w})\Omega^{(4)}_{T_1}(x,q,v,w) = \Omega^{(4)}_{T_2\cdot T1}(x,q,v,w)\,, \tag{2.71}$$

where $(\tilde{x},\tilde{q},\tilde{v},\tilde{w})$ are the transformed moduli under $T_1$. In the case of $\tau$ we had, trivially

$$\Omega^{(4)}_\tau(\tilde{x},\tilde{q},\tilde{v},\tilde{w})\Omega^{(4)}_\tau(x,q,v,w) = \mathbb{I}_4\,. \tag{2.72}$$

because $\Omega^{(4)}_\tau(x,q,v,w)$ only depends on the $v^I$ and these are invariant under $\tau$.

[17] Simultaneous rescalings of the dilaton and the time coordinate $t$ are necessary to eliminate the constant value of the dilaton at infinity and to get an asymptotically TNbh metric.

### 2.3.5. Transformation of the Charges under S Duality

The transformation of the electric, magnetic, dilaton and axion charges under S duality has been previously studied in Ref. [88, 71]. Here we are considering more charges and we are choosing initial configurations with vanishing asymptotic values of the axion and dilaton. In general, S duality generates non-vanishing values of these constants and we will remove them by applying further S duality transformations.

Let us first see the effect of general classical S duality transformations on arbitrary configurations. It is easy to see that the transformation (2.20) acts on the asymptotic value of $\hat{\lambda}$ as follows [88, 71]

$$\hat{\lambda}_0' = \frac{a\hat{\lambda}_0 + b}{c\hat{\lambda}_0 + d}\,, \tag{2.73}$$

and on its complex charge as follows

$$\Upsilon' = \left(\frac{c\bar{\hat{\lambda}}_0 + d}{c\hat{\lambda}_0 + d}\right)\Upsilon\,. \tag{2.74}$$

The factor multiplying $\Upsilon$ is just a $\hat{\lambda}_0$-dependent complex phase and, thus the axion and dilaton charges are simply rotated into one another. It is also easy to see that the additional complex charge that we are considering here $\chi = \mathcal{F} - i\mathcal{W}$ transforms exactly as $\Upsilon$.

The effect on the electric and magnetic charges is a bit more difficult to explain because the electric and magnetic charges that transform in a natural way under S duality, and which are the ones conserved in the quantum theory when the Witten effect [114] is taken into account, are not the ones we have defined. To be precise, the equation of motion and the Bianchi identity tell us that the two charges that are well defined in the quantum theory and obey the Dirac-Schwinger-Zwanziger quantization condition are

$$\left\{\begin{array}{rcccl} q_e^I & \sim & \int_{S^2_\infty} \tilde{\hat{F}}^I & = & e^{-\hat{\phi}_0}\mathcal{Q}_e^I - \hat{a}_0\mathcal{Q}_m^I\,, \\ \\ q_m^I & \sim & \int_{S^2_\infty} \hat{F}^I & = & \mathcal{Q}_m^I\,. \end{array}\right. \tag{2.75}$$

This pair of charges transform under (2.19) as an $SL(2,\mathbb{R})$ doublet

$$\left(q_e^{I\,\prime} \;\; q_m^{I\,\prime}\right) = \left(q_e^I \;\; q_m^I\right)\begin{pmatrix} a & -c \\ \\ -b & d \end{pmatrix}\,, \tag{2.76}$$

which ensures that the DSZ quantization condition, which can be written for two dyons in the form

$$\left(q_e^{I\,(1)} \;\; q_m^{I\,(1)}\right)\begin{pmatrix} 0 & 1 \\ \\ -1 & 0 \end{pmatrix}\begin{pmatrix} q_e^{I\,(2)} \\ \\ q_m^{I\,(2)} \end{pmatrix} = cn\,, \qquad n\in\mathbb{Z}\,, \tag{2.77}$$

where $c$ is some constant, is S duality invariant. From the relation between the charges $\left(q_e^{I\,(1)} \;\; q_m^{I\,(1)}\right)$ and the charges $\mathcal{Q}_e^I, \mathcal{Q}_m^I$ that we are using (2.75) one readily finds

$$\begin{array}{rcl} \mathcal{Q}_e^{I\,\prime} & = & (c\hat{a}_0 + d)\mathcal{Q}_e^I + ce^{-\hat{\phi}_0}\mathcal{Q}_m^I\,, \\ \\ \mathcal{Q}_m^{I\,\prime} & = & -ce^{-\hat{\phi}_0}\mathcal{Q}_e^I + (c\hat{a}_0 + d)\mathcal{Q}_m^I\,. \end{array} \tag{2.78}$$

It is easy to see that the electric and magnetic dipole momenta transform in exactly the same fashion.

Now we have to adapt these formulae to our case in which the original configuration has $\hat{\lambda}_0 = i$ and in which we want the transformed configuration to have also $\hat{\lambda}'_0 = i$. This can be achieved by applying after the general $SL(2,\mathbb{R})$ transformation, two transformations (2.23,2.24) with the appropriate values of $a$ and $b$ to absorb the constant values of the axion and dilaton. This is equivalent to allow only an $SO(2)$ subgroup of $SL(2,\mathbb{R})$ to act on the charges. The result, expressed in terms of the entries of the original $SL(2,\mathbb{R})$ matrix is

$$\begin{pmatrix} \mathcal{Q}_e^{I\,\prime} \\ \\ \mathcal{Q}_m^{I\,\prime} \end{pmatrix} = \begin{pmatrix} \frac{d}{\sqrt{c^2+d^2}} & \frac{c}{\sqrt{c^2+d^2}} \\ \\ \frac{-c}{\sqrt{c^2+d^2}} & \frac{d}{\sqrt{c^2+d^2}} \end{pmatrix} \begin{pmatrix} \mathcal{Q}_e^{I} \\ \\ \mathcal{Q}_m^{I} \end{pmatrix}, \tag{2.79}$$

and similarly for the vector of dipole momenta $\left(\mathcal{P}_m^I\,, \mathcal{P}_e^I\right)$ and

$$\begin{pmatrix} \mathcal{Q}'_d \\ \\ \mathcal{Q}'_a \end{pmatrix} = \begin{pmatrix} \frac{d^2-c^2}{\sqrt{c^2+d^2}} & \frac{2cd}{\sqrt{c^2+d^2}} \\ \\ \frac{-2cd}{\sqrt{c^2+d^2}} & \frac{d^2-c^2}{\sqrt{c^2+d^2}} \end{pmatrix} \begin{pmatrix} \mathcal{Q}_d \\ \\ \mathcal{Q}_a \end{pmatrix}, \tag{2.80}$$

and, analogously for the charge vector $(\mathcal{W}\,, \mathcal{F})$. Observe that the last $SO(2)$ transformation matrix is precisely the square of the former.

It is now clear that the multiplet structure that we built for the T duality transformations is not respected by S duality: the last three components of the "electric" multiplet $\hat{M}$ are rotated into the last three components of the "magnetic" multiplet $\vec{N}$ and vice versa. The same happens with the multiplet $\vec{K}$ defined in Eq. (2.55), whose last three components are rotated into those of the multiplet $\vec{J}$ in exactly the same way, and vice versa (this is the reason why we introduced $K$ and $\vec{K}$ in the first place). To respect the T duality multiplet structure and, at the same time incorporate the S duality multiplet structure it is useful to introduce the complexified multiplets

$$\vec{\mathcal{M}} \equiv \vec{M} + i\vec{N} = \begin{pmatrix} \mathcal{M} \\ i\Upsilon \\ \Gamma^1 \\ \Gamma^2 \end{pmatrix}, \tag{2.81}$$

where

$$\mathcal{M} \equiv M + iN\,, \qquad \Gamma^I \equiv \mathcal{Q}_e^I + i\mathcal{Q}_m^I\,, \tag{2.82}$$

and

$$\vec{\mathcal{J}} \equiv \vec{K} + i\vec{J} = \begin{pmatrix} \mathcal{J} \\ i\chi \\ \Pi^1 \\ \Pi^2 \end{pmatrix}, \tag{2.83}$$

where

$$\mathcal{J} \equiv K + iJ\,, \qquad \Pi^I \equiv \mathcal{P}_e^I + i\mathcal{P}_m^I\,. \tag{2.84}$$

These two complex vectors transform under T duality with exactly the same $\Omega_{ij}^{(4)}$ matrices as the real vectors and, under the above S duality transformations with the complex $\Sigma^{(4)}$ $SO(2)$ matrix

$$\Sigma^{(4)} = \begin{pmatrix} 1 & & & \\ & e^{2i\theta} & & \\ & & e^{i\theta} & \\ & & & e^{i\theta} \end{pmatrix} \qquad \theta = \mathrm{Arg}(d - ic)\,, \tag{2.85}$$

so

$$\vec{\mathcal{M}}' = \Sigma^{(4)}\vec{\mathcal{M}}\,, \qquad \vec{\mathcal{J}}' = \Sigma^{(4)}\vec{\mathcal{J}}\,. \tag{2.86}$$

## 2.4. The Asymptotic Duality Subgroup

We define the Asymptotic Duality Subgroup (ADS) as the subgroup of the full duality group that respects TNbh asymptotics. In the previous section we have identified several one-parameter subgroups of the T duality part of the ADS and we know that the full S duality group is a subgroup of the ADS. However these two subgroups do not commute and, together, generate a large ADS. We proceed to identify it in the next section and later we will use it to study the invariance of the Bogomol'nyi bound relevant for the theory we are considering under it.

### 2.4.1. Identification of the Asymptotic Duality Subgroup

First, we are going to identify the T duality subgroup of the ADS. As we have seen, from the point of view of its action on the charges it has only three non-trivial one-parameter subgroups which we take to be the ones corresponding to the transformations $\Omega^{(4)}_{15}, \Omega^{(4)}_{16}, \Omega^{(4)}_{56}$. To find the group that they generate we first study the algebra of their infinitesimal generators $M^{(4)}_{ij}$

$$\Omega^{(4)}_{ij} = \mathbb{I}_4 - \alpha_{(ij)} M^{(4)}_{(ij)}\,, \tag{2.87}$$

which are given by

$$M^{(4)}_{15} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad M^{(4)}_{16} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}, \tag{2.88}$$

$$M^{(4)}_{56} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

These infinitesimal generators obey the algebra

$$[M^{(4)}_{56}, M^{(4)}_{15}] = M^{(4)}_{16}\,, \quad [M^{(4)}_{56}, M^{(4)}_{16}] = -M^{(4)}_{15}\,, \quad [M^{(4)}_{15}, M^{(4)}_{16}] = -M^{(4)}_{56}\,. \tag{2.89}$$

A small calculation of the Killing metric then show that on the base $\{M^{(4)}_{15}, M^{(4)}_{16}, M^{(4)}_{56}\}$ the metric is diagonal with entries $\eta^{(3)} = \mathrm{diag}(+,-,-)$ thus proving that the algebra is $o(1,2)$ and the group generated by the one-parameter subgroups is $SO^{\uparrow}(1,2)$ and that the T duality part of the ADS (taking into account the discrete transformations) is $O(1,2)$.

This raises now the question as to what is the meaning of the four-component charge vectors. Clearly they transform in the four-dimensional reducible representation of $O(1,2)$ furnished by the matrices $\Omega^{(4)}$. The only representation of this kind is the direct sum of a singlet and a vector (three-dimensional) representation of $O(1,2)$, which in turn means that there is a linear combination of the charges in each charge vector that is invariant under the full T duality part of the ADS. It is easy to see that these combinations are

$$\tfrac{1}{\sqrt{2}}\left(M+\mathcal{Q}_d\right)\,, \qquad \tfrac{1}{\sqrt{2}}\left(N+\mathcal{Q}_a\right)\,, \qquad \tfrac{1}{\sqrt{2}}\left(J+\mathcal{F}\right)\,. \tag{2.90}$$

The triplets over which T duality acts in the vector representation of $SO(1,2)$ are

$$\begin{aligned} \vec{M}^{(3)} &= \begin{pmatrix} \tfrac{1}{\sqrt{2}}\left(M-\mathcal{Q}_d\right) \\ \mathcal{Q}_e^1 \\ \mathcal{Q}_e^2 \end{pmatrix}, \qquad \vec{N}^{(3)} = \begin{pmatrix} \tfrac{1}{\sqrt{2}}\left(N-\mathcal{Q}_a\right) \\ \mathcal{Q}_m^1 \\ \mathcal{Q}_m^2 \end{pmatrix}, \\ \vec{J}^{(3)} &= \begin{pmatrix} \tfrac{1}{\sqrt{2}}\left(J-\mathcal{F}\right) \\ \mathcal{P}_m^1 \\ \mathcal{P}_m^2 \end{pmatrix}, \end{aligned} \tag{2.91}$$

and, on this representation the generators of the algebra are

$$\begin{aligned} M_{15}^{(3)} &= \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M_{16}^{(3)} = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ M_{56}^{(3)} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{2.92}$$

We remark for future use that the four-dimensional matrices $\Omega^{(4)}$ of the $1\oplus 3$ representation of $O(1,2)$ respect the diagonal $O(2,2)$ metric $\eta^{(4)} = \mathrm{diag}(+,+,-,-)$ and are also automatically $O(2,2)$ matrices.

### 2.4.2. The Bogomol'nyi Bound and its Variation

In $N=4$ supergravity there are two Bogomol'nyi (B) bounds, of the form

$$M^2 - |Z_i|^2 \geq 0\,, \qquad i=1,2\,, \tag{2.93}$$

where the $Z_i$'s are the complex skew eigenvalues of the central charge matrix and are combinations of electric and magnetic charges of the six graviphotons. These two bounds can be combined into a single bound by multiplying them and then dividing by $M^2$. One gets, then, a *generalized* B bound

$$M^2 + \frac{|Z_1 Z_2|}{M^2} - |Z_1|^2 - |Z_2|^2 \geq 0\,. \tag{2.94}$$

In regular black-hole solutions the second term can be identified with scalar charges of "secondary" type. The identification is, actually (with zero value for the dilaton at infinity)

$$\frac{|Z_1 Z_2|}{M^2} = \mathcal{Q}_d^2 + \mathcal{Q}_a^2\,, \tag{2.95}$$

and, taking into account the expression of the central charges in terms of the $\mathcal{Q}^I_{e,m}$'s one gets the generalized B bound[18] [88]

$$M^2 + \mathcal{Q}_d^2 + \mathcal{Q}_a^2 - \mathcal{Q}_e^I \mathcal{Q}_e^I - \mathcal{Q}_m^I \mathcal{Q}_m^I \geq 0\,. \tag{2.96}$$

Note however that this bound is valid only for asymptotically flat spaces (i.e. with $N = 0$). This problem can however be overcome by the reasoning of Ref. [72] where it was observed that the NUT charge $N$ does enter in the generalized B bound. With our definitions the B bound for asymptotically TNbh spaces takes the form

$$M^2 + N^2 + \mathcal{Q}_d^2 + \mathcal{Q}_a^2 - \mathcal{Q}_e^I \mathcal{Q}_e^I - \mathcal{Q}_m^I \mathcal{Q}_m^I \geq 0\,. \tag{2.97}$$

Now we want to study the invariance of this bound under the T and S duality pieces of the ADS that preserves TNbh asymptotics. We will not make distinctions between primary and secondary scalar charges since all we are interested in are the transformation rules of the scalar charges which are the same for primary- or secondary-type scalar charges. We will focus on this distinction in the next section.

Before perform a direct check, let us analyze what we can expect the result to be. The T duality piece of the ADS preserves in general unbroken supersymmetries of the low-energy string effective action: one can prove that if one solution admits Killing spinors the dual solution does as well. Equivalent properties can be checked from the world-sheet point of view. The only instances in which T duality seems not to respect unbroken supersymmetries (at least in a *manifest* fashion from the spacetime point of view) is when a Buscher T duality transformation is performed with respect to an isometry with fixed points, like the isometry in the direction $\varphi$ in our axially-symmetric case [7]. However, this transformation does not respect TNbh asymptotics and therefore it does not belong to the ADS. S duality is known to always preserve unbroken supersymmetry [88] and, thus, we can expect the B bound to be invariant under the full ADS.

To study the transformation properties of the B bound under the physical TNbh asymptotics-preserving duality group it is convenient to use the diagonal metric of $SO(2,2)$ $\eta^{(4)} = \mathrm{diag}\,(1,1,-1,-1)$ already introduced at the beginning of this section. Using this metric and the charge vectors defined in Eqs. (2.81,2.83) the B bound can be easily rewritten in this form:

$$\vec{\mathcal{M}}^\dagger \eta^{(4)} \vec{\mathcal{M}} \geq 0\,. \tag{2.98}$$

In this form the B bound of $N = 4, d = 4$ supergravity is manifestly $U(2,2)$-invariant. Observe that $U(2,2) \sim O(2,4)$, although it is not clear if this fact is a mere coincidence or it has a special significance. The T duality piece of the ADS is an $O(1,2)$ subgroup of the $O(2,2)$ canonically embedded in $U(2,2)$ and obviously preserves the B bound. The S duality piece of the ADS is a $U(1)$ subgroup diagonally embedded in $U(2,2)$ through the matrices $\Sigma^{(4)}$ defined in Eq. (2.85) and obviously preserve the B bound.

The charges in the vector $\vec{\mathcal{J}}$ do not appear in the B bound and neither T nor S duality change this fact. It is not possible to constrain the values of any of the charges it (in particular $J$) by using duality and supersymmetry, as was suggested in the Introduction.

Although we are not going to study the full ADS generated by the T duality and the S duality pieces, it is clear that there are transformations in it that rotate the mass $M$ into the NUT charge

[18] A general expression of the same kind for black holes with regular horizons in genera; theories with scalars non-minimally coupled to vector fields has been found in [49].

$N$ and $J$ into $K$: it is enough to perform first a $\tau$ transformation to interchange the first and second components of the $U(2,2)$ vectors $\vec{\mathcal{M}}$ and $\vec{\mathcal{J}}$, then perform an S duality transformation that interchanges the real and imaginary parts of the second component of those vectors and a further $\tau$-transformation to bring this rotated component back to the first position.

### 2.4.3. Primary Scalar Hair and Unbroken Supersymmetry

So far we have not discussed in detail the physical meaning of the charges that define TNbh asymptotics. In particular, we have considered completely unrestricted charges $\mathcal{Q}_d$ and $\mathcal{Q}_a$.

The dilaton charge $\mathcal{Q}_d$, not being protected by a gauge symmetry, is not a conserved charge. In four dimensions the Kalb-Ramond two-form is dual to the pseudoscalar axion and the charge $\mathcal{Q}_a$ is just its charge. Again, $\mathcal{Q}_a$ is not a conserved charge. This may seem contradictory because in the two-form version there is indeed a gauge symmetry. However, the two-form conserved charge is actually associated to one-dimensional extended objects, not to the point-like objects we are considering here. Thus both charges can be considered non-conserved scalar charges (*hair*).

If these scalars were minimally-coupled scalars the standard no-hair theorems would apply to them and any non-vanishing value of $\mathcal{Q}_d$ and $\mathcal{Q}_a$ would imply the presence of naked singularities. The prototype of this kind of singular solution with non-trivial scalar hair (called *primary hair*) is the one given in Refs. [67] for the theory with a massless scalar and action

$$S = \int d^4x \sqrt{|\hat{g}_E|}\ \left[\hat{R}(\hat{g}_E) + \tfrac{1}{2}\partial_{\hat{\mu}}\hat{\phi}\partial^{\hat{\mu}}\hat{\phi}\right] . \tag{2.99}$$

The solutions take the form

$$\left\{ \begin{array}{rcl} d\hat{s}^2_E & = & W^{\frac{M}{r_0}-1} W dt^2 - W^{1-\frac{M}{r_0}} \left[W^{-1}dr^2 + r^2 d\Omega^2\right] , \\ \hat{\phi} & = & \hat{\phi}_0 - \frac{\mathcal{Q}_d}{r_0} \ln W , \end{array} \right. \tag{2.100}$$

where

$$\left\{ \begin{array}{rcl} W & = & 1 - 2r_0/r , \\ r_0^2 & = & M^2 + \mathcal{Q}_d^2 . \end{array} \right. \tag{2.101}$$

The three fully independent parameters that characterize each solution are the mass $M$, the scalar charge[19] $\mathcal{Q}_d$ and the value of the scalar at infinity $\phi_0$. Only when $\mathcal{Q}_d = 0$ one has a regular solution (Schwarzschild). In all other cases there is a singularity at $r = r_0$, where the area of 2-spheres of radius $r$ vanishes.

Before continuing with our discussion a couple of remarks should be made: first, this whole family of solutions belong to the TNbh class and, second, observe that the above family of solutions includes a non-trivial *massless* solution. Setting $M = 0$ above we find

$$\left\{ \begin{array}{rcl} d\hat{s}^2_E & = & dt^2 - dr^2 - W r^2 d\Omega^2 , \\ \hat{\phi} & = & \hat{\phi}_0 - \ln W , \qquad e^{\hat{\phi}-\hat{\phi}_0} = W^{-1} , \end{array} \right. \tag{2.102}$$

with

[19] We use the symbol of the dilaton charge because these solutions (which are written in the Einstein frame) are also solutions of the equations of motion of the low-energy string-effective action Eq. (2.1) with $\hat{\phi}$ identified with the dilaton.

$$W = 1 - \frac{2\mathcal{Q}_d}{r}\,. \tag{2.103}$$

In the full low-energy string effective action, the dilaton and the axion are non-minimally coupled scalars, though, and the existence of black-hole solutions with regular horizons in theories with non-minimally coupled scalars is known [47, 44]. In these solutions, the scalar (dilaton) charge is identical to a certain fixed combinations of the other, conserved, charges:

$$\mathcal{Q}_d \sim \frac{\mathcal{Q}_m^I \mathcal{Q}_m^I - \mathcal{Q}_e^I \mathcal{Q}_2^I}{2M}\,. \tag{2.104}$$

The same is true of the axion charge in solutions with non-trivial axion hair and regular horizons [108, 88, 71, 73]. The axion charge is in those cases given by

$$\mathcal{Q}_a \sim \frac{\mathcal{Q}_e^I \mathcal{Q}_m^I}{2M}\,. \tag{2.105}$$

This kind of scalar hair, whose existence does not imply the presence of naked singularities is called *secondary hair*. It is clear that the existence of secondary hair does not preclude the existence of primary hair. In fact, the solutions above can be interpreted in the framework of string theory with primary but no secondary hair and there are solutions which have both kinds of hair at the same time [1].

Primary scalar hair always seems to imply the presence of naked singularities, and the no-hair theorem (if it existed such a general theorem) should probably be called *no-primary hair theorem*.

So, what can duality and supersymmetry tell us about primary scalar hair? At first sight, nothing. In the standard derivations of the different B bound formulae only conserved electric and magnetic charges appear and only when all the scalar hair is secondary and given by the above formulae one can derive the generalized B bounds of the previous section in which the scalar charges appear.

Nevertheless, let us consider a simple example: Schwarzschild's solution (given above just by setting $\mathcal{Q}_d = 0$). This solution has no unbroken supersymmetries, which can be understood in terms of non-saturation of the B bound ($M \geq 0$). A Buscher T duality transformation in the time direction belongs to the physical duality group and should preserve the supersymmetry properties and asymptotic behaviour of the solution and so it should yield a new solution with no unbroken supersymmetries and TNbh asymptotics. A short calculation shows that the dual solutions is exactly the massless solution with primary scalar hair written above in Eqs. (2.102,2.103)! It is easy to check that this solution admits no $N = 4$ Killing spinors and so it has no unbroken supersymmetries[20]. However, the fact that this solution has no unbroken supersymmetries would not have been clear from the B bound point of view , had we used the once-standard form in which primary hair should not added to it, since its mass and all the other conserved charges are zero, meaning that the bound would be trivially saturated.

All that happened in this transformation is that the mass $M$, which does appear in the B bound has completely transformed in primary dilaton charge $\mathcal{Q}_d$ which in principle does not.

After our study of the transformation of charges it is clear that to reconcile these two results one has to admit that the generalized B bound formula Eq. (2.97) does apply to all kinds of scalar charge and not only to the secondary-type one. Only in this way the invariance of the B bound becomes consistent with the covariance of the Killing spinor equations.

[20] The dilatino supersymmetry transformation rule would be equal to $\delta_\epsilon \lambda^I \sim \not{\partial}\hat{\phi}\epsilon^I$ which only vanishes for $\epsilon^I = 0$. ($I$ is an $SU(4)$ index here).

Although our reasoning is completely clear when we look on specific solutions one should be able to derive B bounds including primary scalar charges using a Nester construction based on the supersymmetry transformation laws of the fermions of the supergravity theory under consideration. To be able to do this one has to be able to manage more general boundary conditions including the seemingly unavoidable naked singularities that primary hair implies.

Although we have kept this discussion strictly four-dimensional it is easy to generalize these arguments to higher dimensions. In fact, solutions generalizing the one above to higher ($d$) dimensions can be straightforwardly found

$$\left\{\begin{array}{rcl} ds^2 & = & -W^{\frac{M}{r_0}-1}W dt^2 + W^{\frac{1}{d-3}\left(1-\frac{M}{r_0}\right)}\left[W^{-1}d\rho^2 + \rho^2 d\Omega^2_{(d-2)}\right]\,, \\ \phi & = & \phi_0 + \frac{\mathcal{Q}_d}{r_0}\ln W\,. \end{array}\right. \tag{2.106}$$

where

$$W = 1 - \frac{2r_0}{\rho^{d-3}}\,, \tag{2.107}$$

and now

$$r_0^2 = M^2 + 2\left(\tfrac{d-3}{d-2}\right)\mathcal{Q}_d\,. \tag{2.108}$$

For $\mathcal{Q}_d = 0$ we recover the $d$-dimensional Schwarzschild solution. In all other cases we have metrics with naked singularities either at $\rho = 0$ or $\rho^{d-3} = 2r_0$.

A further example can be useful to fix these ideas.

Using our conventions, it is possible to write the stringy RN solution in the following form:

$$\left\{\begin{array}{rcl} d\hat{s}^2_E & = & -H^{-2}W dt^2 + H^2\left[W^{-1}dr^2 + r^2 d\Omega^2\right]\,, \\ e^{-\hat{\phi}} & = & H/H = 1\,, \\ \hat{A}^{(1)}{}_t & = & 2\alpha_1\frac{|Q|}{M-r_0}\left(H^{-1}-1\right)\,, \\ \hat{A}^{(2)}{}_\varphi & = & -2\alpha_2|Q|\cos\theta\,, \end{array}\right. \tag{2.109}$$

where $H$ and $W$ are (not independent) harmonic functions

$$H = 1 + \frac{M - r_0}{r}\,, \qquad W = 1 - \frac{2r_0}{r}\,, \tag{2.110}$$

and the constants are:

$$\alpha_i^2 = 1\,, \qquad r_0^2 = M^2 - 2Q^2\,, \tag{2.111}$$

where we have set

$$\mathcal{Q}_e^1 = \alpha_1|Q|\,, \qquad \mathcal{Q}_m^2 = \alpha_2|Q|\,. \tag{2.112}$$

The dilaton charge is identically zero for this family. Observe also that $M - r_0 \geq 0$ always, and thus $H$ never vanishes and so it never gives rise to any singularities in the metric apart from the one at $r = 0$, which is the curvature singularity. The metric is also singular at the horizon $r = 2r_0 > 0$ where $W$ vanishes, covering the physical singularity at $r = 0$.

The extremal limit is $r_0 = 0$, $M = \sqrt{2}|Q|$, which makes $W$ disappear and $H$ becomes an unrestricted harmonic function (we could describe many BHs if we wanted). In this limit the horizon is placed at $r = 0$, which is th locus of a two-sphere instead of a point, as can be seen by a coordinate change. The curvature singularity is not covered by these coordinates.

Th B bound for this family of solutions is

$$M^2 - 2Q^2 = M^2 - (\mathcal{Q}_e^1)^2 - (\mathcal{Q}_m^2)^2 \geq 0\,, \tag{2.113}$$

with the equality satisfied in the extreme $r_0 = 0$ limit. Performing the $\tau$ transformation on the above family of solutions we get the dual family of solutions

$$\left\{\begin{array}{rcl} d\tilde{\hat{s}}_E^2 & = & -H^{-1}K^{-1}W dt^2 + HK\left[W^{-1}dr^2 + r^2 d\Omega^2\right]\,, \\ e^{\tilde{\hat{\phi}}} & = & H/K\,, \\ \tilde{\hat{A}}^1{}_t & = & 2\alpha_1 \frac{|Q|}{M-r_0}\left(K^{-1} - 1\right)\,, \\ \tilde{\hat{A}}^2_\varphi & = & -2\alpha_2 |Q| \cos\theta\,, \end{array}\right. \tag{2.114}$$

where

$$K = 1 - \frac{M + r_0}{r}\,, \tag{2.115}$$

The above metric has several singularities: there is a curvature singularity at $r = 0$ and the would-be horizon singularity at $r = 2r_0$ but both lie beyond another physical singularity at $r = M + r_0 \geq 2r_0$ which is where the function $K$ vanishes and where 2-spheres of radius $r$ have zero area. This is, therefore, a naked singularity.

Now the mass of the dual solution is clearly equal to the dilaton charge of the original RN solution $\tilde{M} = \mathcal{Q}_d = 0$ and vice-versa $\tilde{\mathcal{Q}}_d = M$. The electric and magnetic charges have the same values.

This is a non-extreme massless "black hole" where the non-extremality is provided by primary scalar hair.

Now, if one takes the "extreme limit" $r_0 = 0$ (that is, the extreme limit in the original solution) which is also the limit in which all the primary scalar hair vanishes and all the dilaton charge is completely determined by the electric and magnetic charges[21] $\tilde{\mathcal{Q}}_d^2 = 2\mathcal{Q}^2$ so the B bound is saturated

$$\left\{\begin{array}{rcl} d\tilde{\hat{s}}_E^2 & = & -H^{-1}K^{-1}dt^2 + HK\left[dr^2 + r^2 d\Omega^2\right]\,, \\ e^{\tilde{\hat{\phi}}} & = & H/K\,, \\ \tilde{\hat{A}}^1{}_t & = & -\sqrt{2}\alpha_1\left(K^{-1} - 1\right)\,, \\ \tilde{\hat{A}}^2_\varphi & = & -2\alpha_2 |Q| \cos\theta\,, \end{array}\right. \tag{2.116}$$

[21]The situation parallels the usual situation in which there is unconstrained "primary mass" and "secondary mass" which is completely fixed by the electric and magnetic charges through the B bound.

which is one of the extreme massless black holes in Refs. [12], identified as composite objects in the sense of Ref. [96] in Ref. [89] and further studied in Refs. [40].

Observe that, while primary scalar hair should be included in the B bound, the primary scalar hair completely disappears in the saturated B bound. Thus, unbroken supersymmetry acts as a *cosmic hairdresser* and it is not possible to find solutions with unbroken supersymmetry and primary scalar hair.

As a last example we consider the well-known Kerr spacetime metric which in Boyer-Lindquist coordinates reads:

$$\begin{aligned} d\hat{s}_E^2 &= -\frac{r^2 - 2Mr + a^2}{r^2 + a^2\cos^2\theta}(dt - a\sin^2\theta d\phi)^2 \\ &\quad + \frac{\sin^2\theta}{r^2 + a^2\cos^2\theta}\left[(r^2 + a^2)d\phi - adt\right]^2 \\ &\quad + \frac{r^2 + a^2\cos^2\theta}{r^2 - 2Mr + a^2}dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2\,, \end{aligned} \tag{2.117}$$

where $a = J/M$. This metric belongs to a more general class of metrics which can be written in appropriate coordinates as:

$$d\hat{s}_E^2 = -G(dt - \omega d\phi)^2 + Adr^2 + Bd\theta^2 + Cd\phi^2\,, \tag{2.118}$$

where $G, \omega, A, B$ and $C$ are arbitrary functions of $r$ and $\theta$, conveying the adapted character of the coordinates employed.

The T dual with respect to the isometry with Killing vector $\frac{\partial}{\partial t}$ is easily found to be, in the Einstein frame,

$$d\tilde{\hat{s}}_E^2 = -G^{-1}dt^2 + Adr^2 + Bd\theta^2 + Cd\phi^2\,, \tag{2.119}$$

There is also a two-form present, given by

$$\hat{B} = -\omega dt \wedge d\phi\,, \tag{2.120}$$

as well as a dilaton, namely

$$\hat{\phi} = -\tfrac{1}{2}\log|G|\,, \tag{2.121}$$

It is well known that in the static Schwarzschild case [29] what appear as horizons in one metric, look as singularities in the T dual of it. In the more general, stationary case considered here, there are two related concepts: the infinite redshift surface, (also called the "static limit") that is, the stationary limit surface bordering the region in which the Killing $\frac{\partial}{\partial t}$ is timelike; and the event horizon; that is the hypersurface where $r = constant$ becomes null ; the region between those two surfaces being the ergosphere.

In the $\tilde{Kerr}$ metric presented above, there is no "infinite redshift surface", and before the surface $r = const$ becomes null, a singularity develops, located at

$$G \equiv \frac{r^2 + a^2\cos^2\theta - 2mr}{r^2 + a^2\cos^2\theta} = 0\,. \tag{2.122}$$

The metric is easily seen to be asymptotically flat, and the 2-form goes to zero at infinity as

$$\hat{B} = 2ma\sin^2\theta\,\frac{1}{r}\left[1 + \frac{2m}{r} + \mathcal{O}(r^{-2})\right]dt \wedge d\phi\,. \tag{2.123}$$

## 2.5. Conclusions

The results of the present chapter concerning the transformation of the charges under duality leave unanswered the question posed at the beginning of this chapter: why the angular momentum appears in the definition of extremality (defining the borderline between regular horizon and a naked singularity, with zero Hawking temperature) but not in the Bogomol'nyi bound (whose saturation guarantees absence of quantum corrections, as well as a "zero force condition", allowing superposition of static solutions).

We hold this to be due to the fact that stationary (as opposed to static) black holes possess a specific decay width, which can even be seen classically by scattering waves off the black hole. This process is known as "superradiance" ([113]; see also [42]) in the black hole literature.

The way this appears is that the amplitude for reflected waves is greater than the corresponding incident amplitude, for low frequencies, up to a given frequency cutoff, $m\Omega_H$, depending on the angular momentum of the hole, and such that $\Omega_H(a=0)=0$. The angular momentum of the hole decreases by this mechanism until a static configuration is reached. The physics underlying this process is similar to the one supporting Penrose's energy extraction mechanism, namely, the fact that energy can be negative in the ergosphere. This, in turn, is an straightforward consequence of the mathematical fact that the Energy of a test particle is defined as $E = p.k$, where $p$ is the momentum of the particle, and $k$ is the Killing vector (which has spacelike character precisely in the ergosphere); and the product of a spacelike vector with a timelike one does not have a definite sign.

Quantum mechanically, this means that there are two competing mechanisms of decay for a rotating (stationary) black hole: spontaneous emission (the quantum effect associated to the superradiance), which is not thermal (and disappears when the angular momentum goes to zero) and Hawking radiation, which is thermal.

The first one is most efficient for massive black holes, but its width is never zero even for small masses, until the black hole has lost all its angular momentum.

This clearly shows that even if the black hole is extremal, it cannot be stable quantum mechanically as long as its angular momentum is different from zero. This argument taken literally would suggest that it is not possible to have BPS states with non zero angular momentum, unless they are such that no ergosphere exists. This is the case of the supersymmetric Kerr-Newman solutions which are singular and, therefore, do not have ergosphere. What is not clear is why supersymmetry signals as special that singular case and not the usual extremal Kerr-Newman black hole[22].

It could well be that Supergravity is not capable to give that answer but String Theory is: from the String Theory point of view, given an extreme Reissner-Nordström black hole, if we want to add angular momentum, we can only do it at the expense of adding mass at the same time. Thus, according to the String Theory black-hole building rules, one can get extreme Kerr-Newman black holes but never a supersymmetric (singular) object with non-zero angular momentum. In this sense, while Supergravity acts as a cosmic censor only in static cases, String Theory seems to act as a true cosmic censor in all cases. The singular solutions cannot be built in the theory.

A similar argument could also be enough to prove a no-hair theorem in String Theory: it could happen that it is impossible to build String Theory states with primary scalar hair because there is no primary source for scalar hair in it. In this sense String Theory would act

[22]This argument seems to be valid only in four dimensions, though, since rotating charged black holes which are BPS states exist in five dimensions [28]. The existence of two Casimirs for the five-dimensional angular momentum seems to play an important role.

as a *cosmic hairdresser*. Here the situation is, though, a bit different. First of all, we are clearly a long way from proving that there are no microscopic configurations in String Theory that result in macroscopic primary scalar hair. In fact, the situation resembles a bit the situation of the "primary mass" (the mass that exceeds the Bogomol'nyi identity) since it is not clear what the microscopic configuration that manifests itself as that primary mass is and, thus, there is no String Theory model for the Schwarzschild black hole. It is, in fact, conceivable that both quantities have the similar origins, as T duality seems to be indicating. This would be a more attractive scenario since then we would have a tool ins String Theory to understand no-hair theorems from first principles.

There is, yet, another, more speculative, possibility that we could like to mention. Since extreme and non-extreme massless "black holes" seem to have the same kind of singularities as their regular T dual counterparts (null and spacelike, respectively) one could, in principle, use the spacetime of the massless black hole to patch up the spacetime of the massive one, gluing them at the singularity. This would be a non-analytic continuation through the singularity with the help of T duality much in the same spirit as T duality at finite temperature can relate high and low-temperature regimes of the heterotic string even though, in between the free energy diverges at the Hagedorn temperature [99].

From the point of view of String Theory this possibility looks more plausible when one takes into account the lower sensitivity of strings to spacetime singularities, as compared to point particles [61],

# Chapter 3

# Massive Supergravity in $D = 10$, *i.e.* IIAm

As we have seen in the chapter (1), type IIA string theory contains D(2n)-branes. It is also widely known that the *low-energy-effective* action describing this theory is the so-called type IIA supergravity, in which the $D0$-, $D2$-branes are represented by form-fields. The $D4$- and $D6$-branes can then be introduced by *Hodge* duality. The only $D$-brane not fitting into the usual IIA supergravity is the $D8$-brane.

The $D8$-brane in supergravity would be represented by a nine-form field or rather a 10-form fieldstrength, which carries no degrees of freedom. dualizing this field strength, one expects a scalar field, which, due to the Bianchi identity, is readily seen to be a constant. As was first noted by Polchinski [93], in ten dimensions there actually exists a supergravity Lagrangian which contains a free, constant, parameter and which can be truncated to type IIA, when taking this parameter to zero. It is this *massive IIA*, also called *Romans' theory* [98], which is interpreted as the low energy effective action describing type IIA strings with $D8$ brane contributions.

In this chapter we are going to study the massive IIA sugra, its relation to IIB by means of T-duality, the brane solutions and intersections of the D8 with a fundamental string and a D6 brane and the possible 11-dimensional origin of this action.

## 3.1. Massive $D = 10$ IIA Supergravity

The action of the massive IIA, Romans' for short, theory, reads

$$\begin{aligned}\mathcal{S}_{Romans} \;=\; & \int d^{10}x\sqrt{g}\,\Big[e^{-2\phi}\left\{R\,-\,4(\partial\phi)^2\,+\,\tfrac{1}{2\cdot 3!}H^2\right\}\\ & -\tfrac{1}{2}m^2\,-\,\tfrac{1}{2\cdot 2!}G_{(2)}^2\,-\,\tfrac{1}{2\cdot 4!}G_{(4)}^2\\ & -\tfrac{1}{144\sqrt{g}}\epsilon\left(\partial C^{(3)}\partial C^{(3)}B\,+\,\tfrac{m}{2}\partial C^{(3)}B^3\,+\,\tfrac{9m^2}{80}B^5\right)\Big]\;,\end{aligned} \tag{3.1}$$

where we have defined

$$\left\{\begin{array}{lcl} H & = & 3\partial B\;,\\[2ex] G_{(2)} & = & 2\partial C^{(1)}\,+\,mB\;,\\[2ex] G_{(4)} & = & 4\partial C^{(3)}\,+\,4C^{(1)}H\,+\,3mB^2\;.\end{array}\right. \tag{3.2}$$

The above action is invariant under the following transformations

$$\begin{cases} \delta C^{(1)} &= \partial\Lambda^{(0)} \;-\; m\Lambda^{(1)} \;, \\ \delta B &= 2\partial\Lambda^{(1)} \;, \\ \delta C^{(3)} &= 3\partial\Lambda^{(2)} \;-\; 3m\Lambda^{(1)}B \;+\; 3\partial\Lambda^{(0)}B \end{cases} \tag{3.3}$$

The variations[1] containing $\Lambda^{(0)}$ and $\Lambda^{(2)}$ are nothing but the usual gauge invariances of the 1- and 3-form fields, although adapted to the case at hand. The real surprise are the variations of the $B$ field, as propagated by $\Lambda^{(1)}$: As one can see from the symmetry rules, we can absorb the $U(1)$ field into the Kalb-Ramond field. This has as a consequence that the fieldstrength for $C^{(1)}$, $G_{(2)}$, gets converted into a mass term for $B$. The explicit action reads

$$\begin{aligned} \mathcal{S}_{Romans} \;&=\; \int d^{10}x\sqrt{g}\left[e^{-2\phi}\left\{R \;-\; 4(\partial\phi)^2 \;+\; \tfrac{1}{2\cdot 3!}H^2\right\} - \tfrac{1}{2}m^2 \;-\; \tfrac{1}{2\cdot 2!}m^2B^2 \;-\; \tfrac{1}{2\cdot 4!}G_{(4)}^2 \right. \\ &\quad \left. -\tfrac{1}{144\sqrt{g}}\epsilon\left(\partial C^{(3)}\partial C^{(3)}B \;+\; \tfrac{m}{2}\partial C^{(3)}B^3 \;+\; \tfrac{9m^2}{80}B^5\right)\right] \;, \end{aligned} \tag{3.4}$$

A field which has the transformation laws as the $C^{(1)}$ in this case, is generically called a 'Stückelberg' field.

In [19] a stringy interpretation was given for the occurrence of the mass parameter as the dual fieldstrength of a D8-brane. The argument goes as follows: A constant is nothing but a function constrained to be constant. If we then introduce this function, $M(x)$ say, we can impose a Bianchi identity on it

$$dM(x) = 0 \;, \tag{3.5}$$

which clearly states that the function $M$ must be constant. Now, interpreting this fact in the, by now, standard manner, we would say that $M$ is a 0-form field strength, and as a $p$-form field strength signals the existence of a $(p-2)$-brane, we conclude that the bugger signals the existence of a $(-2)$-brane. However, since there are (-2)-branes, the massparameter can only signal the presence of its Hodge dual brane: A D8-brane. Note that the absence of a dilaton factor for $m$ in Eq. (3.1) supports the fact that one is indeed dealing with a D-brane.

Note that type IIA string theory states that the theory admits to D8-branes, although there is no representation of it, by means of its field strength or its dual, in the usual type IIA Supergravity. This then means that, knowing no other type IIA supergravity representing D8-branes than Romans' theory, Romans' theory ought to be considered the true low energy effective action of type IIA string theory.

Making a conformal rescaling as to switch to the Einstein frame one gets the action

$$\begin{aligned} \mathcal{S} \;&=\; \int d^{10}x\sqrt{g}\left\{R \;+\; \tfrac{1}{2}(\partial\phi)^2 \;+\; \tfrac{1}{2\cdot 3!}e^{-\phi}H^2 \;-\; \tfrac{1}{2\cdot 4!}e^{\phi/2}G_{(4)}^2 \;-\; \tfrac{1}{2\cdot 2!}e^{3\phi/2}G_{(2)}^2 \;-\; \tfrac{1}{2}e^{5\phi/2}m^2 \right. \\ &\quad \left. -\tfrac{1}{144\sqrt{g}}\epsilon\left[\partial C^{(3)}\partial C^{(3)}B \;+\; \tfrac{m}{2}\partial C^{(3)}B^3 \;+\; \tfrac{9m^2}{80}B^5\right]\right\} \;. \end{aligned} \tag{3.6}$$

The susy rules are the same as for the usual type IIA supergravity, see Eq. (1.81), but now taking into account the presence of the zero- and ten-fieldstrengths. Since we are interested in

[1] The most compact way of writing the RR field strengths is by introducing the sum of all RR-fields into one big form, $C \equiv C^{(0)} + C^{(2)} \ldots$, and to define $G \equiv dC - H \wedge C + me^B$. The variations are then compactly written as $\delta C = d\Lambda\, e^B - m\Lambda^{(1)}e^B$.

supersymmetric configurations, we must discuss the supersymmetry variations of the fermionic fields.[2] The susy variation of the gravitino and the dilatino read

$$\left\{\begin{array}{rcl} \delta_\epsilon\psi_\mu &=& \left\{\partial_\mu - \frac{1}{4}\left(\not{\omega}_\mu + \frac{1}{4}\Gamma_{11}\not{H}_\mu + \frac{1}{2\cdot 7!}e^{2\phi}\Gamma_{\mu\nu_1\cdots\nu_7}H^{(7)\,\nu_1\cdots\nu_7}\right)\right\}\epsilon \\ && +\frac{i}{16}e^{\phi}\Sigma_{n=0}^{n=4}\frac{1}{(2n)!}\not{G}^{(2n)}\Gamma_\mu\left(-\Gamma_{11}\right)^n\epsilon\,, \\ \delta_\epsilon\lambda &=& \left[\not{\partial}\phi + \frac{1}{4}\left(\frac{1}{3!}\Gamma_{11}\not{H} - \frac{1}{7!}e^{2\phi}\not{H}^{(7)}\right)\right]\epsilon + \frac{i}{8}e^{\phi}\sum_{n=0}^{n=4}\frac{5-2n}{(2n)!}\not{G}^{(2n)}\left(-\Gamma_{11}\right)^n\epsilon\,. \end{array}\right. \tag{3.7}$$

### 3.1.1. Equations of Motion *etc.*

Here we give the bosonic equations of motion and the Bianchi identities of massive type IIA Sugra in the string frame including the dual RR and NSNS potentials. Many of the general expressions are also valid for the IIB theory. Due to the explicit occurrence of potentials in the action, they can only be dualized on-shell. The dual potentials are defined by the relations between field strengths

$$\left\{\begin{array}{rcl} G^{(10-n)} &=& (-1)^{[n/2]\star}G^{(n)}\,, \\ H^{(7)} &=& e^{-2\phi\star}H\,, \end{array}\right. \tag{3.8}$$

plus the Bianchi identities

$$\left\{\begin{array}{rcl} dG - H\wedge G &=& 0\,, \\ dH &=& 0\,, \\ dH^{(7)} + \frac{1}{2}{}^\star G\wedge G &=& 0\,, \end{array}\right. \tag{3.9}$$

and the equations of motion

$$\left\{\begin{array}{rcl} d^\star G + H\wedge{}^\star G &=& 0\,, \\ d\left(e^{-2\phi\star}H\right) + \frac{1}{2}{}^\star G\wedge G &=& 0\,, \\ d\left(e^{2\phi\star}H^{(7)}\right) &=& 0\,, \end{array}\right. \tag{3.10}$$

where we are using the notation [39, 54, 25] in which which forms of different degrees are formally combined into a single entity:

$$\left\{\begin{array}{rcl} C &=& C^{(0)} + C^{(1)} + C^{(2)} + \ldots\,, \\ G &=& G^{(0)} + G^{(1)} + G^{(2)} + \ldots\,. \end{array}\right. \tag{3.11}$$

These expressions are valid both for the type IIB and for the massive type IIA theory if one selects respectively odd and even rank and odd rank RR differential form field strengths and one makes the identification

$$G^{(0)} = m\,. \tag{3.12}$$

[2]For type IIA Sugra, the spinors are real 32-component spinors and the gamma matrices are purely imaginary satisfying $\left\{\gamma^a,\gamma^b\right\} = 2\eta^{ab}$ and $\gamma_{11} = -\gamma^0\ldots\gamma^9$.

The field strengths that correspond to these Bianchi identities and equations of motion are given by

$$\left\{\begin{array}{rcl} G & = & dC - H\wedge C + me^{B}\,, \\ H^{(7)} & = & dB^{(6)} + \frac{m}{2}C\wedge e^{-B} - \frac{1}{2}{}^{\star}G\wedge C\,, \\ \mathcal{H}^{(7)} & = & d\mathcal{B}^{(6)} + \frac{1}{2}\sum_{n=1}^{n=4}{}^{\star}\mathcal{G}^{(2n+3)}\wedge \mathcal{C}^{(2n)}\,, \end{array}\right. \tag{3.13}$$

where calligraphic fields belong to the IIB theory.

These equations have to be supplemented by the dilaton equation of motion

$$R + 4\,(\partial\phi)^{2} - 4\nabla^{2}\phi + \tfrac{1}{2\cdot 3!}H^{2} = 0\,, \tag{3.14}$$

and the Einstein equation of motion (where we have already eliminated $R$ with the use of the dilaton equation of motion)

$$R_{\mu\nu} - 2\nabla_{\mu}\nabla_{\nu}\phi + \tfrac{1}{4}H_{\mu}{}^{\rho\sigma}H_{\nu\rho\sigma} - \tfrac{1}{4}e^{2\phi}\sum_{n}\tfrac{(-1)^{n}}{(n-1)!}T^{(n)}{}_{\mu\nu}\,, \tag{3.15}$$

where $T^{(n)}{}_{\mu\nu}$ are the energy-momentum tensor of the RR fields:

$$T^{(n)}{}_{\mu\nu} = G^{(n)}{}_{\mu}{}^{\rho_1\cdots\rho_{n-1}}G^{(n)}{}_{\nu\rho_1\cdots\rho_{n-1}} - \tfrac{1}{2n}g_{\mu\nu}G^{(n)\,2}\,, \tag{3.16}$$

when $n\neq 0$ and

$$T^{(0)}{}_{\mu\nu} = -\tfrac{1}{2}m^{2}g_{\mu\nu}\,. \tag{3.17}$$

These equations are also valid both for the type IIA and IIB theories. Observe that the contributions of the energy-momentum tensors of dual fields add up, except in the $n=5$ case.

The field strengths as displayed in Eq. (3.13), and therefore also the Bianchi identities and the equations of motion, are invariant under

$$NS\ :\ \left\{\begin{array}{lcl} \delta B & = & d\chi^{(1)} \\ \delta C & = & -m\chi^{(1)}\wedge e^{B} \\ \delta B^{(6)} & = & d\chi^{(5)} + \frac{m}{2}\chi^{(1)}\wedge C\wedge e^{-B} \end{array}\right. \tag{3.18}$$

$$RR\ :\ \left\{\begin{array}{lcl} \delta B & = & 0 \\ \delta C & = & d\Lambda - H\wedge\Lambda \\ \delta B^{(6)} & = & -\frac{m}{2}\Lambda\wedge e^{-B} + \frac{1}{2}{}^{\star}G\wedge\Lambda \end{array}\right. \tag{3.19}$$

As explained in the introduction, the mass parameter occurs in the form of a cosmological constant (or, in the Einstein frame, of an unbound potential for the dilaton). Furthermore, the field strengths of $C^{(1)}$ and $B^{(6)}$, see Eqs. (3.13), contain the terms

$$\left\{\begin{array}{rcl} G^{(2)} & = & dC^{(1)} + mB\,, \\ H^{(7)} & = & dB^{(6)} + mC^{(7)} + \ldots \end{array}\right. \tag{3.20}$$

associated to these terms there are two *massive gauge transformations*

$$\left\{\begin{array}{rcl} \delta C^{(1)} & = & -m\chi^{(1)}\,, \\ \delta B & = & d\chi^{(1)}\,, \end{array}\right. \qquad \left\{\begin{array}{rcl} \delta B^{(6)} & = & -m\Lambda^{(6)}\,, \\ \delta C^{(7)} & = & d\Lambda^{(6)}\,, \end{array}\right. \tag{3.21}$$

using which one can completely eliminate $C^{(1)}$ and $B^{(6)}$ everywhere. Then, the kinetic terms of these Stückelberg fields become mass terms for $B$ and $C^{(7)}$ respectively. Only these two terms are massive.

Now, solutions describing $p$-branes in massive type IIA Sugra are automatically solutions describing the intersection of those $p$-branes with a D8-brane associated to the mass parameter. General solutions for the intersection of $p_1$ and $p_2$ branes have been found in the literature using a generic model whose action contains only kinetic terms for the dilaton and the $(p_1+2)$- and $(p_2+2)$-form field strengths (See section (3.4.1) for a better introduction). Therefore, those solutions can potentially describe correctly intersections involving a D8-brane and a D0-, D2- and D4-brane, associated to massless fields. However, they cannot correctly describe the intersections of a D8-brane and a fundamental string, a solitonic 5-brane or a D6-brane. Study of the supersymmetry algebra reveals that these solutions should exist and preserve 1/4 or the supersymmetries [100]. In sections (3.4.2) and (3.4.3) we will present the corresponding solutions and will comment on some of their unusual features.

## 3.2. $Sl(2,\mathbb{R})$ Covariant $D=9$ Sugra and T-duality

T-duality between type IIB and ordinary type IIA, in the sugra limit is achieved, by making use of dimensional reduction. It is however clear that simple dimensional reduction will not lead to similar duality between massive IIA and IIB. It was shown in [19] that in order to establish such a duality, one needs to use *Generalized Scherk-Schwarz* reduction [101]. The underlying idea is that when an action is invariant under some global symmetry, one can introduce a local transformation, depending only on the coordinate we are reducing over, such that the resulting action is independent of this coordinate, but introducing mass parameters.

Type IIB sugra is $Sl(2,\mathbb{R})$ invariant, and although one only needs a $U(1)$ in order to achieve T-duality between type IIB and massive IIA, we will have a look at the complete GSS reduction and discuss some of its results.

### 3.2.1. Generalized Dimensional Reduction: A Toy Model

We consider the following toy model which exhibits the general features of generalized dimensional reduction associated to global symmetries with no geometrical origin[3]:

$$\hat{S} = \int d^d\hat{x}\sqrt{|\hat{g}|}\ \left[\hat{R} + \tfrac{1}{2}\left(\partial\hat{\phi}\right)^2\right] . \tag{3.22}$$

This action is invariant under constant shifts of the scalar $\hat{\phi}$, the reason being that $\hat{\phi}$ only occurs through its derivatives. The presence of this global symmetry allows us to extend the general Kaluza-Klein Ansatz (i.e. all fields, and in particular $\hat{\phi}$, are independent of some coordinate, say $z$) to a more general Ansatz in which $\hat{\phi}$ depends on $z$ in a particular way:

$$\hat{\phi}(x,z) = \hat{\phi}^{\rm b}(x) + mz\,, \qquad \hat{x}^{\hat{\mu}} = (x^{\mu}, z)\,, \tag{3.23}$$

where the superscript $^b$ stands for *bare*, or $z$-independent.

This dependence on $z$ can be produced by a *local* shift of $\hat{\phi}(x)$ with a parameter linear in $z$. The invariance of the action under constant shifts ensures that the action will not depend on $z$.

This is only a practical recipe to write a good Ansatz. To understand better what one is doing, one has to recall that $z$ is a coordinate on a circle $S^1$ subject to the identification

[3] In this section we use hats for $d$-dimensional objects and no hats for $(d-1)$-dimensional objects.

$z \sim z + 2\pi l$. In standard Kaluza-Klein reduction one only considers single-valued fields, so that the needed Fourier decomposition of the fields living on $\mathcal{M} \otimes S^1$, reads

$$\hat{\phi}\,(\hat{x}) \;=\; \sum_{n \in \mathbb{Z}} e^{2\pi n z/l}\, \phi^{(n)}(x)\;. \tag{3.24}$$

Dimensional reduction then means keeping the massless modes, *i.e.* $\phi^{(0)}$, only. Some fields can be multivalued, however. If the scalar $\hat{\phi}$ is such that $\hat{\phi} = \hat{\phi} + 2\pi m$, the above Fourier expansion is enhanced to

$$\hat{\phi}\,(\hat{x}) \;=\; \frac{mNz}{l} + \sum_{n \in \mathbb{Z}} e^{2\pi n z/l}\, \phi^{(n)}(x)\;, \tag{3.25}$$

where $N \in \mathbb{Z}$ labels the different *topological sectors*. Now, the action for a field living on an $S^1$ is always invariant under arbitrary shifts of the field, even if the field is to be identified under discrete shifts. This then ensures that the lower dimensional theory does not depend on $z$, the dimensional reduction, if only $\frac{mNz}{l} + \phi^{(0)}$ is kept. Each topological sector is characterized by the charge

$$N \;=\; \lim_{x \to \infty} \frac{1}{2\pi l m} \oint d\hat{\phi}\;, \tag{3.26}$$

which is nothing but the *winding number.*

A more physical interpretation of the technical description of the generalized dimensional reduction recipe will be given later on.

Making use of the standard KK Ansatz for the Vielbein

$$\left(\hat{e}_{\hat{\mu}}{}^{\hat{a}}\right) = \begin{pmatrix} e_\mu{}^a & kA_{(1)\ \mu} \\ 0 & k \end{pmatrix}\;, \qquad \left(\hat{e}_{\hat{a}}{}^{\hat{\mu}}\right) = \begin{pmatrix} e_a{}^\mu & -A_{(1)\ a} \\ 0 & k^{-1} \end{pmatrix}\;, \tag{3.27}$$

we readily obtain the $(d-1)$-dimensional action

$$S = \int d^{d-1}x\sqrt{|g|}\ k\ \left[R - \tfrac{1}{4}k^2 F_{(2)}^2 + \tfrac{1}{2}\left(\mathcal{D}\phi\right)^2 - \tfrac{1}{2}m^2 k^{-2}\right]\;, \tag{3.28}$$

where the field strengths are defined by

$$\left\{\begin{array}{rcl} F_{(2)\ \mu\nu} & = & 2\partial_{[\mu}A_{(1)\ \nu]}\;, \\ \mathcal{D}_\mu\phi & = & \partial_\mu\phi - mA_{(1)\ \mu}\;, \end{array}\right. \tag{3.29}$$

and

$$\phi \equiv \hat{\phi}^{\mathrm{b}}\;. \tag{3.30}$$

A further rescaling of the metric

$$g_{\mu\nu} \to k^{-2/(d-3)} g_{\mu\nu}\;, \tag{3.31}$$

brings us to the final form of the action:

$$S = \int d^{d-1}x\sqrt{|g|}\ \left[R + \tfrac{1}{2}\left(\partial\varphi\right)^2 - \tfrac{1}{4}e^{-a\varphi}F_{(2)}^2 + \tfrac{1}{2}\left(\mathcal{D}\phi\right)^2 - \tfrac{1}{2}m^2 e^{a\varphi}\right]\;, \tag{3.32}$$

where

$$k = e^{-\varphi/2a}\,, \qquad a = -\sqrt{\frac{2(d-2)}{(d-3)}}\,. \tag{3.33}$$

This action and the field strengths are invariant under the following massive gauge transformations:

$$\left\{\begin{array}{rcl} \delta\phi & = & m\chi\,, \\ \\ \delta A_{(1)\;\mu} & = & \partial_\mu \chi\,. \end{array}\right. \tag{3.34}$$

These transformations correspond in the $d$-dimensional theory to the $z$-independent reparametrizations of $z$:

$$\delta z = -\chi(x)\,. \tag{3.35}$$

This is the theory resulting from the standard recipe for generalized dimensional reduction [19].

There is another way of getting the same result in this toy model: We gauge the translation $\hat{\phi} \to \hat{\phi}+m$ and impose that the gauge field is non-vanishing and constant in the internal direction only (a Wilson line). Since the metric does not transform, it is sufficient to demonstrate this on the kinetic term for $\hat{\phi}$.

In order to gauge the translation invariance on $\hat{\phi}$ we introduce the gauge field by minimal coupling

$$\partial_{\hat{\mu}}\hat{\phi} \;\to\; \mathcal{D}_{\hat{\mu}}\hat{\phi} \;=\; \partial_{\hat{\mu}}\hat{\phi} \;+\; \hat{\mathcal{E}}_{\hat{\mu}}\,, \tag{3.36}$$

so that under a local transformation $\hat{\phi} \to \hat{\phi} + \Lambda(\hat{x})$ the gauge field transforms in an Abelian manner, i.e.

$$\hat{\mathcal{E}}'_{\hat{\mu}} \;=\; \hat{\mathcal{E}}_{\hat{\mu}} \;+\; \partial_{\hat{\mu}}\Lambda(\hat{x})\;. \tag{3.37}$$

Making then the *standard* KK Ansatz and imposing that $\hat{\mathcal{E}}_{\hat{\mu}}$ is non-vanishing and constant, with value $m$, in the compact direction only, one finds

$$\left\{\begin{array}{rcl} \mathcal{D}_a\phi & = & e_a{}^{\mu}\left(\partial_\mu\phi \;-\; mA_{(1)\;\mu}\right) \;\equiv\; e_a{}^{\mu}\mathcal{D}_\mu\phi\;, \\ \\ \mathcal{D}_z\phi & = & k^{-1}m\;, \end{array}\right. \tag{3.38}$$

leading to

$$\int d^dx\sqrt{|\hat{g}|}\,\tfrac{1}{2}\,(\partial\phi)^2 \;=\; \int d^{d-1}x\sqrt{|g|}\,k\left[\tfrac{1}{2}\,(\mathcal{D}\phi)^2 \;-\; \tfrac{1}{2}k^{-2}m^2\right]\,. \tag{3.39}$$

Comparing this result with Eq. (3.28), one sees that, at least in this toy-model, generalized Scherk-Schwarz reduction leads to the same result as the above algorithm.

We will also use this method in the context of type IIB supergravity and check that one gets the same results as well.

Observe that the field content looks the same as in the standard dimensional reduction: There is a vector and two scalars (apart from the metric). The symmetries and couplings are different, though. The massive gauge symmetry allows us to eliminate one scalar (the Stückelberg field Ref. [110]) and give mass to the vector field. The number of degrees of freedom is exactly the same. So, what is it we have done? To shed some light on the meaning of this procedure we are going to perform the "standard" dimensional reduction of the action (3.22) but Poincaré-dualizing first the scalar into a $(d-2)$-form potential[4] $\hat{A}_{(d-2)\ \hat{\mu}_1\cdots\hat{\mu}_{(d-1)}}$:

$$\partial\hat{\phi} = {}^{\star}\hat{F}_{(d-1)}\,. \tag{3.40}$$

The dual action is

$$\tilde{\hat{S}} = \int d^dx\sqrt{|\hat{g}|}\ \left[\hat{R} + \tfrac{(-1)^{(d-2)}}{2\cdot(d-1)!}\hat{F}^2_{(d-1)}\right]\,. \tag{3.41}$$

*Standard* dimensional reduction with the same Vielbein Ansatz gives

$$\tilde{S} = \int d^{d-1}x\sqrt{|g|}\ k\ \left[R - \tfrac{1}{4}k^2F^2_{(2)} + \tfrac{(-1)^{(d-2)}}{2\cdot(d-1)!}F^2_{(d-1)} + \tfrac{(-1)^{(d-3)}}{2\cdot(d-2)!}k^{-2}F^2_{(d-2)}\right]\,, \tag{3.42}$$

where

$$\left\{\begin{array}{rcl} F_{(d-1)} & = & (d-1)\partial A_{(d-2)} + (-1)^{(d-1)}A_{(1)}F_{(d-2)}\,, \\ \\ F_{(d-2)} & = & (d-2)\partial A_{(d-3)}\,, \end{array}\right. \tag{3.43}$$

are the field strengths of the $(d-2)$- and $(d-3)$-form potentials of the $(d-1)$-dimensional theory.

We can now dualize the potentials. A $(d-2)$-form potential in $(d-1)$ dimensions is dual to a constant that we call $m$. Adding the term

$$-\tfrac{1}{(d-1)!}\int d^{d-1}x\ m\epsilon\left[F_{(d-1)} + (-1)^d(d-1)A_{(1)}F_{(d-2)}\right]\,, \tag{3.44}$$

to the action (3.42), and eliminating $F_{(d-1)}$ using its equation of motion

$$m = k^{\star}F_{(d-1)}\,, \tag{3.45}$$

in the action we get

$$\begin{array}{rcl} \tilde{S} & = & \int d^{d-1}x\sqrt{|g|}\ \left\{k\ \left[R - \tfrac{1}{4}k^2F^2_{(2)} + \tfrac{(-1)^{(d-3)}}{2\cdot(d-2)!}k^{-2}F^2_{(d-2)} - \tfrac{1}{2}m^2k^{-2}\right]\right. \\ \\ & & \left. +\tfrac{1}{(d-2)!}\tfrac{\epsilon}{\sqrt{|g|}}F_{(d-2)}\left[-mA_{(1)}\right]\right\}\,. \end{array} \tag{3.46}$$

Now we dualize into a scalar field the $(d-3)$-form potential: We add to the above action the term

$$\tfrac{1}{(d-2)(d-2)!}\int d^{d-1}x\ \epsilon F_{(d-2)}\partial\phi\,, \tag{3.47}$$

and eliminate $F_{(d-2)}$ by substituting in the action its equation of motion

$$F_{(d-2)} = (-1)^{(d-2)}k\ {}^{\star}\mathcal{D}\phi\,, \tag{3.48}$$

obtaining, perhaps surprisingly, Eq. (3.28).

[4]When indices are not explicitly shown we assume all indices to be antisymmetrized with weight one. This is slightly different from differential form notation.

What we have done is represented in figure 3.1.

The translation to brane language is obvious: Generalized dimensional reduction, which is essentially applied to scalars, is a way of keeping track of the dual $(d-3)$- and $(d-4)$-branes which should arise had we started with the dual of the scalar field.

Observe that in the generalized dimensional reduction Ansatz, Eq. (3.23), the scalar is not single-valued in the compact coordinate: $\hat{\phi}(z+1) = \hat{\phi}(z) + m$. The charge of the $(d-3)$-brane can be associated to the monodromy of $\hat{\phi}$ and to the $(d-1)$-dimensional vector mass:

$$q \sim \int {}^{\star}\hat{F}_{(d-1)} \sim \int d\hat{\phi} \sim m \,. \tag{3.49}$$

The implication of these results is obvious: The standard recipe for generalized dimensional reduction is just a way of performing a dimensional reduction taking into account all the possible fields (i.e. branes) that can arise in $(d-1)$ dimensions. In particular, the presence of $(d-3)$-branes is associated to the dependence on the internal coordinate and the charge of the background $(d-3)$-branes is proportional to the mass parameter. Generalized dimensional reduction should, from this point of view, be considered the standard full dimensional reduction, while the standard dimensional reduction is incomplete and there is an implicit truncation. The reason why this has not been realized before is that the missing fields only carry discrete degrees of freedom. The mass parameters are to be considered fields, although one can equally consider them as expectation values of those fields.

Figure 3.1: This diagram represents two different ways of obtaining the same result: Generalized dimensional reduction and "dual" standard dimensional reduction.

### 3.2.2. The $Sl(2,\mathbb{R})$-Covariant Generalized Dimensional Reduction of Type IIB Supergravity: An S Duality Multiplet of $N=2, d=9$ Massive Supergravities

In this Section we perform the complete generalized dimensional reduction of type IIB supergravity in the direction parametrized by $y$ using the ideas of Ref. [19] as they were generalized in Ref. [78]. As we are going to explain, in the end we will obtain a three-parameter family (a triplet) of type II 9-dimensional supergravities connected by $SL(2,\mathbb{R})$ transformations (in the adjoint representation).

We are going to perform the generalized dimensional reduction in a manifestly $SL(2,\mathbb{R})$-covariant way. $SL(2,\mathbb{R})$ symmetry is manifest in the Einstein-frame. However, T duality, being a stringy symmetry, is better described in string frame. Thus we will spend some time relating the fields appearing in both frames. Since reducing an action is easier than reducing equations of motion, we are going to use the *non-self-dual* (NSD) action introduced in Ref. [17]. We study these two points in the following subsection and we perform the actual reduction in the next section.

### Generalized Dimensional Reduction

Now that we have set up the action we want to reduce, we can proceed. First, we will explain the generalized KK Ansatz. In this point we will follow the recipe of Ref. [78] adapted to our conventions. Then we will reduce the action and the self-duality constraint and finally we will eliminate the constraint, obtaining the action of the 9-dimensional theory.

The fields of the Einstein-frame 9-dimensional theory are the same as in the massless case:

$$\{g_{E\ \mu\nu}, A_{(3)\ \mu\nu\rho}, \vec{A}_{(2)\ \mu\nu}, \vec{A}_{(1)\ \mu}, A_{(1)\ \mu}, K, \mathcal{M}\}\,, \tag{3.50}$$

and only the couplings and symmetries will be different.

As usual in dimensional reductions, we assume the existence of a Killing vector $\hat{s}^{\hat{\mu}}\partial_{\hat{\mu}} = \partial_{\underline{y}}$ associated to the coordinate $y$. We choose adapted coordinates $\hat{x}^{\hat{\mu}} = (x^\mu, y)$ so that the metric does not depend on $y$. We normalize the coordinate $y$ such that it takes values in the interval $[0,1]$ and so $y \sim y+1$. Our Ansatz for the Einstein-frame Zehnbeins is then that of Eq. (3.27) adapted to ten dimensions and with the scalar $k$, the length of the (spacelike) Killing vector, relabeled

$$|\hat{s}^{\hat{\mu}}\hat{s}_{\hat{\mu}}|^{1/2} = K^{-3/4}\,, \tag{3.51}$$

for convenience.

Now, instead of assuming that all the other fields in our theory have vanishing Lie derivatives with respect to $\hat{s}^{\hat{\mu}}$, we assume that the remaining fields depend on $y$ but in a very specific way: All the $y$-dependence is introduced by a local $SL(2,\mathbb{R})$ transformation with parameters linear in $y$, $\Lambda(y)$:

$$\left\{\begin{array}{rcl} \hat{\mathcal{M}}(\hat{x}) & \equiv & \Lambda(y)\hat{\mathcal{M}}^{\mathrm{b}}(x)\Lambda^T(y)\,, \\ \hat{\vec{\mathcal{B}}}(\hat{x}) & \equiv & \Lambda(y)\vec{\mathcal{B}}^{\mathrm{b}}(x)\,, \\ \hat{D}(\hat{x}) & = & \hat{D}^{\mathrm{b}}(x)\,, \end{array}\right. \tag{3.52}$$

where we have denoted by a superscript $^b$ the *bare* $y$-independent fields.

Obviously, the Ansatz for $\hat{\mathcal{M}}$ is equivalent, in terms of $\hat{\lambda}(\hat{x})$ to

$$\hat{\lambda}(\hat{x}) = \frac{a(y)\hat{\lambda}^{\mathrm{b}}(x) + b(y)}{c(y)\hat{\lambda}^{\mathrm{b}}(x) + d(y)}\,. \tag{3.53}$$

In this scheme $\hat{D}$ cannot depend on $y$ because it is inert under $SL(2,\mathbb{R})$, but it is worth stressing that the string-frame metric does depend on $y$. The *bare* fields are $y$-independent and will become the 9-dimensional fields. On the other hand, they transform under $SL(2,\mathbb{R}$ as the real fields do.

The meaning of this kind of Ansatz is the following: We are constructing a non-trivial line bundle over the circle parametrized by $y$ with fiber $\hat{\lambda}$ (or, equivalently $\hat{\mathcal{M}}$) and structure group $SL(2,\mathbb{R})$ (we will later study the restriction to $SL(2,\mathbb{Z})$). Going once around the circle we go back to the same $\hat{\mathcal{M}}$ up to a global $SL(2,\mathbb{R})$ transformation that we can describe by an $SL(2,\mathbb{R})$ monodromy matrix $M$. The explicit form of $M$ depends on the explicit form of $\Lambda(y)$.

Let us now describe more precisely the form of $\Lambda(y)$. If

$$T_1 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_2 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_3 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.54}$$

are the generators of $SL(2,\mathbb{R})$, then the most general $SL(2,\mathbb{R})$ transformation with local parameters linear in $y$ can be written in the form

$$\Lambda(y) = \exp\left\{\tfrac{1}{2} y m^i T_i\right\}. \tag{3.55}$$

The three real parameters $m^i$ fully determine $\Lambda(y)$ and therefore the particular compactification. These parameters are going to become masses in the lower-dimensional theory. We define the *mass matrix* $m$

$$m \equiv \left(\partial_{\underline{y}}\Lambda\right)\Lambda^{-1} = \tfrac{1}{2} m^i T_i = \tfrac{1}{2}\begin{pmatrix} m^1 & m^2 + m^3 \\ m^2 - m^3 & -m_1 \end{pmatrix}. \tag{3.56}$$

This matrix belongs to the Lie algebra $sl(2,\mathbb{R})$ and therefore it transforms in the (irreducible) adjoint representation:

$$m' = \Lambda m \Lambda^{-1}, \tag{3.57}$$

and thus the three $m^i$ transform as a triplet (a vector of $SO(2,1) \sim SL(2,\mathbb{R})$). The expression

$$\alpha^2 = \mathrm{Tr}\ (m^2) = \tfrac{1}{4} m^i m^j h_{ij}, \qquad h_{ij} = \mathrm{diag}(++-), \tag{3.58}$$

where $h_{ij}$ is the Killing metric, is thus $SL(2,\mathbb{R})$-invariant. Furthermore, the mass matrix satisfies

$$\eta m \eta^{-1} = -m^T. \tag{3.59}$$

Observe that the parameters $m^1, m^2$ are associated to non-compact generators of $SL(2,\mathbb{R})$, while $m^3$ is associated to the maximal compact subgroup of $SL(2,\mathbb{R})$ ($SO(2)$). Thus, we are bound to get mass terms with the wrong sign (for instance in terms like Eq. (3.58)) but we must keep the three mass parameters in order to have full $SL(2,\mathbb{R})$-covariance and the most general 9-dimensional massive type II supergravity.

Our Ansatz generalizes that of Ref. [78], which only had two independent parameters: $m^1, m^2 = m^3$. The authors argued that generalized dimensional reduction using $SL(2,\mathbb{R})$ y-dependent transformations in the stability subgroup $SO(2)$ (i.e. those generated by $T_3$ and associated to $m^3$ in our conventions) would have no effect. As we discussed in the previous Section, there is no stability subgroup for the coset scalars. Furthermore, since the three mass parameters we just defined transform irreducibly, the three of them are required to obtain $SL(2,\mathbb{R})$-covariant families of theories. Finally, the $SL(2,\mathbb{R})$ transformation $S = \eta$ is inside the excluded $SO(2)$ and this is one of the generators of the quantum S duality group $SL(2,\mathbb{Z})$.

$\Lambda(y)$ will only manifest itself through the mass matrix in the lower-dimensional theory. However, in order to reconstruct the 10-dimensional fields we need to know it explicitly. The explicit form of $\Lambda(y)$ reads

$$\Lambda(y) = \begin{pmatrix} \cosh\alpha y + \frac{m^1}{2\alpha}\sinh\alpha y & \frac{m^2+m^3}{2\alpha}\sinh\alpha y \\ \\ \frac{m^2-m^3}{2\alpha}\sinh\alpha y & \cosh\alpha y - \frac{m^1}{2\alpha}\sinh\alpha y \end{pmatrix}, \tag{3.60}$$

where $\alpha$ was defined in Eq. (3.58).

It is easy to see from our definition of $\Lambda(y)$ that the monodromy matrix will be

$$M(m^i) = \exp\{\tfrac{1}{2}m^i T_i\} = \Lambda(y=1)\,, \tag{3.61}$$

and

$$\left\{ \begin{array}{rcl} \hat{\mathcal{M}}(x, y+1) & = & M\hat{\mathcal{M}}(x,y)M^T\,, \\ \\ \hat{\vec{\mathcal{B}}}(x, y+1) & = & M\vec{\mathcal{B}}(x,y)\,. \end{array} \right. \tag{3.62}$$

Quantum-mechanically, the monodromy matrices can only be $SL(2,\mathbb{Z})$ matrices. It is convenient to describe the most general $SL(2,\mathbb{Z})$ monodromy matrix by for integers $n^i, n\,,\ i=1,2,3$ subject to the constraint

$$n^i n_i = n^2 - 1\,. \tag{3.63}$$

Given that this constraint is satisfied, then we simply make the identifications

$$\alpha = \cosh^{-1} n\,, \qquad m^i = \frac{2\alpha}{\sqrt{n^2-1}} n^i\,, \tag{3.64}$$

and write the monodromy matrix as follows:

$$M = \begin{pmatrix} n+n^1 & n^2+n^3 \\ \\ n^2-n^3 & n-n^1 \end{pmatrix}. \tag{3.65}$$

Thus, in our conventions, the mass parameters $m^i$ will be naturally quantized in terms of the three integers $n^i$ which also transform in the "adjoint" of $SL(2,\mathbb{Z})$. $n$ is $SL(2,\mathbb{Z})$-invariant.

In Section 3.5 we will relate the integers $n^i$ to the charges of 7-branes.

We can now perform the dimensional reduction.

### Dimensional Reduction

Using the standard techniques [101] we get with the just-described Ansatz the NSD 9-dimensional action

$$
\begin{aligned}
S_{\rm NSD} &= \\
\int d^9x\sqrt{|g|}\ \Big\{K^{-3/4}\ \Big[\ R(g)\ &+\tfrac{1}{4}{\rm Tr}\left(D\mathcal{M}\mathcal{M}^{-1}\right)^2 - \tfrac{1}{4}K^{-3/2}F_{(2)}^2 \\
&-\tfrac{1}{4}K^{3/2}\vec{F}_{(2)}^{\,T}\mathcal{M}^{-1}\vec{F}_{(2)} + \tfrac{1}{2\cdot 3!}\vec{F}_{(3)}^{\,T}\mathcal{M}^{-1}\vec{F}_{(3)} - \tfrac{1}{4\cdot 4!}K^{3/2}F_{(4)}^2 \\
&+\tfrac{1}{4\cdot 5!}F_{(5)}^2 - K^{3/2}\mathcal{V}(\mathcal{M})\Big] \\
&+\tfrac{1}{2^7\cdot 3^2\cdot 5}\tfrac{1}{\sqrt{|g|}}\ \epsilon\left\{\left(F_{(5)}-5A_{(1)}F_{(4)}\right)\times\right. \\
&\times\left[2\left(\vec{F}_{(3)}-3A_{(1)}\vec{F}_{(2)}\right)^T\eta\vec{A}_{(1)}+3\vec{F}_{(2)}^{\,T}\eta\vec{A}_{(2)}\right] \\
&\left.-5F_{(4)}\left(\vec{F}_{(3)}-3A_{(1)}\vec{F}_{(2)}\right)^T\eta\vec{A}_{(2)}\right\}\ ,
\end{aligned}
\tag{3.66}
$$

and the 9-dimensional duality constraint

$$
F_{(5)} = -K^{3/4}\ {}^\star F_{(4)}\ , \tag{3.67}
$$

where the field strengths are defined as follows:

$$
\left\{
\begin{aligned}
\mathcal{D}\mathcal{M} &= \partial\mathcal{M} - \left(m\mathcal{M}+\mathcal{M}m^T\right)A_{(1)}\ , \\
F_{(2)} &= 2\partial A_{(1)}\ , \\
\vec{F}_{(2)} &= 2\partial\vec{A}_{(1)} - m\vec{A}_{(2)}\ , \\
\vec{F}_{(3)} &= 3\partial\vec{A}_{(2)} + 3A_{(1)}\vec{F}_{(2)}\ , \\
F_{(4)} &= 4\partial A_{(3)} - 3\vec{A}_{(2)}^{\,T}\ \eta\ \vec{F}_{(2)} + 2\vec{A}_{(1)}^{\,T}\ \eta\ \vec{F}_{(3)} + 6A_{(1)}\vec{A}_{(1)}^{\,T}\ \eta\ \vec{F}_{(2)}\ , \\
F_{(5)} &= 5\partial A_{(4)} - 5\vec{A}_{(2)}^{\,T}\ \eta\ \vec{F}_{(3)} + 15A_{(1)}\vec{A}_{(2)}{}^T\eta\vec{F}_{(2)} + 5A_{(1)}F_{(4)}\ ,
\end{aligned}
\right.
\tag{3.68}
$$

and

$$
\mathcal{V}(\mathcal{M}) = \tfrac{1}{2}{\rm Tr}\left(m^2 + m\mathcal{M}m^T\mathcal{M}^{-1}\right)\ , \tag{3.69}
$$

is the scalar potential.

The 10- and 9-dimensional fields are related as follows:

$$
\begin{aligned}
\hat{\mathcal{M}}^{\rm b} &= \mathcal{M}\,, & \hat{D}_{\mu_1\mu_2\mu_3\underline{y}} &= -A_{(3)\ \mu_1\mu_2\mu_3}\ , \\
\vec{\mathcal{B}}^{\rm b}{}_{\mu\underline{y}} &= -\vec{A}_{(1)\ \mu}\ , & \hat{D}_{\mu_1\cdots\mu_4} &= A_{(4)\ \mu_1\cdots\mu_4}\ . \\
\vec{\mathcal{B}}^{\rm b}{}_{\mu\nu} &= \vec{A}_{(2)\ \mu\nu}\ ,
\end{aligned}
\tag{3.70}
$$

In order to eliminate the self-duality constraint Eq. (3.67) we first Poincaré-dualize the NSD action with respect to the 4-form potential. First, we add the Lagrange multiplier term

$$
\begin{aligned}
\frac{1}{2^5 \cdot 3^2} \int d^9x\ \epsilon \partial \tilde{A}_{(3)} \partial A_{(4)} \quad &= \\
\frac{1}{2^5 \cdot 3^2} \int d^9x\ \epsilon \partial \tilde{A}_{(3)} \Big[ F_{(5)} + 5 \vec{A}^{\,T}_{(2)}\ \eta\ \vec{F}_{(3)} - 15 A_{(1)} \vec{A}_{(2)}{}^{T} \eta \vec{F}_{(2)} - 5 A_{(1)} F_{(4)} \Big]\ ,
\end{aligned}
\tag{3.71}
$$

to the NSD action (3.66). The equation of motion of the Lagrange multiplier field $\tilde{A}_{(3)}$ enforces the Bianchi identity of $F_{(5)}$ and we can consider the new action as a functional of $F_{(5)}$ instead of $A_{(4)}$ which does not occur explicitly. The equation of motion for $F_{(5)}$ is nothing but

$$
F_{(5)} = -K^{3/4}\ {}^{\star}\tilde{F}_{(4)}\ , \tag{3.72}
$$

where $\tilde{F}_{(4)}$ is like $F_{(4)}$ but with $A_{(4)}$ replaced by $\tilde{A}_{(4)}$. This equation is purely algebraic and we can use it to eliminate $F_{(5)}$ in the NSD action (3.66) plus the Lagrange multiplier term. The result is an action the depends both on $A_{(4)}$ and $\tilde{A}_{(4)}$. Now, we simply observe that the equation of motion for $F_{(5)}$ has the same form as the self-duality constraint Eq. (3.67) and therefore, eliminating the self-duality constraint amounts to the simple identification

$$
F_{(4)} = \tilde{F}_{(4)}\ . \tag{3.73}
$$

The result of these manipulations plus a Weyl rescaling to go to the Einstein frame (the metric $g$ is neither the string metric nor Einstein's)

$$
g_{\mu\nu} = K^{3/14} g_{E\ \mu\nu}\ . \tag{3.74}
$$

is the action of the type II massive supergravity:

$$
\begin{aligned}
S \quad = \quad & \int d^9x\ \sqrt{|g_E|}\ \Big\{ R_E + \tfrac{9}{14} \left(\partial \log K\right)^2 + \tfrac{1}{4} \mathrm{Tr}\left(\mathcal{D}\mathcal{M}\mathcal{M}^{-1}\right)^2 - \tfrac{1}{4} K^{-12/7} F^2_{(2)} \\
& - \tfrac{1}{4} K^{\frac{9}{7}} \vec{F}^{\,T}_{(2)} \mathcal{M}^{-1} \vec{F}_{(2)} + \tfrac{1}{2 \cdot 3!} K^{-3/7} \vec{F}^{\,T}_{(3)} \mathcal{M}^{-1} \vec{F}_{(3)} - \tfrac{1}{2 \cdot 4!} K^{6/7} F^2_{(4)} - K^{12/7} \mathcal{V}\left(\mathcal{M}\right) \\
& - \tfrac{1}{2^7 \cdot 3^2} \tfrac{1}{\sqrt{|g_E|}}\ \epsilon \left\{ 16 (\partial A_{(3)})^2 A_{(1)} \right. \\
& + 24 \partial A_{(3)} \Big[ \partial \vec{A}^{\,T}_{(2)} \eta \vec{A}_{(2)} - \Big( 4 \vec{A}^{\,T}_{(2)} \eta \partial \vec{A}_{(1)} + 2 \vec{A}^{\,T}_{(1)} \eta \partial \vec{A}_{(2)} - \vec{A}^{\,T}_{(2)} \eta m \vec{A}_{(2)} \Big) A_{(1)} \Big] \\
& - 36 \Big( \vec{A}^{\,T}_{(2)} \eta \partial \vec{A}_{(1)} + \vec{A}^{\,T}_{(1)} \eta \partial \vec{A}_{(2)} \Big) \partial \vec{A}^{\,T}_{(2)} \eta \vec{A}_{(2)} \\
& - 36 \Big( \vec{A}^{\,T}_{(2)} \eta \partial \vec{A}_{(1)} - \vec{A}^{\,T}_{(1)} \eta \partial \vec{A}_{(2)} \Big)^2 A_{(1)} \\
& + 9 \vec{A}^{\,T}_{(2)} \eta m \vec{A}_{(2)} \Big[ \partial \vec{A}^{\,T}_{(2)} \eta \vec{A}_{(2)} - 4 \Big( \vec{A}^{\,T}_{(2)} \eta \partial \vec{A}_{(1)} - \vec{A}^{\,T}_{(1)} \eta \partial \vec{A}_{(2)} \Big) A_{(1)} \\
& \left. + (\, \vec{A}^{\,T}_{(2)} \eta m \vec{A}_{(2)}\, ) A_{(1)} \Big] \right\} \Big\}\ .
\end{aligned}
\tag{3.75}
$$

whose topological term, in order to facilitate comparison with the results of Section 3.3, was rewritten in terms of potentials only (no field strengths) by integrating several times by parts and using algebraic properties like

$$
\left( \vec{A}^{\,T}_{(1)} \eta \partial \vec{A}_{(2)} \right) \left( \partial \vec{A}^{\,T}_{(2)} \eta \vec{A}_{(2)} \right) = -\tfrac{1}{2} \left( \vec{A}^{\,T}_{(1)} \eta \vec{A}_{(2)} \right) \left( \partial \vec{A}^{\,T}_{(2)} \eta \partial \vec{A}_{(2)} \right)\ . \tag{3.76}
$$

**Gauge and Global Symmetries of the 9-Dimensional Theory**

The local symmetries of the 9-dimensional theory (3.75) have three different origins: The gauge transformations of the 2-form fields:

$$\delta\hat{\vec{\mathcal{B}}} = 2\partial\hat{\vec{\Sigma}}\,, \tag{3.77}$$

the gauge transformations of the 4-form

$$\delta\hat{D} = 4\partial\hat{\Delta} - \tfrac{2}{5}\hat{\vec{\Sigma}}^{T}\eta\hat{\vec{\mathcal{H}}}\,, \tag{3.78}$$

and the $y$-independent reparametrizations of the compact coordinate $y$

$$\delta\hat{x}^{\hat{\mu}} = \delta^{\hat{\mu}\underline{y}}\chi(x)\,. \tag{3.79}$$

The dependence of the 10-dimensional fields on $y$, inexistent in standard dimensional reduction, induces new terms (the transport terms) in the $\chi$-transformations.

The 9-dimensional fields have the following *infinitesimal* $\chi$ gauge transformations and finite $\Sigma_{(0)}, \vec{\Sigma}_{(0)}, \vec{\Sigma}_{(1)}, \Sigma_{(3)}$ gauge transformations:

$$\left\{\begin{array}{rcl}
\delta\mathcal{M} & = & \chi\left(m\mathcal{M} + \mathcal{M}m^{T}\right)\,,\\[2mm]
\delta A_{(1)} & = & \partial\chi\,,\\[2mm]
\delta\vec{A}_{(1)} & = & \partial\vec{\Sigma}_{(0)} + m\vec{\Sigma}_{(1)} + \chi m\vec{A}_{(1)}\,,\\[2mm]
\delta\vec{A}_{(2)} & = & 2\partial\vec{\Sigma}_{(1)} + 2\partial\chi\vec{A}_{(1)} + \chi m\vec{A}_{(2)}\,,\\[2mm]
\delta A_{(3)} & = & 3\partial\Sigma_{(2)} + \tfrac{3}{2}\vec{\Sigma}_{(1)}{}^{T}\eta\vec{F}_{(2)} - \tfrac{3}{2}\vec{\Sigma}_{(0)}{}^{T}\eta\partial\vec{A}_{(2)}\,,\\[2mm]
\delta A_{(4)} & = & 4\partial\Sigma_{(3)} + 6\vec{\Sigma}_{(1)}\eta\partial\vec{A}_{(2)} + 4\partial\chi A_{(3)}\,.
\end{array}\right. \tag{3.80}$$

The $\chi$-transformations can be exponentiated:

$$\left\{\begin{array}{rcl}
V' & = & e^{\chi m}V\,,\\[2mm]
\mathcal{M}' & = & e^{\chi m}\mathcal{M}\,e^{\chi m^{T}}\,,\\[2mm]
A'_{(1)} & = & A_{(1)} + \partial\chi\,,\\[2mm]
\vec{A}'_{(1)} & = & e^{\chi m}\vec{A}_{(1)}\,,
\end{array}\right.
\qquad
\left\{\begin{array}{rcl}
\vec{A}'_{(2)} & = & e^{\chi m}\left(\vec{A}_{(2)} + 2\partial\chi\vec{A}_{(1)}\right)\,,\\[2mm]
A'_{(3)} & = & A_{(3)}\,,\\[2mm]
A'_{(4)} & = & 4\partial\chi A_{(3)}\,.
\end{array}\right. \tag{3.81}$$

Under the $\chi$-transformations, the field strengths transform covariantly instead of being invariant:

$$\left\{\begin{array}{rcl}
(D\mathcal{M})' & = & e^{\chi m}D\mathcal{M}\,e^{\chi m^{T}}\,,\\[2mm]
\vec{F}'_{(2,3)} & = & e^{\chi m}\vec{F}_{(2,3)}\,,\\[2mm]
F'_{(4,5)} & = & F_{(4,5)}\,.
\end{array}\right. \tag{3.82}$$

We could easily define field strengths invariant under $\chi$-transformations: For instance

$$\tilde{\vec{F}}_{(2,3)} = V^{-1}\vec{F}_{(2,3)}\,, \tag{3.83}$$

as was done in Ref. [19], but we will choose not to do so.

It is trivial to check the invariance of the action (3.75) under the above gauge transformations.

The action Eq. (3.75) enjoys some global invariances as well, namely rescalings of $K$ and $SL(2,\mathbb{R})$ transformations. The latter are the most interesting. Their action on the fields $\mathcal{M}, \vec{A}_{(1)\ \mu}, \vec{A}_{(2)\ \mu\nu}$ is

$$\mathcal{M}' = \Lambda\mathcal{M}\Lambda^T\,, \qquad \vec{A}'_{(1,2)} = \Lambda\vec{A}_{(1,2)}\,. \tag{3.84}$$

As was said before, the mass matrix belongs to the Lie algebra $sl(2,\mathbb{R})$ and transforms in the adjoint representation:

$$m' = \Lambda m\Lambda^{-1}\,, \tag{3.85}$$

and thus the three $m^i$ transform as a triplet (a vector of $SO(2,1) \sim SL(2,\mathbb{R})$).

Finally, the theory is also invariant under constant rescalings of the fields:

$$\begin{array}{rclcrcl} K & \rightarrow & e^{14\alpha}K\,, & & m & \rightarrow & e^{-12\alpha}m\,, \\ A_{(1)} & \rightarrow & e^{12\alpha}A_{(1)}\,, & & \vec{A}_{(1)} & \rightarrow & e^{-9\alpha}\vec{A}_{(1)}\,, \\ A_{(3)} & \rightarrow & e^{-6\alpha}A_{(3)}\,, & & \vec{A}_{(2)} & \rightarrow & e^{3\alpha}\vec{A}_{(2)}\,. \end{array} \tag{3.86}$$

### 3.2.3. An Alternative Recipe for Generalized Dimensional Reduction: Gauging of Global Symmetries

In this Section we will apply an alternative recipe for generalized dimensional reduction to type IIB supergravity. The general idea is that gauging the global symmetry and imposing that the gauge field takes non-vanishing and constant values in the internal direction only, is equivalent to applying generalized Scherk-Schwarz reduction. In order to demonstrate this, the algorithm will be applied to the NSD IIB action, albeit written in terms of forms. The conventions for forms are the ones used in Ref. [87] and in particular we need

$$\int F_{(p)}{}^{\star}F_{(p)} \;=\; \int d^dx\sqrt{|g|}\ \tfrac{1}{p!}F_{(p)\ \mu_1\cdots\mu_p}F_{(p)}{}^{\mu_1\cdots\mu_p}\,, \tag{3.87}$$

The NSD IIB action written in forms reads

$$\begin{aligned} S_{IIB} \;=\; & \int d^{10}x\sqrt{|\hat{g}|}\ \Big[\hat{R}(\hat{g}) - \tfrac{1}{4}\,\mathrm{Tr}\Big(\partial_{\hat{\mu}}\hat{\mathcal{M}}\cdot\partial^{\hat{\mu}}\hat{\mathcal{M}}^{-1}\Big)\Big] \\ & + \int_{10}\Big\{\tfrac{1}{2}\hat{\vec{H}}^T\hat{\mathcal{M}}^{-1}\,{}^{\star}\hat{\vec{H}} + \tfrac{1}{4}\hat{F}_{(5)}{}^{\star}\hat{F}_{(5)} + \tfrac{1}{4}\hat{F}_{(5)}\hat{\vec{B}}^T\eta\hat{\vec{H}}\Big\}\,, \end{aligned} \tag{3.88}$$

where we have defined

$$\left\{\begin{array}{rcl} \hat{\vec{H}} & = & d\hat{\vec{B}}\,, \\ \hat{F}_{(5)} & = & d\hat{D} - \tfrac{1}{2}\hat{\vec{B}}^T\eta\hat{\vec{H}}\,, \end{array}\right. \tag{3.89}$$

which are nothing else than the definitions in Eqs. (1.90), but written in terms of forms.

In order to follow through the above procedure, we start by gauging the $SL(2,\mathbb{R})$ symmetry. We introduce a covariant derivative through

$$\left\{\begin{array}{lcl} \partial_{\hat{\mu}}\hat{\mathcal{M}} & \rightarrow & \mathcal{D}_{\hat{\mu}}\hat{\mathcal{M}} \; = \; \partial_{\hat{\mu}}\hat{\mathcal{M}} + \hat{\mathcal{E}}_{\hat{\mu}}\hat{\mathcal{M}} + \hat{\mathcal{M}}\hat{\mathcal{E}}_{\hat{\mu}}^{T} \,, \\ \\ d\hat{\vec{B}} & \rightarrow & \mathcal{D}\hat{\vec{B}} \; = \; d\hat{\vec{B}} \,+\, \hat{\mathcal{E}}\wedge\hat{\vec{B}} \,, \end{array}\right. \tag{3.90}$$

and one finds that $\hat{\mathcal{E}}$ has to transform as a gauge field

$$\hat{\mathcal{E}} \rightarrow \Lambda^{-1}\hat{\mathcal{E}}\Lambda \,+\, \Lambda^{-1}d\Lambda \,. \tag{3.91}$$

Now, applying the same KK Ansatz for the metric as was used in the preceding section, one sees that the covariant derivatives on $\hat{\mathcal{M}}$ get transformed into, changing notation such that $\mathcal{E}$ is the constant matrix in the internal direction,

$$\left\{\begin{array}{lcl} \mathcal{D}_a\hat{\mathcal{M}} & = & \partial_a\mathcal{M} - A_{(1)a}\left[\mathcal{E}\mathcal{M} + \mathcal{M}\mathcal{E}^T\right] \,, \\ \mathcal{D}_y\mathcal{M} & = & K^{3/4}\left(\mathcal{E}\mathcal{M} + \mathcal{M}\mathcal{E}^T\right) \,. \end{array}\right. \tag{3.92}$$

Clearly $\mathcal{E}$ is going to be the mass matrix $m$. This then means that we can write down

$$\left\{\begin{array}{lcl} \mathrm{Tr}\left(\partial\hat{\mathcal{M}}\partial\hat{\mathcal{M}}^{-1}\right) & \rightarrow & \mathrm{Tr}\left(\partial\mathcal{M}\partial\mathcal{M}^{-1}\right) \\ \\ & & +2A_{(1)\mu}\mathrm{Tr}\left[\mathcal{M}^{-1}\partial^{\mu}\mathcal{M}\left(\mathcal{M}^{-1}\mathcal{E}\mathcal{M} + \mathcal{E}^T\right)\right] \\ \\ & & +2\left(K^{\frac{3}{2}} - A_{(1)}^2\right)\,\mathrm{Tr}\left(\mathcal{M}^{-1}\mathcal{E}\mathcal{M}\mathcal{E}^T + \mathcal{E}^2\right) \end{array}\right. \tag{3.93}$$

One will readily acknowledge that this is exactly the result found in Section 3.2.2 with $\mathcal{E} = m$.

Decomposing $\hat{\vec{B}}$ as

$$\hat{\vec{B}} \;=\; \vec{A}_{(2)} \;-\; \vec{A}_{(1)}d\underline{y} \,, \tag{3.94}$$

one finds that the reduction of $\hat{\vec{H}}$ leads to

$$\left\{\begin{array}{lcl} \hat{\vec{H}} & = & \vec{F}_{(3)} \;-\; K^{\frac{3}{4}}\vec{F}_{(2)}dy \,, \\ \\ \vec{F}_{(2)} & = & d\vec{A}_{(1)} \;-\; \mathcal{E}\vec{A}_{(2)} \,, \\ \\ \vec{F}_{(3)} & = & d\vec{A}_{(2)} \;+\; A_{(1)}\vec{F}_{(2)} \,. \end{array}\right. \tag{3.95}$$

This then allows us to reduce the $\hat{\vec{H}}$ term in the action as

$$\int_{10}\hat{\vec{H}}^T\mathcal{M}^{-1}\,{}^\star\hat{\vec{H}} \;=\; \int_9\left[K^{-\frac{3}{4}}\vec{F}_{(3)}^T\mathcal{M}^{-1}\,{}^\star\vec{F}_{(3)} - K^{\frac{3}{4}}\vec{F}_{(2)}^T\mathcal{M}^{-1}\,{}^\star\vec{F}_{(2)}\right] \,. \tag{3.96}$$

Doing the same thing on the 5-form field strength, we find that

$$\int_{10}\hat{F}_{(5)}{}^\star\hat{F}_{(5)} \;=\; \int_9\left[K^{-\frac{3}{4}}F_{(5)}{}^\star F_{(5)} - K^{\frac{3}{4}}F_{(4)}{}^\star F_{(4)}\right] \,, \tag{3.97}$$

where we have used

$$\begin{cases} \hat{D} \;=\; A_{(4)} \;-\; A_{(3)} d\underline{y} \;, \\ F_{(4)} \;=\; dA_{(3)} \;+\; \tfrac{1}{2}\vec{A}^{\,T}_{(1)}\eta\vec{F}_{(3)} \;-\; \tfrac{1}{2}\vec{A}^{\,T}_{(2)}\eta\vec{F}_{(2)} \;+\; \tfrac{1}{2}A_{(1)}\vec{A}^{\,T}_{(1)}\eta\vec{F}_{(2)} \;, \\ F_{(5)} \;=\; dA_{(4)} \;+\; A_{(1)}F_{(4)} \;+\; \tfrac{1}{2}A_{(1)}\vec{A}^{\,T}_{(2)}\eta\vec{F}_{(2)} \;-\; \tfrac{1}{2}\vec{A}^{\,T}_{(2)}\eta\vec{F}_{(3)} \,. \end{cases} \tag{3.98}$$

Now, reducing the CS-term and dualizing the $d=9$ 5-form field strength we end up with the following contribution to the $d=9$ action

$$\begin{aligned} S_{(4)} \;=\; & \int_9 \Big\{ -\tfrac{1}{2}K^{\frac{3}{4}}F_{(4)}{}^{\star}F_{(4)} - \tfrac{1}{2}F_{(4)}F_{(4)}A_{(1)} + \tfrac{1}{2}F_{(4)}\vec{A}^{\,T}_{(2)}\eta\left(\vec{F}_{(3)} - A_{(1)}\vec{F}_{(2)}\right) \\ & + \; \tfrac{1}{8}\left[\vec{A}^{\,T}_{(2)}\eta\vec{F}_{(2)} - A^{T}_{(1)}\eta\left(\vec{F}_{(3)} - A_{(1)}\vec{F}_{(2)}\right)\right]\vec{A}^{\,T}_{(2)}\eta\left(\vec{F}_{(3)} - A_{(1)}\vec{F}_{(2)}\right)\Big\} \;. \end{aligned} \tag{3.99}$$

Comparing the above results with the results in Eq. (3.75) one can see that both ways of reducing lead to the same thing.

**Derivation of the Massive Transformations**

Before the gauging, in $d=10$, we have the invariance

$$\delta\hat{\vec{B}} \;=\; d\hat{\vec{\mathcal{N}}} \,, \tag{3.100}$$

and we want to find the effect of these transformations after the gauging and the reduction: These will turn out to be related to *some of* the massive transformations.

When gauging the action, we have to covariantize the corresponding transformations. Since the $SL(2,\mathbb{R})$ acts on the $\hat{\vec{B}}$ fields, it is only natural to introduce the covariantized transformation rules

$$\delta\hat{\vec{B}} \;=\; d\hat{\vec{\mathcal{N}}} \;\rightarrow\; \delta\hat{\vec{B}} \;=\; \mathcal{D}\hat{\vec{\mathcal{N}}} \;=\; d\hat{\vec{\mathcal{N}}} \;+\; \hat{\mathcal{E}}\wedge\hat{\vec{\mathcal{N}}} \,, \tag{3.101}$$

under which the field strength for the $\hat{\vec{B}}$ field transforms as

$$\delta\hat{\vec{H}} \;=\; F(\hat{\mathcal{E}})\wedge\hat{\vec{\mathcal{N}}} \,, \tag{3.102}$$

where we have defined $F(\hat{\mathcal{E}}) = d\hat{\mathcal{E}} + \hat{\mathcal{E}}\wedge\hat{\mathcal{E}}$. This looks worse than it actually is: Since we take the gauge field to be constant and in one direction only, the field strength for the gauge field $\hat{\mathcal{E}}$ is identically zero, rendering the variation for $\hat{\vec{H}}$ nil.

Splitting the $\hat{\vec{B}}$ fields then as before, and defining

$$\hat{\vec{\mathcal{N}}} \;=\; \vec{\Sigma}_{(1)} \;-\; \vec{\Sigma}_{(0)} d\underline{y} \,, \tag{3.103}$$

one finds the following massive transformations

$$\begin{cases} \delta\vec{A}_{(2)} \;=\; d\vec{\Sigma}_{(1)} \,, \\ \delta\vec{A}_{(1)} \;=\; d\vec{\Sigma}_{(0)} \;+\; \mathcal{E}\vec{\Sigma}_{(1)} \,. \end{cases} \tag{3.104}$$

One can then see that the field strengths for the $d=9$ fields $\vec{A}_{(2)}$ and $\vec{A}_{(1)}$ are indeed invariant under these transformations, and are $SL(2,\mathbb{R})$ invariant.

Under the $d=10$ transformation $\delta\hat{\vec{B}} = d\hat{\vec{\mathcal{N}}}$ one finds that

$$\delta\hat{F}_{(5)} \;=\; d\delta\hat{D} - \tfrac{1}{2}(\delta\hat{\vec{B}})^T\eta\hat{\vec{H}}\;, \tag{3.105}$$

because $\hat{\vec{H}}$ is invariant. Now, using the facts

$$d\hat{\mathcal{E}} = 0\;,\; \hat{\mathcal{E}}\wedge\hat{\mathcal{E}} = 0\;,\; (\hat{\mathcal{E}}\wedge\hat{\vec{\mathcal{N}}})^T = -\hat{\vec{\mathcal{N}}}^T\wedge\hat{\mathcal{E}}^{\;T}\;,\; \hat{\mathcal{E}}^{\;T}\eta \;=\; -\eta\hat{\mathcal{E}}\;, \tag{3.106}$$

one finds that the variation reads

$$\delta\hat{F}_{(5)} \;=\; d\delta\hat{D} - \tfrac{1}{2}d\left(\hat{\vec{\mathcal{N}}}^{\;T}\eta\hat{\vec{H}}\right)\;. \tag{3.107}$$

This then means that iff

$$\delta\hat{D} \;=\; d\hat{\Delta}^{(3)} + \tfrac{1}{2}\hat{\vec{\mathcal{N}}}^{\;T}\eta\hat{\vec{H}}\;, \tag{3.108}$$

the 5-form field strength is invariant.

Dimensional reduction of the above transformation rule, leads to the variation rule for the 3-form, i.e.

$$\delta A_{(3)} \;=\; d\Delta^{(2)} + \tfrac{1}{2}\vec{\Sigma}_{(1)}^{\;T}\eta\vec{F}_{(2)} - \tfrac{1}{2}\vec{\Sigma}_{(0)}{}^{T}\eta\left[\vec{F}_{(3)} - A_{(1)}\vec{F}_{(2)}\right]\;. \tag{3.109}$$

Clearly, these transformations correspond to the non-$\chi$ transformations found in the preceding subsection.

## 3.3. The Eleven-Dimensional Origin of IIAm

In this Section we construct an 11-dimensional action which, upon dimensional reduction (zero-mode compactification) over a 2-torus gives the massive 9-dimensional type II supergravity action Eq. (3.75). In Section 3.2.2 it was important for us to keep $SL(2,\mathbb{R})$-covariance throughout the dimensional reduction and as a result we got a general action which describes a 3-parameter family of massive 9-dimensional type II supergravities. The three mass parameters transform in the adjoint representation of $SL(2,\mathbb{R})$ and thus, an $SL(2,\mathbb{R})$ transformation takes us from one member of the family (a supergravity theory) to another one.

Thus, in order to make contact with that result from an 11-dimensional (that is, from a type IIA/M-theoretical) starting point, it is important to have full control over the $SL(2,\mathbb{R})\subset GL(2,\mathbb{R})$ symmetry that arises in the dimensional reduction in two dimensions. This symmetry in the type IIA side exactly corresponds to the S duality of the type IIB side [15, 16, 18, 5]. Thus, we will first reduce standard 11-dimensional supergravity making this symmetry manifest.

### 3.3.1. Compactification of 11-Dimensional Supergravity on $T^2$ and $Sl(2,\mathbb{R})$ Symmetry

The bosonic fields of $N=1, d=11$ supergravity [32] are the Elfbein and a 3-form potential

$$\left\{\hat{\hat{e}}_{\hat{\hat{\mu}}}{}^{\hat{\hat{a}}}, \hat{\hat{C}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}\hat{\hat{\rho}}}\right\}\;. \tag{3.110}$$

The field strength of the 3-form is

$$\hat{\hat{G}} = 4\partial\hat{\hat{C}}\,, \tag{3.111}$$

and is obviously invariant under the gauge transformations

$$\delta\hat{\hat{C}} = 3\partial\hat{\hat{\chi}}\,, \tag{3.112}$$

where $\hat{\hat{\chi}}$ is a 2-form. The action for these bosonic fields is

$$\hat{\hat{S}} = \int d^{11}x\,\sqrt{|\hat{\hat{g}}|}\;\left[\hat{\hat{R}} - \tfrac{1}{2\cdot 4!}\hat{\hat{G}}^2 - \tfrac{1}{6^4}\frac{1}{\sqrt{|\hat{\hat{g}}|}}\hat{\hat{\epsilon}}\partial\hat{\hat{C}}\partial\hat{\hat{C}}\hat{\hat{C}}\right]\,. \tag{3.113}$$

We have 2 mutually commuting Killing vectors $\{\hat{\hat{k}}_{(m)}{}^{\hat{\hat{\mu}}}\}$ and use coordinates adapted to both of them: $\{\hat{\hat{x}}^{\hat{\hat{\mu}}}\} = \{x^\mu, x^m\}$ with $m = 9, 10$ and $x^9 = x, x^{10} = z$ and

$$\hat{\hat{k}}_{(m)}{}^{\hat{\hat{\mu}}}\frac{\partial}{\partial\hat{\hat{x}}^{\hat{\hat{\mu}}}} = \frac{\partial}{\partial x^m}\,. \tag{3.114}$$

In these coordinates

$$\hat{\hat{k}}_{(m)}{}^{\hat{\hat{\mu}}}\hat{\hat{k}}_{(n)}{}^{\hat{\hat{\nu}}}\hat{\hat{g}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}} = \hat{\hat{g}}_{mn}\,. \tag{3.115}$$

This is the internal space metric and it is in general non-diagonal, so the Killing vectors are not mutually orthogonal in general.

The standard KK Ansatz is[5]

$$\left(\hat{\hat{e}}_{\hat{\hat{\mu}}}{}^{\hat{\hat{a}}}\right) = \begin{pmatrix} e_\mu{}^a & e_m{}^i A^{(m)}{}_\mu \\ & \\ 0 & e_m{}^i \end{pmatrix}\,, \qquad \left(\hat{\hat{e}}_{\hat{\hat{a}}}{}^{\hat{\hat{\mu}}}\right) = \begin{pmatrix} e_a{}^\mu & -A^{(m)}{}_a \\ & \\ 0 & e_i{}^m \end{pmatrix}\,, \tag{3.116}$$

where $A^{(m)}{}_a = e_a{}^\mu A^{(m)}{}_\mu$. For the metric, this means the following decomposition in 9-dimensional fields:

$$\left\{\begin{array}{rcl} \hat{\hat{g}}_{\mu\nu} & = & g_{\mu\nu} + G_{mn}A^{(m)}{}_\mu A^{(n)}{}_\nu\,, \\ && \\ \hat{\hat{g}}_{\mu m} & = & G_{mn}A^{(n)}{}_\mu = \hat{\hat{k}}_{(m)\;\mu}\,, \\ && \\ \hat{\hat{g}}_{mn} & = & G_{mn} = \hat{\hat{k}}_{(m)}{}^{\hat{\hat{\mu}}}\hat{\hat{k}}_{(n)\;\hat{\hat{\mu}}}\,. \end{array}\right. \tag{3.117}$$

The inverse relations are given in Appendix B.1.

From now on we will write the internal metric in matrix form and the two KK vectors in a column vector form:

$$G \equiv \begin{pmatrix} G_{\underline{xx}} & G_{\underline{xz}} \\ G_{\underline{zx}} & G_{\underline{zz}} \end{pmatrix}\,, \qquad \vec{A}_\mu \equiv \begin{pmatrix} A^{(\underline{x})}{}_\mu \\ \\ A^{(\underline{z})}{}_\mu \end{pmatrix}\,. \tag{3.118}$$

Under global transformations in the internal space

$$x^{m\,\prime} = \left(R^{-1\;T}\right)^m{}_n\,x^n + a^m\,, \qquad R \in GL(2,\mathbb{R})\,, \tag{3.119}$$

objects with internal space indices transform as follows:

[5]This is not exactly the standard KK Ansatz, which includes a rescaling of the lower-dimensional metric to end up in the Einstein conformal frame. We will perform the rescaling as a second step for pedagogical reasons.

$$G' = RGR^T\,, \qquad \vec{A}'_\mu = (R^{-1})^T\vec{A}_\mu\,. \tag{3.120}$$

We know that $GL(2,\mathbb{R})$ can be decomposed in $SL(2,\mathbb{R})\times\mathbb{R}^+\times\mathbb{Z}_2$ and any matrix $R$ can therefore be decomposed into

$$R = a\Lambda(\sigma^1)^\alpha\,, \quad \Lambda\in SL(2,\mathbb{R})\,, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \alpha = 0,1\,, \quad a\in\mathbb{R}^+\,. \tag{3.121}$$

The effect of a $\mathbb{Z}_2$ transformation $\sigma^1$ is the relabeling of the two internal coordinates and we will ignore it. Thus, we will focus on $GL(2,\mathbb{R})/\mathbb{Z}_2 \sim SL(2,\mathbb{R})\times\mathbb{R}^+$. We want to separate fields that transform under the different factors. First we define the symmetric $SL(2,\mathbb{R})$ matrix[6]

$$\mathcal{M} = -G/|\det\, G|^{1/2}\,, \tag{3.122}$$

and the scalar

$$K = |\det\, G|^{1/2}\,. \tag{3.123}$$

Now, under $SL(2,\mathbb{R})$ only $\mathcal{M}$ and $\vec{A}_\mu$ transform:

$$\mathcal{M}' = \Lambda\mathcal{M}\Lambda^T\,, \quad \vec{A}'_\mu = (\Lambda^{-1})^T\vec{A}_\mu\,, \tag{3.124}$$

that is, $\vec{A}_\mu$ transforms contravariantly, while under $\mathbb{R}^+$ rescalings only $K$ and $\vec{A}_\mu$ transform:

$$K' = aK\,, \quad \vec{A}'_\mu = a\vec{A}_\mu\,. \tag{3.125}$$

It is convenient for our purposes to use a slightly different set of vector fields $\vec{A}_{(1)\ \mu}$ transforming covariantly under $SL(2,\mathbb{R})$, defined as follows:

$$\vec{A}_{(1)\ \mu} = \eta\vec{A}_\mu\,, \quad \vec{F}_{(2)\ \mu\nu} = 2\partial_{[\mu}\vec{A}_{(1)\ \nu]}\,, \quad \vec{A}'_{(1)\ \mu} = a\Lambda\vec{A}_{(1)\ \mu}\,. \tag{3.126}$$

Using the standard techniques, the above Elfbein Ansatz and rescaling the resulting 9-dimensional metric to the Einstein frame

$$g_{\mu\nu} = K^{-2/7}g_{E\ \mu\nu}\,, \tag{3.127}$$

one finds

$$\begin{aligned} \int d^{11}\hat{x}\,\sqrt{|\hat{\hat{g}}|}\,\left[\,\hat{\hat{R}}\,\right] \;=\; & \int d^9x\,\sqrt{|g_E|}\,\Big[R_E + \tfrac{9}{14}\left(\partial\log K\right)^2 \\ & +\tfrac{1}{4}\mathrm{Tr}\left(\partial\mathcal{M}\mathcal{M}^{-1}\right)^2 - \tfrac{1}{4}K^{\frac{9}{7}}\vec{F}^{\,T}_{(2)}\mathcal{M}^{-1}\vec{F}_{(2)}\Big]\,. \end{aligned} \tag{3.128}$$

The 3-form term can be reduced along the same lines and we decompose the 11-dimensional 3-form potential into the 9-dimensional fields $A_{(3)\ \mu\nu\rho}, \vec{A}_{(2)\ \mu\nu}$ and $A_{(1)\ \mu}$ as follows:

[6]The minus sign is due to our mostly minus signature which makes the internal metric negative definite. We want $\mathcal{M}$ to be positive definite.

$$
\left\{
\begin{array}{rcl}
\hat{\hat{C}}_{\mu\nu\rho} & = & A_{(3)\ \mu\nu\rho} + \frac{3}{2}\vec{A}^{\,T}_{(1)\ [\mu}\eta\vec{A}_{(2)\ \nu\rho]} + 3A_{(1)\ [\mu}\vec{A}^{\,T}_{(1)\ \nu}\eta\vec{A}_{(1)\rho]}\,, \\
\left(\begin{array}{c} \hat{\hat{C}}_{\mu\nu\underline{x}} \\ \hat{\hat{C}}_{\mu\nu\underline{z}} \end{array}\right) & = & \vec{A}_{(2)\ \mu\nu} - 2A_{(1)\ [\mu}\vec{A}_{(1)\ \nu]}\,, \\
\left(\begin{array}{cc} 0 & \hat{\hat{C}}_{\mu\underline{xz}} \\ \hat{\hat{C}}_{\mu\underline{zx}} & 0 \end{array}\right) & = & +\eta A_{(1)\ \mu}\,,
\end{array}
\right.
\tag{3.129}
$$

The corresponding 9-dimensional field strengths $F_{(4)}, \vec{F}_{(3)}$ and $F_{(2)}$ are defined exactly by the massless limit of Eq. (3.68). The relation with the 11-dimensional field strength $\hat{\hat{G}}$ is

$$
\left\{
\begin{array}{rcl}
\hat{\hat{G}}_{\mu\nu\rho\sigma} & = & F_{(4)\ \mu\nu\rho\sigma} - 4\vec{A}^{\,T}_{(1)\ [\mu}\eta\vec{F}_{(3)\ \nu\rho\sigma]} \\
 & & +5\vec{A}^{\,T}_{(1)\ [\mu}\eta\vec{A}_{(1)\ \nu}F_{(2)\ \rho\sigma]}\,, \\
\left(\begin{array}{c} \hat{\hat{G}}_{\mu\nu\rho\underline{x}} \\ \hat{\hat{G}}_{\mu\nu\rho\underline{z}} \end{array}\right) & = & \vec{F}_{(3)\ \mu\nu\rho} - 3\vec{A}_{(1)\ [\mu}F_{(2)\ \nu\rho]}\,, \\
\left(\begin{array}{cc} 0 & \hat{\hat{G}}_{\mu\nu\underline{xz}} \\ \hat{\hat{G}}_{\mu\nu\underline{zx}} & 0 \end{array}\right) & = & \eta F_{(2)\ \mu\nu}\,.
\end{array}
\right.
\tag{3.130}
$$

This allows us to decompose the kinetic term as follows:

$$
\sqrt{|\hat{\hat{g}}|}\tfrac{-1}{2\cdot 4!}\hat{\hat{G}}^2 = \sqrt{|g_E|}\left\{\tfrac{-1}{2\cdot 4!}K^{6/7}F^2_{(4)} + \tfrac{1}{2\cdot 3!}K^{-3/7}\vec{F}^{\,T}_{(3)}\mathcal{M}^{-1}\vec{F}_{(3)} - \tfrac{1}{4}K^{-12/7}F^2_{(2)}\right\}\,. \tag{3.131}
$$

and the topological term as follows:

$$
\begin{array}{rcl}
\frac{1}{(144)^2}\,\hat{\hat{\epsilon}}\hat{\hat{G}}\hat{\hat{G}}\hat{\hat{C}} & = & \frac{1}{3^2\cdot 2^8}\epsilon\epsilon^{mn}\left\{\hat{\hat{G}}\hat{\hat{G}}\hat{\hat{C}}_{mn} + 4\hat{\hat{G}}\hat{\hat{G}}_m\hat{\hat{C}}_n\right\} \\
 & = & \frac{1}{3^2\cdot 2^7}\epsilon\left[F_{(4)} - 4\vec{A}_{(1)}{}^T\eta\vec{F}_{(3)} + 6\vec{A}_{(1)}{}^T\eta\vec{A}_{(1)}F_{(2)}\right]\times \\
 & & \times\left\{\left[F_{(4)} - 4\vec{A}_{(1)}{}^T\eta\vec{F}_{(3)} + 6\vec{A}_{(1)}{}^T\eta\vec{A}_{(1)}F_{(2)}\right]A_{(1)}\right. \\
 & & \left.+2\left[\vec{F}_{(3)} - 3\vec{A}_{(1)}F_{(2)}\right]^T\eta\left[\vec{A}_{(2)} + 2\vec{A}_{(1)}A_{(1)}\right]\right\}\,.
\end{array}
\tag{3.132}
$$

Putting all our partial results together, Eqs. (3.128,3.131,3.132), we arrive at the action of type II 9-dimensional supergravity in Einstein frame, Eq. (3.75), which we obtained through generalized dimensional reduction of the 10-dimensional type IIB theory with the mass matrix set to zero [15].

The fact that upon dimensional reduction the type IIA and type IIB *supergravity theories* are identical in nine dimensions is nothing but the manifestation at the level of the massless modes of the T duality existing between the type IIA and type IIB *superstring theories* when they are compactified in circles of dual radii [34, 38].

There are four important points we would like to stress:

1. There is no "hidden symmetry" of the 9-dimensional type II theory corresponding to this T duality.

2. To obtain two *identical* actions it is crucial that the two topological terms come with the same global sign. In the M/type IIA side the sign can be changed by the 11-dimensional transformation $\hat{\hat{C}} \rightarrow -\hat{\hat{C}}$ which is not a symmetry. In the type IIB side, flipping the sign of the 4-form $\hat{D}$ does not work because it changes the definition of its field strength. Changing the signs of $\hat{D}$ and, say, $\hat{B}^{(1)}$ leaves $\hat{F}$ invariant but also leaves invariant the topological term. Thus, at first sight, there seems to be no IIB-side version of this rather trivial M/IIA-side transformation.

   It is, however, easy to see that the sign of the topological term in the NSD 10-dimensional type IIB action is directly related to the self-duality of the 5-form field strength. Had we considered an anti-self-dual 5-form the sign would have been exactly the opposite in ten and nine dimensions. The (anti-) self-duality of the 5-form is related to the chirality of the theory.

   The picture that emerges is therefore the following: There are two (otherwise equivalent) 11-dimensional supergravity theories and two 10-dimensional type IIA theories that differ only in the sign of the action's topological term. Upon dimensional reduction to nine dimensions they are related to the two type IIB theories of opposite chiralities.

   In the decompactification limit, each of these two 9-dimensional (and, thus, non-chiral) theories knows to which chiral 10-dimensional type IIB theory it should decompactify.

3. The above observation solves in part the puzzle found in Ref. [90] where it was argued that approximately half of all extreme black holes are not supersymmetric in type II theories. Clearly, those which are not supersymmetric in one of the 11-dimensional supergravities are supersymmetric in the 11-dimensional supergravity with the sign of the 3-form $\hat{\hat{C}}$ reversed. As suggested also in Ref. [76], the whole picture begs for both 11-dimensional supergravities to be integrated into a higher-dimensional supergravity from which also the type IIB would be derivable, perhaps one of those with the algebras studied in Ref. [9]. (This argument is completely different from the one in Ref. [10] and, in fact, it is in disagreement with it).

4. The fact that the two theories (A and B) are identical allows us to relate the 10-dimensional fields of the two type II theories. This relation provides a generalization of Buscher's T duality rules [30]. These type II Buscher rules were found in Ref. [15] and they are determined again in Appendices B.1, B.2 and B.3 in our (more systematic) conventions and extended to the massive case at hands.

### 3.3.2. $Sl(2,\mathbb{R})$-Covariant Massive 11-Dimensional Supergravity

So much for the massless case. Now, it is clear that the picture seems to break down whenever the mass matrix does not vanish. In Ref. [19] the particular case with mass matrix with $m^1 = 0, m^2 = m^3 = m$

$$m_{\rm BRGPT} = \left(\begin{array}{cc} 0 & m \\ 0 & 0 \end{array}\right)\,, \tag{3.133}$$

was considered. As will be discussed in Section 3.5 this particular choice of mass matrix corresponds to compactification of the type IIB on a background with different species of 7-branes. Since the T dual of a D-7-brane in a direction orthogonal to its worldvolume is a type IIA D-8-brane, one expects the theory with mass matrix $m_{\rm BRGPT}$ to correspond to the type IIA theory on a background with D-8-branes.

While it is not possible to write the 10-dimensional type IIB theory in presence of D-7-branes in a covariant fashion (there is dependence on the compactifying coordinate $y$) it is possible to write in a covariant fashion the action for the type IIA theory in presence of D-8-branes. As was first first realized in Ref. [94], this theory has long been known as Romans' massive type IIA supergravity [98]. The precise identification, leading to a further generalization of Buscher's rules was carried out in Ref. [19]. We stress that these T duality rules are essentially identical to the original type II T duality rules of Ref. [15] but are deformed in a $y$-dependent fashion in the type IIB side of the equations.

Our task in the remainder of this Section will be to generalize the results of Ref. [19]. It is clear from the setting that this generalization amounts to its $SL(2,\mathbb{R})$-covariantization: We start from the compactification of the type IIB theory on a background containing D-7-branes and their S duals and, after T duality, we expect to find a type IIA theory on a background of the T duals of D-7-branes and their S duals. We will not repeat here the discussion of the Introduction where we concluded that we must look for a non-covariant generalization of Romans' type IIA supergravity.

As a matter of fact, it is easier to generalize the 11-dimensional theory that gives Romans', given in Ref. [25]. From our point of view this theory would correspond to 11-dimensional supergravity with a KK-9M-brane in the background. To find in 9-dimensions an $SL(2,\mathbb{R})$-covariant result we must consider a theory describing 11-dimensional supergravity with two KK-9M-branes in the background.

In what follows we will construct such a theory along the same lines as Ref. [25] and show that it gives the massive 9-dimensional type II theory constructed in Section 3.2.2.

Since each KK-9M-brane is associated to a Killing vector we assume the presence of the two mutually commuting Killing vectors of the previous Section and also assume that the Lie derivatives of all fields with respect to both of them vanishes.

Next, we define the 11-dimensional massive transformations. For a general tensor, except for $\hat{\hat{C}}$ whose transformation law will be defined below, they are

$$\delta_{\hat{\chi}} L_{\hat{\mu}_1\cdots\hat{\mu}_r} = \hat{\hat{\lambda}}^{(n)}{}_{\hat{\mu}_1}\hat{\hat{k}}_{(n)}{}^{\hat{\nu}}\hat{\hat{L}}_{\hat{\nu}\hat{\mu}_2\cdots\hat{\mu}_r} + \ldots + \hat{\hat{\lambda}}^{(n)}{}_{\hat{\mu}_r}\hat{\hat{k}}_{(n)}{}^{\hat{\nu}}\hat{\hat{L}}_{\hat{\mu}_1\cdots\hat{\mu}_{r-1}\hat{\nu}}\,, \tag{3.134}$$

where we have defined

$$\hat{\hat{\lambda}}^{(n)} \equiv -i_{\hat{\hat{k}}_{(m)}}\hat{\chi}Q^{nm}\,, \qquad Q^{nm} = \left(m^T\eta\right)^{mn} = \tfrac{1}{2}\left(\begin{array}{cc} -(m^2+m^3) & m^1 \\ & \\ m^1 & m^2-m^3 \end{array}\right)\,. \tag{3.135}$$

The contraction of a space tensor with the Killing vectors will bear an $SL(2,\mathbb{R})$ index: The extension of the above rule for incorporating $SL(2,\mathbb{R})$ indices is found by defining the inclusion to commute with the massive transformations.

In particular we find that the 11-dimensional metric and $r$-forms $\hat{\hat{S}}$ transform as

$$\left\{\begin{array}{rcl}\delta_{\hat{\hat{\chi}}}\hat{\hat{g}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}} & = & 2\hat{\hat{\lambda}}^{(n)}{}_{(\hat{\hat{\mu}}}\,\hat{\hat{k}}_{(n)}{}^{\hat{\hat{\rho}}}\hat{\hat{g}}_{\hat{\hat{\nu}})\hat{\hat{\rho}}}\ , \\ \delta_{\hat{\hat{\chi}}}\hat{\hat{S}}_{\hat{\hat{\mu}}_1\ldots\hat{\hat{\mu}}_r} & = & (-)^{r-1}r\hat{\hat{\lambda}}^{(n)}{}_{[\hat{\hat{\mu}}_1}\,\hat{\hat{k}}_{(n)}{}^{\hat{\hat{\rho}}}\hat{\hat{S}}_{\hat{\hat{\mu}}_2\ldots\hat{\hat{\mu}}_r]\hat{\hat{\rho}}}\ .\end{array}\right. \tag{3.136}$$

Observe that these rules imply that

$$\left\{\begin{array}{rcl}\delta_{\hat{\hat{\chi}}}\sqrt{|\hat{\hat{g}}|} & = & 0\ , \\ \delta_{\hat{\hat{\chi}}}\hat{\hat{S}}^2 & = & 0\ ,\end{array}\right. \tag{3.137}$$

where the latter holds due to the fact that also the metric varies under the massive transformations, and the former holds due to the fact that the matrix $Q = m^T\eta$ is symmetric.

The 3-form field $\hat{\hat{C}}$ is going to play the role of a connection-field with respect to the massive transformations and, as such, does not transform covariantly

$$\delta_{\hat{\hat{\chi}}}\hat{\hat{C}} \ = \ d\hat{\hat{\chi}} \ + \ \hat{\hat{\lambda}}^{(n)} \wedge \left(i_{\hat{\hat{k}}_{(n)}}\hat{\hat{C}}\right)\ . \tag{3.138}$$

The generalization of the field strength for $\hat{\hat{C}}$, denoted as before by $\hat{\hat{G}}$, is then found by requiring that the field strength does transform covariantly. One can see that this implies that

$$\hat{\hat{G}} \ = \ d\hat{\hat{C}} \ - \ \tfrac{1}{2}\left(i_{\hat{\hat{k}}_{(n)}}\hat{\hat{C}}\right) Q^{nm}\left(i_{\hat{\hat{k}}_{(n)}}\hat{\hat{C}}\right)\ . \tag{3.139}$$

Comparing this with a torsionful covariant derivative acting on a 3-form, one sees that the above equation states that the massive transformations induce a torsion term in our spacetime connection. This then means that if we want our $d = 11$ theory to be invariant under the massive transformations, we have to define our theory in terms of the torsionful connection.

The torsion we need is given by

$$\hat{\hat{T}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}}{}^{\hat{\hat{\rho}}} = -\left(i_{\hat{\hat{k}}_{(n)}}\hat{\hat{C}}\right)_{\hat{\hat{\mu}}\hat{\hat{\nu}}} Q^{nm}\hat{\hat{k}}_{(m)}{}^{\hat{\hat{\rho}}}\,. \tag{3.140}$$

The torsionful connection $\hat{\hat{\Omega}}$ is then defined in the standard way, by adding the so-called contorsion-torsion tensor,

$$\hat{\hat{K}}_{\hat{\hat{a}}\hat{\hat{b}}\hat{\hat{c}}} \ = \ \tfrac{1}{2}\left(\hat{\hat{T}}_{\hat{\hat{a}}\hat{\hat{c}}\hat{\hat{b}}} \ + \ \hat{\hat{T}}_{\hat{\hat{b}}\hat{\hat{c}}\hat{\hat{a}}} \ - \ \hat{\hat{T}}_{\hat{\hat{a}}\hat{\hat{b}}\hat{\hat{c}}}\right)\ , \tag{3.141}$$

to the Levi-Cività connection $\hat{\hat{\omega}}$, i.e.

$$\hat{\hat{\Omega}}_{\hat{\hat{a}}}{}^{\hat{\hat{b}}\hat{\hat{c}}} \ = \ \hat{\hat{\omega}}_{\hat{\hat{a}}}{}^{\hat{\hat{b}}\hat{\hat{c}}} \ + \ \hat{\hat{K}}_{\hat{\hat{a}}}{}^{\hat{\hat{b}}\hat{\hat{c}}}\ . \tag{3.142}$$

From the above equation we can obtain the non-vanishing components of the torsion written directly in 9-dimensional Lorentz coordinates for future use

$$\left\{\begin{array}{rcl}\hat{\hat{T}}_{abi} & = & -A_{(2)(n)ab}\eta^{np}m_p{}^q e_{qi}\ , \\ \hat{\hat{T}}_{aij} & = & A_{(1)a}e_i{}^p m_p{}^q e_{qi}\ .\end{array}\right. \tag{3.143}$$

Having all this, one can see that the 11-dimensional theory invariant under the massive transformation reads[7]

$$
\begin{aligned}
\hat{\hat{S}} \;=\; & \int d^{11}\hat{\hat{x}}\sqrt{|\hat{\hat{g}}|}\,\Big\{\hat{\hat{R}}\left(\hat{\hat{\Omega}}\right)+\left(d\hat{\hat{k}}_{(n)}\right)_{\hat{\hat{\mu}}\hat{\hat{\nu}}}Q^{nm}\left(i_{\hat{\hat{k}}_{(m)}}\hat{\hat{C}}\right)^{\hat{\hat{\mu}}\hat{\hat{\nu}}}-\tfrac{1}{2\cdot 4!}\hat{\hat{G}}^{2} \\
& -2\hat{\hat{K}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}\hat{\hat{\rho}}}\hat{\hat{K}}^{\hat{\hat{\nu}}\hat{\hat{\rho}}\hat{\hat{\mu}}}+\tfrac{1}{2}\left(\hat{\hat{k}}_{(n)\;\hat{\hat{\mu}}}Q^{nm}\hat{\hat{k}}_{(m)}{}^{\hat{\hat{\mu}}}\right)^{2}-\left(\hat{\hat{k}}_{(n)\;\hat{\hat{\mu}}}Q^{nm}\hat{\hat{k}}_{(m)\;\hat{\hat{\nu}}}\right)^{2} \\
& -\tfrac{1}{6^4}\frac{\hat{\hat{\epsilon}}}{\sqrt{|\hat{\hat{g}}|}}\Big\{\partial\hat{\hat{C}}\partial\hat{\hat{C}}\hat{\hat{C}}-\tfrac{9}{8}\partial\hat{\hat{C}}\hat{\hat{C}}\left(i_{\hat{\hat{k}}_{(n)}}\hat{\hat{C}}\right)Q^{nm}\left(i_{\hat{\hat{k}}_{(m)}}\hat{\hat{C}}\right) \\
& +\tfrac{27}{80}\hat{\hat{C}}\left[\left(i_{\hat{\hat{k}}_{(n)}}\hat{\hat{C}}\right)Q^{nm}\left(i_{\hat{\hat{k}}_{(m)}}\hat{\hat{C}}\right)\right]^{2}\Big\}\Big\}\;,
\end{aligned}
\tag{3.144}
$$

For the dimensional reduction of the above theory, the fields will be split in the same way as in the preceding subsection; The only thing that changes, is the torsion part of the connection and some terms in the 11-dimensional Chern-Simons term.

Let us first consider the reduction of the curvature term, evaluated using the connection in Eq. (3.142). Using Palatini's identity for torsionful connections

$$
\begin{aligned}
\int_d \sqrt{|g|}\,e^{-2\phi}\,R(\Omega) \;=\; & -\int_d \sqrt{|g|}\,e^{-2\phi}\Big\{\Omega_b{}^{ba}\Omega_c{}^c{}_a \;+\; \Omega_a{}^{bc}\Omega_{bc}{}^a \;+\; 4\Omega_b{}^{ba}\partial_a\phi \\
& -2\Omega_b{}^{ba}K_c{}^c{}_a \;-\; 2\Omega_a{}^{bc}K_{bc}{}^a\Big\}\;,
\end{aligned}
\tag{3.145}
$$

the facts

$$
\left\{
\begin{aligned}
\hat{\hat{K}}_{\hat{\hat{a}}}{}^{\hat{\hat{a}}b} \;&=\; A^b_{(1)}\hat{\hat{\eta}}^{ij}e_i{}^m e_{nj}m_m{}^n \;=\; A^b_{(1)}\mathrm{Tr}\,(m) \;=\; 0\;, \\
\hat{\hat{K}}_{\hat{\hat{a}}}{}^{\hat{\hat{a}}i} \;&=\; 0\;,
\end{aligned}
\right.
\tag{3.146}
$$

and the fact that the second term in Eq. (3.144) annihilates the $\Omega K$-terms whilst applying Palatini's identity to the case at hand, one can write

$$
\begin{aligned}
& \int d^{11}\hat{\hat{x}}\sqrt{|\hat{\hat{g}}|}\Big\{\hat{\hat{R}}\left(\hat{\hat{\Omega}}\right)+\left(d\hat{\hat{k}}_{(n)}\right)_{\hat{\hat{\mu}}\hat{\hat{\nu}}}Q^{nm}\left(i_{\hat{\hat{k}}_{(m)}}\hat{\hat{C}}\right)^{\hat{\hat{\mu}}\hat{\hat{\nu}}}-2\hat{\hat{K}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}\hat{\hat{\kappa}}}\hat{\hat{K}}_{\hat{\hat{\nu}}\hat{\hat{\kappa}}\hat{\hat{\mu}}}\Big\} \;=\; \\
& \quad = \int d^9x\sqrt{|g|}\;K\;\Big[R(g)-(\partial\log K)^2\;+\;\tfrac{1}{4}\left(F_{iab}+\hat{T}_{abi}\right)^2 \\
& \quad +\tfrac{1}{4}\left(e_i{}^n e_j{}^m\partial_a G_{nm}\;+\;2\hat{T}_{a(ij)}\right)^2\Big)\;.
\end{aligned}
\tag{3.147}
$$

Using now our previous partial results Eqs. (3.122,3.126,3.143) and rescaling to the Einstein frame, Eq. (3.127), this can be written as

[7] Note that the cosmological constant part is, apart from correspondence with the massive $d=9$ theory, arbitrary. However, supersymmetry should completely determine it.

$$= \int d^9x\sqrt{|g_E|}\ \left[R(g_E) + \tfrac{9}{14}(\partial\log K)^2 + \tfrac{1}{4}\mathrm{T}r\left(\mathcal{D}\mathcal{M}\mathcal{M}^{-1}\right)^2 - \tfrac{1}{4}K^{9/7}\vec{F}^{\,T}_{(2)}\mathcal{M}^{-1}\vec{F}_{(2)}\right]\ , \tag{3.148}$$

where the field strengths and covariant derivative are the same as the ones used in Section 3.2.2.

The cosmological constant part is readily reduced by using the well-known identity

$$\eta^{mn}\eta^{pq}\ =\ -\eta^{np}\eta^{mq}\ -\ \eta^{pm}\eta^{nq}\,, \tag{3.149}$$

and it follows that

$$\begin{aligned}\tfrac{1}{2}\int d^{11}\hat{\hat{x}}\sqrt{|\hat{\hat{g}}|}\ \left[\left(\hat{\hat{k}}_{(n)\ \hat{\hat{\mu}}}Q^{nm}\hat{\hat{k}}_{(m)}{}^{\hat{\hat{\mu}}}\right)^2 - \left(\hat{\hat{k}}_{(n)\ \hat{\hat{\mu}}}Q^{nm}\hat{\hat{k}}_{(m)\ \hat{\hat{\nu}}}\right)^2\right]\ &= \\ = -\tfrac{1}{2}\int d^9x\sqrt{|g_E|}\ K^{12/7}\mathrm{Tr}\left(m^2\ +\ \mathcal{M}m\mathcal{M}^{-1}m^T\right)\ &,\end{aligned} \tag{3.150}$$

which is just the result obtained in the $d=9$ theory.

The effect of the torsion included in definition (3.139), can readily be seen to promote the field strengths to their massive equivalents Eq. (3.68). As such, it will be no surprise at all to see that

$$\begin{aligned}\int_{11} -\tfrac{1}{2}\hat{\hat{G}}{}^\star\hat{\hat{G}}\ \ &=\ \ \int_9\Big\{-\tfrac{1}{2}K^{6/7}F_{(4)}{}^\star F_{(4)}\ +\ \tfrac{1}{2}K^{-3/7}\vec{F}^{\,T}_{(3)}\mathcal{M}^{-1\star}\vec{F}_{(3)} \\ &\qquad -\tfrac{1}{2}K^{-12/7}F_{(2)}{}^\star F_{(2)}\Big\}\ .\end{aligned} \tag{3.151}$$

From the fact that we do not change the decomposition of the fields while doing the reduction, it is clear that the $d\hat{\hat{C}}d\hat{\hat{C}}\hat{\hat{C}}$ will lead to the same result as in Eq. (3.132). The other terms can easily be seen to result in

$$\begin{aligned}\tfrac{1}{6^4}\int_{11}\hat{\hat{\epsilon}}\,\tfrac{9}{8}\partial\hat{\hat{C}}\hat{\hat{C}}\left[\left(i_{\hat{\hat{k}}_{(n)}}\hat{\hat{C}}\right)\,Q^{nm}\,\left(i_{\hat{\hat{k}}_{(m)}}\hat{\hat{C}}\right)\right]\ \ &=\ \ \tfrac{1}{3^2 2^7}\int_9\epsilon\,\Big[6F_{(4)}\,\vec{A}^T_{(2)}Q\vec{A}_{(2)}\,A_{(1)} \\ &\qquad -9\left(\vec{A}^T_{(2)}Q\vec{A}_{(2)}\right)\left(\vec{A}^T_{(2)}\eta\partial\vec{A}_{(2)}\right)\Big]\ , \\ \tfrac{1}{6^4}\int_{11}\hat{\hat{\epsilon}}\,\tfrac{27}{80}\hat{\hat{C}}\left[\left(i_{\hat{\hat{k}}_{(n)}}\hat{\hat{C}}\right)\,Q^{nm}\,\left(i_{k_{(m)}}\hat{\hat{C}}\right)\right]^2\ \ &=\ \ \tfrac{1}{3^2 2^7}\int_9\epsilon\,9\left(\vec{A}^T_{(2)}Q\vec{A}_{(2)}\right)^2A_{(1)}\ .\end{aligned} \tag{3.152}$$

Adding the above equations to Eq. (3.132) we find that the effect of the torsion is, once again, precisely to turn the massless CS term, into the massive CS term of the massive 9-dimensional type II theory we got by generalized dimensional reduction of the type IIB theory. Thus, we have achieved our second goal.

The T duality rules that one can immediately deduce from this relation between 10-dimensional theories will be worked out in the Appendices.

## 3.4. BPS Solutions and Intersections

The aim of this section is to introduce the solutions to the low-energy-effective actions, which are interpreted as the fundamental objects in string theory. The duality relations between these objects will be mentioned briefly, and are depicted in Fig. (B.1) for convenience.

After the basic objects have been introduced, a class of solutions, the so-called *intersections*, will be dealt with. There it will be shown that the intersections known to the literature, are based on a generic action, which is however unable to capture the essence of the massive IIA supergravity, in case of an intersection of a D8 with a string or a D6. Solutions, breaking susy to one fourth, describing the aforementioned intersections do exist and are dealt with in sections (3.4.2,3.4.3).

Thinking about the various extended objects, predicted by string theory, in the '*static*' gauge, one sees that they break Poincaré invariance, as does every solution. This then means that for a given solution a possible interpretation is closely related to the symmetry the solution presents. If we then also think about supersymmetry, and notice that solutions will definitely break translation invariance, so that one ought to expect a breaking of supersymmetry. This then leads to the question of how much supersymmetry is broken by a given string state, and whether we can find supergravity solutions with the appropriate properties.

The first solution one can think of is of course the string, in this ambiance called fundamental string. The solution, in the sigma frame, is given by[8]

$$\begin{aligned} ds^2 &= H^{-1}\left(dt^2 - dy^2_{(1)}\right) - d\vec{x}^2_{(8)} , && (3.153)\\ B_{ty} &= \pm\left\{H^{-1} - 1\right\} , && (3.154)\\ e^{-2\phi} &= H , && (3.155)\end{aligned}$$

where $H$ is a harmonic function of the $\vec{x}_{(8)}$, *i.e.* it satisfies $\vec{\partial}^2 H = 0$. This solution breaks half of the available supersymmetry, the necessary projector being

$$\mathcal{P}^{\mp}_{F1} = \tfrac{1}{2}\left[\mathbb{I} \mp \gamma^{01}\mathcal{O}\right] \quad \longrightarrow \quad \epsilon = H^{1/4}\mathcal{P}^{\pm}_{F1}\epsilon_0 , \tag{3.156}$$

where 0 (1) is the tangentspace coordinate associated to $t$ ($y$ *resp.*), $\epsilon_0$ is an arbitrary, constant, spinor and $\mathcal{O}$ is $\gamma_{11}$ ($\sigma_3$) for the type IIA (IIB *resp.*).

As one can see from Eqs. (1.107), applying T-duality in a coordinate transverse to the string, *i.e.* in one of the coordinates $\vec{x}_{(8)}$, leads once again to a string. Applying the T-duality rules in the $y_{(1)}$ direction however, one ends up with a pure gravitational solution, called the wave. The metric for wave is given by

$$ds^2_\sigma = (2-H)\,dt^2 \mp 2\,(1-H)\,dtdy - Hdy^2 - d\vec{x}^2_{(8)} , \tag{3.157}$$

and in this case $H$ is a function of $t \mp y$ and $\vec{x}_{(8)}$.

One can also find a 'brane-like' solution coupled magnetically to the Kalb-Ramond field, and is called the '*solitonic five-brane*', or the '*H monopole*'. The solution is given by

$$\begin{aligned} ds^2 &= dt^2 - d\vec{y}^2_{(5)} - H\,d\vec{x}^2_{(4)} ,\\ e^{-2\phi} &= H^{-1} ,\\ B^{(6)}_{ty_1\cdots y_5} &= \pm H^{-1} , && (3.158)\end{aligned}$$

[8]In all the solutions dealt with in this thesis, spacelike worldvolume coordinates will be called $y_{(p)}$, where $(p)$ denotes the number of worldvolume coordinates, and the transverse coordinates will be labeled $x_{(q)}$, where $(q)$ denotes the number of transverse coordinates. Furthermore, unless stated otherwise the functions $H$ will only depend on the transverse coordinates and is harmonic, *i.e.* $\vec{\partial}^2_{(q)}H = 0$. It is also assumed that the spinors that occur, are the ones appropriate for the theory we are considering, *e.g.* a doublet of Majorana-Weyl spinors for the IIB. Furthermore we abstain from giving the exact expressions for the covariant spinors, when this is not needed for the sequel.

where one should note that a possible constant term in $B^{(6)}$ can be gotten rid of, using a gauge transformation. This configuration also breaks half of the available supersymmetry and leads to the projector

$$\mathcal{P}^{\pm}_{S5} \;=\; \tfrac{1}{2}\left(\mathbb{I} \pm \gamma^{0\ldots 5}\mathcal{O}\right) \quad\longrightarrow\quad \epsilon \;=\; \mathcal{P}^{\mp}_{S5}\,\epsilon_0 \;, \tag{3.159}$$

where $\mathcal{O}$ is $\mathbb{I}$ ($\sigma_3$) for the type IIA (IIB *resp.*).

The solution for the D-branes, in the sigma-frame, reads

$$\begin{aligned} ds^2_p &= H^{-1/2}\left[dt^2 \,-\, H^{1/2}d\vec{y}^{\,2}_{(p)}\right] \,-\, d\vec{x}^{\,2}_{(9-p)} \;, && (3.160)\\ C^{(p+1)}_{t\,y_1\ldots y_p} &= H^{-1} \;, && (3.161)\\ e^{-2\phi} &= H^{\frac{p-3}{2}} \;, && (3.162) \end{aligned}$$

where $H$ is once again an harmonic function in the transverse coordinates. By applying the Buscher rules, Eq. (1.107), on the above solutions, one can see that doing T-duality in the transverse direction[9] one ends up with a $(p+1)$-brane solution, whereas applying it in a worldvolume direction one ends up with a $(p-1)$-brane solution. In this case also half of the supersymmetry is broken and the respective projectors read

$$\begin{aligned} \textbf{IIA} \;&:\; \mathcal{P}_{2n} \;=\; \tfrac{1}{2}\left(\mathbb{I} \mp i\gamma^{0\ldots 2n}\left(-\gamma_{11}\right)^{n+1}\right) \;, && (3.163)\\ \textbf{IIB} \;&:\; \mathcal{P}_{2n+1} \;=\; \tfrac{1}{2}\left(\mathbb{I} \pm \gamma^{0\ldots 2n+1}\Sigma_n\right) \;, && (3.164) \end{aligned}$$

where in the last equation

$$\Sigma_n \;=\; \left\{ \begin{array}{lcl} \sigma^1 & : & n \text{ even} \\ i\sigma^2 & : & n \text{ odd} \end{array}\right. \tag{3.165}$$

The covariant spinor then read $\epsilon = H^{-1/8}\epsilon_0$, where $\epsilon_0$ is a constant spinor which is annihilated by the appropriate projector in Eqs. (3.163,3.164).

### M branes

As by now must be obvious, the basic extended objects in '*M effective theory*' is the two- and the five-brane. By dimensional reduction these objects will transform into the fundamental string, the D2-, the D4-brane and the solitonic five brane. The solution for the '*M2*' is

$$\begin{aligned} ds^2_{11} &= H^{-2/3}\left\{dt^2 - d\vec{y}^{\,2}_{(2)}\right\} \,-\, H^{1/3}d\vec{x}^{\,2}_{(8)} \;, \\ C_{ty_1y_2} &= H^{-1} \;, \end{aligned} \tag{3.166}$$

whereas the one for the '*M5*' reads

$$\begin{aligned} ds^2_{11} &= H^{-1/3}\left[dt^2 - d\vec{y}^{\,2}_{(5)}\right] - H^{2/3}d\vec{x}^{\,2}_{(5)} \;, \\ G_{(4)\mu_1\ldots\mu_4} &= \pm\epsilon_{\mu_1\ldots\mu_4\mu_5}\partial_{\mu_5}H \;, \end{aligned} \tag{3.167}$$

where the $\mu$'s lie in the subspace spanned by the coordinates $\vec{x}_{(5)}$ and $\epsilon_{\ldots}$ is the Levi-Cività symbol on this (Euclidean) subspace.

[9] Note that this also means that one needs to impose an isometry in our solution, *i.e.* one needs to take $H$ to be independent of this direction.

There are more things to be found in 11 dimensions however. One of these things is the ever present *gravitational wave*. Since the wave is a pure gravitational solution, there is no problem in oxidising it to $D = 11$. The '*M-wave*' reads

$$ds^2 \;=\; (2-H)\,dt^2 \mp 2\,(1-H)\,dtdy - Hdy^2 - d\vec{x}^2_{(9)} \;, \tag{3.168}$$

and in this case $H$ is a function of $t \mp y$ and $\vec{x}_{(9)}$.

There is also something as a Kaluza-Klein monopole in $D = 11$, called '*KK7*',[10] and the metric reads

$$ds^2 \;=\; dt^2 \;-\; dy_{(6)} \;-\; H^{-1}\,(dz-A)^2 \;-\; Hdx^2_{(3)} \;, \tag{3.169}$$

where $H$ is a function of the $\vec{x}_{(3)}$ only and $A$ is a one-form depending and lying only in the $\vec{x}_{(3)}$ directions. Furthermore, $A$ must satisfy

$$\partial_m A_n \;-\; \partial_n A_m \;=\; \epsilon_{mnp}\partial_p H \;. \tag{3.170}$$

Note that when one dimensionally reduce this solution over the coordinate $z$, one finds the D6, and that by dimensionally reducing over another direction one finds solutions of the same form, and are correspondingly denoted by '*KK6A*' or '*KK7A*'. Needless to say, the wave and the KK's exists in a multitude of dimensions.

Up to now we have not been able to obtain the D8-brane from 11 dimensions. There is a proposed 'M9' brane, which is supposed to do the job. Having a look at the D8 brane, we can oxidize it to $D = 11$ where it takes the form

$$ds^2 \;=\; H^{1/3}\left[dt^2 - d\vec{y}^2_{(8)}\right] \;-\; H^{-5/4}dz^2 \;-\; H^{4/3}dx^2_{(1)} \;, \tag{3.171}$$

where $H$ is harmonic and only depends on $x_{(1)}$. Using then the above nomenclature, it is also denoted '*KK9*'.

All of the solutions presented, and solutions resembling the ones, presented in this section, are connected by the various dualities. Figure (B.1) depicts the duality- and reduction relations between the solutions.

### 3.4.1. Standard Intersections

An intersection is a solution to a sugra, which can be interpreted as consisting of several extended objects and is BPS. A *conditio sine qua non* for this to occur is, of course, that the various susy projectors commute, so that there is any residual supersymmetry that will ensure stability. There exists a vast literature on the subject from which we will take the most important results.

The first classification one can make is just to determine when the projectors, Eqs. (3.156,3.163,3.164) commute. Due to the great homogeneity of the D-brane projectors, the classification of D-branes intersection is straightforward [68]. Denoting a general intersection of two D-branes as $(q|D(q+r), D(q+s))$, which means that we are dealing with a $D(q+r)$- and a $D(q+s)$-brane, which have $q$ world-volume dimensions in common, one can see that supersymmetry implies that $r + s \;=\; 0 \; mod\, 4$.[11] By making use of the various duality relations between the solutions one can then generate the rest: *E.g.* Look at the intersection of a D1- and a D3-brane. In this case, the only supersymmetric intersection is $(0|D1, D3)$, which after S-duality should lead to

[10] The name KK7 comes from the fact that one takes $\vec{y}_{(6)}$ and $z$ as the 'worldvolume directions, since the function $H$ does not depend on them.

[11] Note that the case $r + s = 0$, is nothing but two D-branes of the same kind lying in the same direction, and therefore only breaks half of the available supersymmetry.

the intersection of a D3-brane with a Fundamental string, also written as, abusing notation a bit, $(0|F1, D3)$. Now, since an F1 stays an F1 if we use T-duality in a direction transverse to the string, we can use the invariance of the class of D-brane solutions to see that $(0|F1, Dp)$ should also lead to a supersymmetric, *i.e.* remembering one fourth of the available supersymmetry, intersection.

Following Tseytlin's [111] ideas, we can write down an Ansatz for these Following the *Harmonic Superposition rule*, we can write an Ansatz for these intersections: Just overlap the forms for the metric, multiply the expressions for the dilaton of each solution, and add some factor to, if necessary for the equation of motion, the form-fields.

Using this idea, the $(0|D1, D3)$ Ansatz, for example, reads

$$
\begin{aligned}
ds^2_\sigma &= H_{D1}^{-1/2} H_{D3}^{-1/2} dt^2 \;-\; H_{D1}^{-1/2} H_{D3}^{1/2} dy^2_{(1)} \;-\; H_{D1}^{1/2} H_{D3}^{-1/2} d\vec{z}^{\,2}_{(3)} \;-\; H_{D1}^{1/2} H_{D3}^{1/2} d\vec{x}^{\,2}_{(5)} \,, \\
e^{-2\phi} &= H_{D1}^{-1} \,, \\
C^{(2)}_{ty} &= H_{D1}^{-1} H_{D3}^{\alpha} \,, \\
C^{(4)}_{tz_1z_2z_3} &= H_{D1}^{\beta} H_{D3}^{-1} \,,
\end{aligned}
\tag{3.172}
$$

where $\alpha$ and $\beta$ are to be determined, $H_{D1} = H_{D1}(\vec{z}_{(3)}, \vec{x}_{(5)})$ and $H_{D3} = H_{D3}(y_{(1)}, \vec{x}_{(5)})$. Plugging this kind of Ansätze into the equations of motions derived from the generic action[12]

$$
\mathcal{S}_{p,q} \;=\; \int d^{10}x \sqrt{|g|} \left\{ e^{-2\phi} \left[ R(g) - 4\,(\partial\phi)^2 \right] + (-)^{p+1} \tfrac{1}{2\cdot(p+2)!} G^2_{(p+2)} + (-)^{q+1} \tfrac{1}{2\cdot(q+2)!} G^2_{(q+2)} \right\}
\tag{3.173}
$$

where $G_{(p+2)} = dC^{(p+1)}$, one finds that the Harmonic superposition works modulo some constraints [21, 23, 68, 116, 117, 79]: Both harmonic functions depend on the *overall transverse coordinates*, $\vec{x}_{(5)}$ in the example, only one of the function can depend on its *relative transverse coordinates*, *e.g* $H_{D1}$ can depend on $\vec{z}_{(3)}$ <u>or</u> $H_{D3}$ can depend on $y_{(1)}$. In the above example, the constraints are

$$
0 \;=\; \vec{\partial}^2_x H_{D1} \;+\; H_{D3} \vec{\partial}_z H_{D1} \,, \tag{3.174}
$$

$$
0 \;=\; \vec{\partial}^2_x H_{D3} \;+\; H_{D1} \vec{\partial}_y H_{D3} \,, \tag{3.175}
$$

$$
0 \;=\; \partial_z H_{D1} \partial_y H_{D3} \,. \tag{3.176}
$$

Furthermore, if $r + s = 4$ one finds $\alpha = \beta = 1$, and if $r + s = 8$ one must have $\alpha = \beta = 1$.

Note that the above rules apply to D-brane intersections only: By means of the duality rules the intersections can be generalized to include the various other objects.

It is however paramount that the above intersection Ansatz possibly will not work for the intersection of an F1 with a D8: The action describing the general intersection ignores the fact that the $F_{(2)}$ becomes massive, thus ignoring this contribution. This is of course related to the Stückelberg invariance, which allows for the Kalb-Ramond field to become massive. Looking at the fieldstrengths and the invariances of the, manifestly duality invariant, massive type IIA, one sees that the same thing can happen to the $C^{(7)}$. This then means that there are possibly two intersections where the harmonic superposition Ansatz might not work: $(0|F1, D8)$ and $(5|D6, D8)$. Just by examination of the dilatino variation one can see that the harmonic superposition Ansatz will not lead to a solution breaking susy to one fourth.

[12] One might ask oneself, why there is no Chern-Simons term in this generic action: As one can see from the actions (1.79,1.82), the Cern-Simons term is only important when we have more than 2 objects activated. In IIA such a configuration would be (D2,D2,D1), and in IIB it would be (D3,D1,F1).

### 3.4.2. Massive String

As was seen above, the harmonic superposition intersections automatically satisfy the equations of motion. Note however that this may be true in general, but when one switches on the D8, the Kalb-Ramond field becomes massive. This then means that in at least the intersection of a D8 with a fundamental string, F1, one may suspect that things do not work out according to the harmonic rule. A look at the supersymmetry rules, one sees that such an intersection is allowed and it can be shown [100] to exist and should lead to one quarter of remembered supersymmetry. By direct inspection one can see that an Ansatz according to the harmonic rule does not satisfy the equations of motion [69], so that one has to find another way of doing things.

This solution is given by

$$\left\{\begin{array}{rcl} ds^2 &=& \Omega^{-1}\left(dt^2 - dy^2\right) - d\vec{x}_8^{\,2}\,, \\ B_{\underline{ty}} &=& \pm\left(\Omega^{-1} - 1\right)\,, \\ C^{(1)}{}_{\underline{t}} &=& \pm my\,, \\ e^{-2\phi} &=& \Omega\,, \end{array}\right. \tag{3.177}$$

where

$$\left\{\begin{array}{rcl} \vec{x}_8 &=& \left(x^1,\ldots,x^8\right) = \left(x^m\right)\,, \\ \partial_{\underline{m}}\partial_{\underline{m}}\Omega &=& -m^2\,, \\ \partial_{\underline{y}}\Omega = \alpha m\,. && \end{array}\right. \tag{3.178}$$

This solution has the following properties:

1. The function $\Omega$ consists of three pieces: a piece linear in $y$ (which is interpreted as the coordinate along the string and perpendicular to the D8-brane), a piece quadratic in $\vec{x}_8$ (which are interpreted as the worldvolume coordinates of the D8-brane, orthogonal to the string) and a harmonic function of $\vec{x}_8$:

$$\Omega = \alpha my - \sum_p M_p x^p x^p + H(\vec{x}_8)\,, \qquad \sum_p M_p = \tfrac{1}{2}m^2\,, \qquad \partial_{\underline{m}}\partial_{\underline{m}}H = 0\,. \tag{3.179}$$

   Thus, it can describe, in principle, several objects in equilibrium.

2. In the massless limit $\Omega(y,\vec{x}_8) = H(\vec{x}_8)$ and for the right choice of $H$ it is just the fundamental string solution [33].

3. The limit in which the string is eliminated is unattainable from this solution. Even if we set $H = 0$ $\Omega$ is still non-trivial and the solution will have only $1/4$ of the supersymmetries unbroken.

4. The $C^{(1)}$ field can be completely gauged away, canceling the $\mp 1$ in $B_{\underline{ty}}$. We have introduced it in order to have $B_{\underline{ty}}$ in the form which corresponds to a fundamental string source. (It can be argued that there is a D0-brane in the intersection between the string and the D8-brane, as we will see when we study the unbroken supersymmetry the solution.)

5. We have a solution for any value of the constant $\alpha$. However, only for $\alpha = \mp 1$ the solution is supersymmetric. This is a quite unusual behavior

Let us now find the unbroken supersymmetries. We will only analyze the dilatino supersymmetry rule to show how it works. In this case

$$\delta_\epsilon \lambda = \left( \not\partial \phi + \tfrac{1}{2 \cdot 3!} \Gamma_{11} \not H \right) \epsilon + \tfrac{5i}{4} m e^{\phi} \epsilon - \tfrac{3i}{8} e^{\phi} \not G^{(2)} \Gamma_{11} \epsilon \,, \tag{3.180}$$

with

$$\left\{ \begin{array}{rcl} \not H & = & \mp 3! \partial_{\underline{m}} \Omega \Gamma^m \Gamma^{0y} \,, \\ \not\partial \phi & = & -\tfrac{1}{2} \Omega^{-1} \partial_{\underline{m}} \Omega \Gamma^m - \tfrac{1}{2} \Omega^{-1/2} \partial_{\underline{y}} \Omega \Gamma^y \,, \\ \not G^{(2)} & = & \pm 2 m \Gamma^{0y} \,. \end{array} \right. \tag{3.181}$$

Substituting into the dilatino supersymmetry rule we find

$$-\tfrac{1}{2} \Omega^{-1} \partial_{\underline{m}} \Omega \Gamma^m \left[ 1 \mp \Gamma^{0y} \Gamma_{11} \right] \epsilon - \tfrac{1}{2} \Omega^{-1/2} \partial_{\underline{y}} \Omega \Gamma^y \epsilon + \tfrac{i}{4} m \Omega^{-1/2} \left[ 5 \mp 3 \Gamma^{0y} \Gamma_{11} \right] \epsilon = 0 \,. \tag{3.182}$$

The first term cancels if we impose

$$\tfrac{1}{2} \left[ 1 \mp \Gamma^{0y} \Gamma_{11} \right] \epsilon = 0 \,, \tag{3.183}$$

which is the condition satisfied by the Killing spinor of the fundamental string. This operator is a projector and therefore has eigenvalues 1 or 0. The trace is 16, one half of the trace of the identity and therefore this condition breaks a half of the supersymmetries. Using this condition also in the third term we get

$$-\partial_{\underline{y}} \Omega \Gamma^y \epsilon + i m \epsilon = 0 \,, \tag{3.184}$$

which is solved by $\alpha = \mp 1$ and

$$m \tfrac{1}{2} \left[ 1 \mp i \Gamma^y \right] \epsilon = 0 \,, \tag{3.185}$$

which is the condition satisfied by the Killing spinor of a D8-brane. For analogous reasons, this second condition breaks a half of the supersymmetries for $m \neq 0$. These two projectors commute and therefore both conditions can be fulfilled simultaneously. Since the trace of the product of both projectors is 8, 1/4 of the supersymmetries are preserved.

The gravitino equation also vanishes if the Killing spinor is

$$\epsilon = \Omega^{1/4} \epsilon_0 \,, \tag{3.186}$$

where $\epsilon_0$ is a constant spinor satisfying the above constraints.

Now, observe that if the Killing spinor is an eigenspinor of the fundamental string and D8-brane projectors, then it obeys automatically

$$\tfrac{1}{2} \left[ 1 \mp i \Gamma^0 \Gamma_{11} \right] \epsilon = 0 \,, \tag{3.187}$$

which is the condition of the D0-brane Killing spinor. This may seem a bit surprising since $C^{(1)}$ is trivial (unless $y$ is a compact coordinate). However, its field strength $G^{(2)}$, which is the meaningful quantity is not trivial.

For all these reasons one can identify this solution with the intersection of fundamental string and a D8-brane over a D0-brane[13].

### 3.4.3. Massive D6-Brane

This solution is given by

$$\left\{\begin{array}{rcl} ds^2 & = & \Omega^{-1/2}\left(dt^2 - d\vec{y}_6^{\,2}\right) - \Omega^{1/2}d\vec{x}_3^{\,2}\,, \\ B_{\underline{mn}} & = & \mp\frac{m}{3}\epsilon_{mnp}x^p\,, \\ B^{(6)}{}_{\underline{ty^2\cdots y^6}} & = & \pm m y^1\,, \\ C^{(7)}{}_{\underline{ty^1\cdots y^6}} & = & \pm\left(\Omega^{-1}-1\right)\,, \\ e^{-2\phi} & = & \Omega^{3/2}\,, \end{array}\right. \tag{3.188}$$

where

$$\left\{\begin{array}{rcl} \vec{y}_6 & = & \left(y^1,\ldots,y^6\right) = \left(y^i\right)\,, \\ \vec{x}_3 & = & \left(x^1,x^2,x^3\right) = \left(x^m\right)\,, \\ \partial_{\underline{m}}\partial_{\underline{m}}\Omega & = & -m^2\,, \\ \partial_{\underline{y^1}}\Omega = \alpha m\,. & & \end{array}\right. \tag{3.189}$$

Some remarks are necessary:

1. As $C^{(1)}$ in the massive string case, $B^{(6)}$ is pure gauge but we have introduced it only for the sake of consistency.

2. $B$ is *not pure* gauge. A non-trivial $C^{(7)}$ (necessary for a D6-brane) implies a non-trivial $H^{(7)}$ and, by Hodge duality, a nontrivial $H$ and a non-trivial $B$. This (plus the constraints of unbroken supersymmetry) will give support to the interpretation that there is a solitonic 5-brane in the intersection.

3. The coordinate $y^1$ has been chosen for simplicity but any other direction in the D6-brane worldvolume (coordinates $\left(t, y^i\right)$) would do as direction orthogonal to the solitonic 5-brane and D8-brane.

4. Again, the function $\Omega$ consists of three pieces: a piece linear in $y^1$ (the coordinate orthogonal to the solitonic 5-brane and the D8-brane), a piece quadratic in $\vec{x}_3$ (which are interpreted as worldvolume coordinates of the D8-brane, orthogonal to both the solitonic 5-brane and the D8-brane) and a harmonic function of $\vec{x}_3$:

$$\Omega = \alpha m y^1 - \sum_p M_p x^p x^p + H(\vec{x}_3)\,, \qquad \sum_p M_p = \tfrac{1}{2}m^2\,, \qquad \partial_{\underline{m}}\partial_{\underline{m}}H = 0\,. \tag{3.190}$$

[13] A string solution to a class of massive supergravity theories was recently given in [109]. However, the mass parameter in that model is of NSNS type and there is no mass term for $B$. Thus it cannot describe the fundamental string of the massive type IIA theory.

Thus, it can describe, in principle, several objects in equilibrium.

5. In the massless limit $\Omega(y^1, \vec{x}_3) = H(\vec{x}_3)$ and the right choice of $H$ it is just the D6-brane solution.

6. We have a solution for any value of the constant $\alpha$. However, only for $\alpha = \mp 1$ the solution is supersymmetric. Actually, one finds that (for those values of $\alpha$) the Killing spinor is

$$\epsilon = \Omega^{-1/8} \epsilon_0 \,, \tag{3.191}$$

where $\epsilon_0$ is a constant spinor which satisfies

$$\left\{ \begin{array}{rcl} \frac{1}{2}\left(1 \mp i\Gamma^{01\cdots 6}\right)\epsilon_0 & = & 0\,, \\ m\frac{1}{2}\left[1 \mp i\Gamma^{y}\right]\epsilon_0 & = & 0\,. \end{array} \right. \tag{3.192}$$

If both equations are satisfied, then the following equation is satisfied

$$\tfrac{1}{2}\left[1 \pm \Gamma^{02\cdots 6}\right]\epsilon = 0\,, \tag{3.193}$$

which is the condition satisfied by the solitonic 5-brane Killing spinor.

It is reasonable to identify these solution with the intersection of a D6- and D8-brane over a solitonic 5-brane.

## 3.5. More on D7-branes

In this Section we want to identify the 10-dimensional background of the type IIB theory that produces the masses of the 9-dimensional theory. The T dual background will be dealt with in Section 3.6.

S duality is (believed to be) a fundamental non-perturbative symmetry of type IIB string theory. This implies that the full spectrum of the theory has to be S duality-invariant and thus all the states can be organized in $SL(2,\mathbb{Z})$ multiplets. Thus, bound states of $q$ fundamental strings and $p$ D-strings, known as $pq$-strings, transform as doublets under $SL(2,\mathbb{Z})$. A general solution describing all possible $pq$-strings was constructed in Ref. [103] and a dual general solution describing all possible $pq$-5-branes was recently constructed in Ref. [82]. The D-3-brane, being self-dual, is an $SL(2,\mathbb{Z})$ singlet. The situation for D-9-branes and D-instantons is unclear, although one expects to have D-9-brane solutions which only differ in the constant value of the dilaton.

It is commonly accepted that there are bound states of $p$ D-7-branes and $q$ NS-NS 7-branes (that we will call Q-7-branes) which transform as doublets. As we are going to see, this is not so clear and we will argue that 7-brane states transform as triplets. We will relate the monodromy matrices of massive 9-dimensional type II supergravity and these 7-brane triplets, showing again in this way that the presence of a background of 7-branes is the origin of the masses.

### 3.5.1. Point-Like (in Transverse Space) 7-Branes

The extreme D-7-brane solution in the string frame is

$$
\left\{
\begin{array}{rcl}
ds^2 & = & H_{D7}^{-1/2}\left[dt^2 - d\vec{y}_7{}^2\right] - H_{D7}^{1/2} d\vec{x}_2{}^2\,, \\
 & & \\
e^{-2(\hat{\varphi}-\varphi_0)} & = & H_{D7}^2\,, \\
 & & \\
\hat{C}^{(8)}{}_{ty^1\cdots y^7} & = & \pm e^{-\hat{\varphi}_0} H_{D7}^{-1}\,,
\end{array}
\right.
\tag{3.194}
$$

where $\vec{y}_7 = \left(y_7^1, y_7^2, \ldots, y_7^7\right)$ are the worldvolume coordinates and $\vec{x}_2 = \left(x_2^1, x_2^2\right)$ are the coordinates of the 2-dimensional transverse space. Any function $H_{D7}$ harmonic in the transverse space provides a D-7-brane-type solution. A harmonic function $H_{D7}$ with a single point-like singularity

$$
\partial_{x_2^i}\partial_{x_2^i} H_{D7} = 2\pi h_{D7}\delta^{(2)}(\vec{x}_2)\,,
\tag{3.195}
$$

describes a single D-7-brane placed at $\vec{x}_2 = 0$. The positive constant $h_{D7}$ is proportional to the D-7-brane charge and mass and later on we will determine the precise relation between them. The two possible signs of the charge are taken care of by the $\pm$ in $\hat{C}^{(8)}$. The standard solution in $\mathbb{R}^2$ to the above equation is (the additive constant is arbitrary and momentarily we set to zero)

$$
H_{D7} = h_{D7}\log|\vec{x}_2|\,.
\tag{3.196}
$$

The 8-form potential $\hat{C}^{(8)}$ is nothing but the dual of the RR scalar $\hat{C}^{(0)}$ that occurs in the type IIB theory (i.e. their field strengths are each other's Hodge dual $\hat{G}^{(1)} = {}^\star\hat{G}^{(9)}$). This dualization can only be done "on shell", i.e. using at the same time $\hat{C}^{(0)}$ and $\hat{C}^{(8)}$ because $\hat{C}^{(0)}$ occurs explicitly in the type IIB action. This gives the standard form of $\hat{G}^{(9)}$ suggested in Refs. [54, 22]. If we ignore all other fields apart from $\hat{\lambda}$ both dualizations are equivalent. Using this relation we find

$$
\partial_i\hat{C}^{(0)} = \pm e^{-\hat{\varphi}_0}\epsilon_{ij}\partial_j H_{D7}\,,
\tag{3.197}
$$

and we can rewrite the solution in terms of just the metric and the two real scalars $\hat{C}^{(0)}, e^{-\hat{\varphi}}$ that we combine into the single complex scalar $\hat{\lambda} = \hat{C}^{(0)} + ie^{-\hat{\varphi}}$. For the single D-7-brane we find

$$
\hat{\lambda} = \left\{
\begin{array}{l}
ie^{-\hat{\varphi}_0} h_{D7}\log\omega\,, \\
 \\
ie^{-\hat{\varphi}_0} h_{D7}\log\overline{\omega}\,,
\end{array}
\right.
\qquad \omega = x_2^1 + ix_2^2\,,
\tag{3.198}
$$

for the upper and lower signs respectively.

The charge of a D-7-brane is just, with our normalizations (in the string frame)

$$
p = \oint_\gamma {}^\star\hat{G}^{(9)} = \oint_\gamma \hat{G}^{(1)} = \oint_\gamma d\hat{C}^{(0)} = \Re\mathrm{e}\oint_\gamma d\hat{\lambda}\,.
\tag{3.199}
$$

The contour $\gamma$ is any circle around the point in the transverse space. Using the residue theorem we find for our case that the imaginary part of the integral is zero and

$$
p = \mp 2\pi e^{-\hat{\varphi}_0} h_{D7}\,,
\tag{3.200}
$$

so the solution indeed describes an anti-D-7-brane (upper sign, $\hat{\lambda} = \hat{\lambda}(\omega)$ a holomorphic function of $\omega$) or D-7-brane (lower sign, $\hat{\lambda} = \hat{\lambda}(\overline{\omega})$ a holomorphic function of $\overline{\omega}$) for

$$h_{D7} = \frac{e^{\hat{\varphi}_0}}{2\pi}\,. \tag{3.201}$$

We stress that the transformation that takes us from the D-7-brane to the anti-D-7-brane with opposite RR charge is

$$\hat{\lambda}_{(p)} \to \hat{\lambda}_{(-p)} = -\overline{\hat{\lambda}_{(p)}}\,, \tag{3.202}$$

and it is not an $SL(2,\mathbb{R})$ transformation.

We have just associated the charge of the D-7-brane to the monodromy properties of the anti-holomorphic function $\hat{\lambda}(\overline{\omega})$: If we place at the origin a D-7-brane of unit charge, described by

$$\hat{\lambda}_{(p=1)} = -\frac{1}{2\pi i}\log\overline{\omega}\,, \tag{3.203}$$

and travel once along the path $\gamma(\xi)\,,\ \xi \in [0,1]$, around the origin

$$\begin{aligned} \hat{\lambda}_{(p=1)}[\gamma(1)] &= \hat{\lambda}_{(p=1)}[\gamma(0)] + 1 = \left(M_{(p=1)}\hat{\lambda}_{(p=1)}\right)[\gamma(0)]\,, \\ M_{(p=1)} &= \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right) = T\,, \end{aligned} \tag{3.204}$$

where $M_{(p=1)}$ is the $SL(2,\mathbb{Z})$ monodromy matrix characterizing the 7-brane with charge $p = 1$. One can then apply $SL(2,\mathbb{Z})$ transformations $\Lambda$ to generate other solutions as done in Ref. [43]. Clearly, the monodromy matrix transforms in the adjoint representation

$$M' = \Lambda M \Lambda^{-1}\,. \tag{3.205}$$

Now, it is usually assumed that there are bound states of two kinds of 7-branes ($pq$-branes) transforming as doublets under $SL(2,\mathbb{Z})$. In particular, the charge vector of $pq$-7-branes transforms *covariantly* under $SL(2,\mathbb{Z})$, that is

$$\left(\begin{array}{c} p' \\ q' \end{array}\right) = \Lambda \left(\begin{array}{c} p \\ q \end{array}\right)\,. \tag{3.206}$$

The charge vector of $pq$-strings transforms contravariantly [103], that is

$$(p'\ q') = (p\ q)\Lambda^{-1}\,, \tag{3.207}$$

and so does the charge vector of $pq$-5-branes [82]. Using the above transformation law, one can generate, starting from the $(p = 1) \equiv (1,0)$ other charge vectors using the $SL(2,\mathbb{Z})$ matrix $\Lambda_{(p,q)}$

$$\Lambda_{(p,q)} = \left(\begin{array}{cc} p & b \\ q & d \end{array}\right)\,, \qquad \Lambda_{(p,q)}\left(\begin{array}{c} 1 \\ 0 \end{array}\right) = \left(\begin{array}{c} p \\ q \end{array}\right)\,. \tag{3.208}$$

With the same transformation we generate the supergravity solution describing the $pq$-7-brane with those charges. The monodromy matrix that characterizes this solution is

$$M_{(p,q)} = \Lambda_{(p,q)} M_{(1,0)} \Lambda_{(p,q)}^{-1} = \begin{pmatrix} 1-pq & p^2 \\ \\ -q^2 & 1+pq \end{pmatrix} . \tag{3.209}$$

Clearly not any pair $(p,q)$ can be generated in this way from $(1,0)$. $p$ and $q$ cannot be even at the same time, to start with. According to the standard lore of S duality $p$ and $q$ have to be coprime in order to correspond to stable bound states, and thus this first objection does not seem serious. Still, there is no proof that all pairs corresponding to stable states can be generated in this way.

A second problem is that this is not (by far) the most general $SL(2,\mathbb{Z})$ matrix. Thus, given a certain monodromy matrix we cannot in general determine to which $(p,q)$ state it corresponds.

But there is a more serious problem: We saw in Eq. (3.202) that the transformation that takes us from the $(1,0)$ state to the $(-1,0)$ state is not an $SL(2,\mathbb{Z})$ transformation. However, if the rule Eq. (3.206) is true the transformation $-\mathbb{I}_{2\times 2}$ does the same job. But this transformation leaves $\hat{\lambda}$ exactly invariant![14]

We conclude that bound states of $p$- and $q$-7-branes cannot transform according to Eq. (3.206), and it is easy to see that they do not transform contravariantly either. Thus, they cannot transform as doublets.

It is evident that D-7-branes are not singlets. Thus, the next possibility to be tested is that 7-branes are triplets, i.e. they transform in the adjoint representation. This possibility looks particularly promising if we stick to the characterization of 7-brane bound states through monodromy matrices, which transform in the adjoint representation. Furthermore, there is no $SL(2,\mathbb{Z})$ transformation taking us from the monodromy matrix of the $(p=1)$ state, $T$, to the monodromy matrix of the $(p=-1)$ state, $T^{-1}$.

To clarify completely this issue we are going to make a precise definition of the charges involved and their relation with the monodromy matrix. First, we observe that the equations of motion for the scalars can be written as (we suppress hats here):

$$\nabla_\mu \mathcal{J}^\mu = 0 \,, \qquad \mathcal{J}_\mu = 2\partial_\mu \mathcal{M}\mathcal{M}^{-1} = 2 \begin{pmatrix} \frac{1}{2} j_\mu^{(\varphi)} & j_\mu \\ \\ j_\mu^{(0)} & -\frac{1}{2} j_\mu^{(\varphi)} \end{pmatrix} , \tag{3.210}$$

where

$$\left\{ \begin{array}{rcl} j_\mu^{(\varphi)} & = & e^{2\varphi}\partial_\mu |\lambda|^2 \,, \\ \\ j_\mu^{(0)} & = & e^{2\varphi}\partial_\mu C^{(0)} \,, \\ \\ j_\mu & = & -C^{(0)} j_\mu^{(\varphi)} + |\lambda|^2 j_\mu^{(0)} \,. \end{array} \right. \tag{3.211}$$

The divergences of the first two currents are the dilaton and RR scalar equations of motion. The divergence of the third current is zero on shell but it is not an equation of motion. These three conserved currents can be associated to the three parameters of $SL(2,\mathbb{R})$. In fact, the Noether current associated to the global $SL(2,\mathbb{R})$ transformation $\Lambda = e^m$ where $m$ is the mass matrix defined in Eq. (3.56) is given by

$$j_\mu^{(m)} = \mathrm{Tr}\left(\mathcal{J}_\mu m\right) \,. \tag{3.212}$$

[14]As we said before, the group acting on $\hat{\lambda}$ is $PSL(2,\mathbb{Z}) \equiv SL(2,\mathbb{Z})/\{\pm\mathbb{I}_{2\times 2}\}$.

Using the current matrix we can define a conserved charge matrix

$$\mathcal{Q} \equiv \begin{pmatrix} \frac{\delta}{2} r & \beta q \\ \gamma p & -\frac{\delta}{2} r \end{pmatrix} \equiv \frac{1}{2} \oint_{S^1} \mathcal{J} = \oint_{S^1} d\mathcal{M}\mathcal{M}^{-1} \,, \tag{3.213}$$

where $p, q, r$ are integer charges and $\delta, \beta, \gamma$ are the adequate normalization constants. $r$ is the charge associated to the dilatation current:

$$2\alpha r = \oint j^{T_1} = 2 \oint j^{(\varphi)} \,, \tag{3.214}$$

$p$ is the charge associated to shifts of the RR scalar

$$2\gamma p = \oint j^{\frac{1}{2}(T_2+T_3)} = 2 \oint j^{(0)} \,, \tag{3.215}$$

and therefore the D-7-brane charge, and $q$ is the charge associated to the remaining independent transformation

$$2\beta q = \oint j^{\frac{1}{2}(T_2-T_3)} = 2 \oint j \,. \tag{3.216}$$

Observe that both the current matrix and charge matrix transform in the adjoint representation under $SL(2,\mathbb{R})$. Let the $S^1$ be parametrized by $\xi \in [0,1]$: We define

$$\mathcal{Q}(\xi) \equiv \int_0^\xi d\mathcal{M}\mathcal{M}^{-1} \,, \quad \Rightarrow \quad d\mathcal{Q}(\xi) = d\mathcal{M}\mathcal{M}^{-1} \,. \tag{3.217}$$

If $\mathcal{Q}(\xi) = \mathcal{Q}\xi$, the differential equation can be integrated giving

$$\mathcal{M}(\xi) = e^{\frac{1}{2}\mathcal{Q}\xi} \mathcal{M}_0 \, e^{\frac{1}{2}\mathcal{Q}^T \xi} \,, \tag{3.218}$$

so that the corresponding monodromy matrix reads

$$M_{(p,q,r)} = e^{\frac{1}{2}\mathcal{Q}} \,. \tag{3.219}$$

The restriction to $M \in SL(2,\mathbb{Z})$ implies the quantization of the charges $(p, q, r)$. In particular it implies that there are no allowed *quantum* states with $p = q = 0, r \neq 0$. This seems to restrict the number of independent charge to just two: $p$ and $q$. But it is not easy to talk about the number of independent integers related by a Diophantic equation: Not any pair $p, q$ is allowed.

The general form Eq. (3.65) for an $SL(2,\mathbb{Z})$ matrix is useful to illustrate our result. Let us take the case $n = 1$. The other three integers $n^i$ are a Pythagorean triplet and can be parametrized by three integers $t, s, l$ with the only restriction that $s$ and $l$ are coprime and one of them is an even number:

$$n^1 = \pm t(s^2 - l^2) \,, \quad n^2 = \pm 2tsl \,, \quad n^3 = \pm t(s^2 + l^2) \,. \tag{3.220}$$

This restricted case already produces a monodromy matrix much more general than the $M_{pq}$ in Eq. (3.209). Only two of the integers are independent and the three of them can be put in one-to-one correspondence with the charges $p$ and $q$.

In any case, the important lesson at this stage is that given the monodromy matrix of a certain 7-brane configuration, the above relation immediately allows us to find the 7-brane charges.

To finish this Section, let us stress that these solutions are just examples of the general class of negative-charge 7-brane-type solutions that we write below in the Einstein frame:

$$\left\{\begin{array}{rcl} ds_E^2 & = & dt^2 - d\vec{y}_7^{\,2} - H_7 d\omega d\overline{\omega}\,, \\ \\ H_7 & = & |h|^2 \Im\mathrm{m}\hat{\lambda}\,, \\ \\ \partial_{\overline{\omega}}\hat{\lambda} & = & \partial_{\overline{\omega}} h = 0\,. \end{array}\right. \tag{3.221}$$

The holomorphic function $h$ is nothing but a holomorphic coordinate change. The solutions with positive charge can be obtained by the transformation in Eq. (3.202).

### Q-7-Branes

We can now generate the S duals of the D-7-brane. The rules found above allow us to identify their charges. However, we need a formulation in terms of 8-form potentials to understand physically whether $r$ represents an independent 7-brane charge or not. We will present such a formulation elsewhere.

First, we will construct the Q-7-brane.

Any $SL(2,\mathbb{Z})$ transformation can be written as a product of $S$ and $T$ transformations

$$S = \eta = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right), \qquad T = \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right), \tag{3.222}$$

raised to positive or negative powers. Under these transformations, the charges $p, q, r$ transform as follows:

$$\left\{\begin{array}{rcl} r & \stackrel{S}{\rightarrow} & -r\,, \\ \\ q & \stackrel{S}{\rightarrow} & -\gamma/\beta p\,, \\ \\ p & \stackrel{S}{\rightarrow} & -\beta/\gamma q\,, \end{array}\right. \qquad \left\{\begin{array}{rcl} r & \stackrel{T}{\rightarrow} & r + 2\gamma/\delta p\,, \\ \\ q & \stackrel{T}{\rightarrow} & q - \delta/\beta r - \gamma/\beta p\,, \\ \\ p & \stackrel{T}{\rightarrow} & p\,. \end{array}\right. \tag{3.223}$$

We see that, as expected, from a configuration with only $p$ charge (a D-7-brane) an $S$ transformation generates a configuration with only $q$ charge. We call the object described by this kind of solution a "Q-7-brane" and, taking the D-7-brane solution in Eq. (3.203) we can immediately find its form:

$$\begin{array}{c} Q7 \\ (7,0,2) \end{array} \quad \left\{\begin{array}{rcl} d\hat{s}_{IIB}^2 & = & \left(H_{D7}^2 + A^2\right)^{1/2}\left[H_{D7}^{-1/2}\left(\eta_{ij}dy^i dy^j - dy^2\right) - H_{D7}^{1/2} d\omega d\overline{\omega}\right], \\ \\ \hat{\lambda} & = & -1/(-A + iH_{D7})\,, \end{array}\right. \tag{3.224}$$

where

$$H_{D7} = \tfrac{1}{4\pi}\log\omega\overline{\omega}\,, \qquad A = \tfrac{1}{4\pi} i \log\omega/\overline{\omega}\,. \tag{3.225}$$

The $T$ transformation generates out of the D-7-brane a configuration with a different constant value for $\hat{C}^{(0)}$. Although this is the only difference with the original D-7-brane solution, this constant value induces $q$-charge through the Witten effect. The presence of both $p$ and $q$ charges induces $r$-charge which here seems not to be independent.

### 3.5.2. 7-Branes with a Compact Transverse Dimension

We want to transform 7-branes under T duality and therefore we need to consider the corresponding solutions with a compact transverse dimension.

If one of the transverse coordinates, say $x_2^1 \equiv y$ is compact $y \sim y + 2\pi\ell$ then the function $H_{D7}$ that solves Eq. (3.195) in $\mathbb{R} \times S^1$ takes a different form (we set to zero the additive constant for simplicity):

$$H_{D7} = \frac{h_{D_7}}{2\ell}|x_2^2| + h_{D7} \log \sqrt{1 - 2e^{-|x_2^2|}\cos y/\ell + e^{-2|x_2^2|}}\,. \tag{3.226}$$

Usually, only the zero-mode in the Fourier expansion of this function is considered when performing T duality transformations because the only T duality rules known (Buscher's [30]) apply only to solutions independent of the compact coordinate (at least the metric has to be). This is a strong limitation which only recently started to be appreciated [56]. Nevertheless, the behavior of this zero-mode seems to be well understood and we will focus on it. In our case, then, we will take, for a single D-7-brane ($h_{D7} = 1/(2\pi e^{-\hat{\varphi}_0})$) and for $\ell = 1/2\pi$ ($y \sim y + 1$)

$$H_{D7} = \frac{h_{D_7}}{2\ell}|x_2^2|\,. \tag{3.227}$$

Restricting ourselves to the region $x_2^2 > 0$ for simplicity we find for the complex scalar $\hat{\lambda}$ the expression

$$\left\{\begin{array}{rcl} \hat{\lambda}_{(p=-1)_0} & = & \frac{1}{2}\omega\,. \\ & & \\ \hat{\lambda}_{(p=+1)_0} & = & -\frac{1}{2}\overline{\omega}\,. \end{array}\right. \qquad \omega = y + ix_2^2 \tag{3.228}$$

Somewhat surprisingly, the solution does depend on the compact coordinate $y$. The metric does not, but, after an $SL(2,\mathbb{R})$ transformation, the string metric will depend on $y$ while the Einstein metric will not.

Again, it is convenient to rewrite $\hat{\lambda}$ as follows:

$$\hat{\lambda}_{(p=1)_0} = \tfrac{1}{2}e^{-\hat{\varphi}_0}\begin{pmatrix} z \\ -\bar{z} \end{pmatrix}, \qquad z = y + ix_2^2\,. \tag{3.229}$$

Let us now start by analyzing the monodromy of the positive charge solution zeromode (the holomorphic one). The above function is regular everywhere: The D-7-brane has been smeared out. The only non-trivial cycle to study is the one along $y$, and one finds that the zeromode is shifted by 1/2. This is not an $SL(2,\mathbb{Z})$ transformation. To understand this result it is convenient to map the cylinder into the Riemann sphere with two punctures by means of the conformal transformation $1/w = e^{2\pi iz}$. $w$ is the coordinate in the patch around infinity. Going around the origin in the $w$ plane is the same as going around the cylinder's $S^1$ parametrized by $y$ in the negative sense. The complex scalar zeromode becomes

$$\hat{\lambda}_{(p=1)_0} = -\tfrac{1}{2}\frac{1}{2\pi i}\log w\,, \tag{3.230}$$

which obviously corresponds to a D-7-brane with charge $-1/2$ placed at infinity in the Riemann sphere, i.e. at infinity in the cylinder (we are considering only the positive $x_2^2$ part of the cylinder). Something analogous happens at minus infinity. Then, the presence of a D-7-brane on a cylinder induces the presence of other D-7-branes at infinity. The D-7-branes at infinity have to have integer charge and thus we can only place a D-7-brane of charge ($p = 2$) to have a consistent picture. The situation is depicted in Figure 3.2 The monodromies along the compact coordinate measure the 7-brane charges at infinity and are, therefore $SL(2,\mathbb{Z})$ matrices as discussed in the previous section (now with $\xi = y$). These are precisely the monodromy matrices that appear in our massive 9-dimensional type II supergravity theory.

Figure 3.2: If we place a 7-brane on a cylinder, one has to take into account that automatically 7-branes are created at the boundaries. This can be easily seen by conformally transforming the cylinder into a punctured sphere. Consistency of the monodromy implies that the total sum of the charges in the sphere is nil.

In the supergravity theory, the monodromy matrices are determined by the mass matrix $m$ and, comparing with the results of the previous Section, this is identical to the $pq$-7-brane charge matrix $m$:

$$m = \tfrac{1}{2}\begin{pmatrix} m^1 & m^2 + m^3 \\ m^2 - m^3 & -m_1 \end{pmatrix} = \mathcal{Q} = \begin{pmatrix} \frac{\delta}{2}r & \beta q \\ \gamma p & -\frac{\delta}{2}r \end{pmatrix} . \tag{3.231}$$

This is the sought for relation between the background of 7-branes and the mass parameters of the massive 9-dimensional type II supergravity theory.

## 3.6. KK-7A- and KK-8A-branes and T Duality

In this Section we are going to check explicitly the dualities between extended objects underlying the generalized T duality between the type IIA and type IIB theories. We will find some of the objects whose existence we conjectured in the Introduction. We will essentially prove the connections shown in Figure 3.3.

It is convenient to start with the 11-dimensional Kaluza-Klein monopole which we refer to as KK-7M-brane. This is a 7-dimensional, purely gravitational object, but one of the spacelike

Figure 3.3: This figure is a magnified and more detailed piece of Figure B.1 in which a general picture of all the known extended objects of M/string theory and their duality relations is given. Only well-established relations are shown, and so no duality connections between the conjectured KK-8B-brane and other objects are drawn. In the triplets $(m, n, p)$ $m$ stands for the number of standard spacelike dimensions of the object, $n$ for the number of special isometric directions $(z)$ and $p$ for the number of standard transverse dimensions. The double arrows indicate on which directions T duality acts.

worldvolume directions, with coordinate $z$ is compactified on a circle. Its metric is given by

$$
\begin{array}{c} KK7M \\ (6,1,3) \end{array}
\left\{
\begin{array}{rcl}
d\hat{\hat{s}}^2 & = & \eta_{ij}dy^i dy^j - H^{-1}\left(dz^2 + A_m dx^m\right)^2 - Hd\vec{x}_3^{\,2}\,, \\
2\partial_{[m}A_{n]} & = & \epsilon_{mnp}\partial_p H\,,
\end{array}
\right.
\tag{3.232}
$$

where $\vec{x}_3 = (x^m) = (x^1, x^2, x^3)$ and $i = 0, 1, \ldots, 6$. The standard solution corresponds to the choice

$$
H = 1 + \frac{h}{|\vec{x}_3|}\,.
\tag{3.233}
$$

We can reduce this solution in three different ways. First, we can reduce in the isometry direction, $z$. It is well-known that the resulting object is the D-6-brane. Reducing on one of the standard spacelike worldvolume directions (double dimensional reduction) trivially gives the KK-6A-brane, which is nothing but the 10-dimensional KK monopole.

Finally, we can reduce it on a transverse coordinate, $x^3$. We obtain

$$
\begin{array}{c} KK7A \\ (6,1,2) \end{array}
\left\{
\begin{array}{rcl}
d\hat{s}^2_{IIA} & = & \left(\frac{H}{H^2+A^2}\right)^{-1/2}\left[\eta_{ij}dy^i dy^j - \frac{H}{H^2+A^2}dz^2 - Hd\omega d\overline{\omega}\right]\,, \\
e^{\hat{\phi}} & = & \left(\frac{H}{H^2+A^2}\right)^{-3/4}\,, \\
\hat{C}^{(1)}{}_{\underline{z}} & = & \frac{A}{H^2+A^2}\,, \\
\partial_\omega A & = & i\partial_\omega H\,,
\end{array}
\right.
\tag{3.234}
$$

where $\omega = x^1 + ix^2$ and $A = A_3$ and the last equation is simply $2\partial_{[m}A_{n]} = \epsilon_{mnp}\partial_p H$ with the assumption that $H$ does not depend on $x^3$ and in the $A_1 = A_2 = 0$ gauge. In complex notation the last equation then reads $\partial_\omega\left(A_3 - iH\right) = 0$, which has as a particular solution

$$
H = \tfrac{h}{2}\log\omega\overline{\omega}\,, \qquad A = \tfrac{h}{2}i\log\omega/\overline{\omega}\,.
\tag{3.235}
$$

This kind of solutions has been previously considered in Refs. [21, 23, 81]. To relate it with type IIB solutions, we further reduce it in the isometry direction $z$. The resulting solution is a 9-dimensional "Q-6-brane":

$$
\begin{array}{c} Q6_9 \\ (6,0,2) \end{array}
\left\{
\begin{array}{rcl}
ds^2_{II} & = & \left(H^2+A^2\right)^{1/2}\left[H^{-1/2}\eta_{ij}dy^i dy^j - H^{1/2}d\omega d\overline{\omega}\right]\,, \\
e^{\phi} & = & \left(\frac{H}{H^2+A^2}\right)^{-1}\,, \\
C^{(0)} & = & \frac{A}{H^2+A^2}\,, \\
\partial_\omega A & = & i\partial_\omega H\,.
\end{array}
\right.
\tag{3.236}
$$

This is a solution of our massive 9-dimensional type II theory with $m^{i=0}$. We are going to show it through duality arguments.

Notice that we have obtained two different solutions by reducing first on $z$ and then on $x^3$ and in the inverse order. The difference is a rotation in internal space $z, x^3$ and, by T duality to an S duality transformation in the type IIB side, as we are going to see.

We can now uplift this solution using the type IIB rules and adding the coordinate $y$. We obtain the Q-7-brane solution Eq. (3.224):

$$
\begin{array}{c} Q7 \\ (7,0,2) \end{array}
\left\{
\begin{array}{rcl}
d\hat{s}^2_{IIB} & = & \left(H^2+A^2\right)^{1/2}\left[H^{-1/2}\left(\eta_{ij}dy^i dy^j - dy^2\right) - H^{1/2}d\omega d\overline{\omega}\right] , \\
\hat{\lambda} & = & -1/(-A+iH) .
\end{array}
\right. \tag{3.237}
$$

This solution is the S dual of the standard D-7-brane solution. In fact, performing the $SL(2,\mathbb{Z})$ transformation $S$ and substituting the explicit expressions for $H$ and $A$ we get

$$
\begin{array}{c} D7 \\ (7,0,2) \end{array}
\left\{
\begin{array}{rcl}
d\hat{s}^2_{IIB} & = & H^{-1/2}\left(\eta_{ij}dy^i dy^j - dy^2\right) - H^{1/2}d\omega d\overline{\omega} , \\
\hat{\lambda} & = & -\frac{h}{i}\log\overline{\omega} ,
\end{array}
\right. \tag{3.238}
$$

which is the (positive charge) D-7-brane solution of Eq. (3.203) if we set $h = 1/2\pi$.

We could have reduced the KK-7A-brane on another transverse direction $x^2$. Equivalently, we could have simultaneously reduced the KK-7M-brane on $x^2$ and $x^3$. We immediately face a problem: if $H$ is a harmonic function that only depends on $x^1$, then $A_1 = 0$ but $A_2$ and/or $A_3$ depend on $x^3$ and/or $x^2$.

The situation is identical to that of the reduction of the Q-7-brane on a transverse coordinate. There, it was impossible to eliminate the dependence on that coordinate and generalized dimensional reduction was necessary. Here, only through generalized dimensional reduction of 11-dimensional supergravity one can find the 9-dimensional solution and the T dual. The T dual must have a special isometric direction and 7 standard spacelike worldvolume coordinates. Such a configuration is what we call a KK-8B-brane. The generalized dimensional reduction of 11-dimensional supergravity must give the same 9-dimensional theory as the reduction of type IIB in presence of KK-8B-branes.

We could have reduced the KK-7A-brane on a standard worldvolume direction $y^i$, getting

$$
\begin{array}{c} KK7_9 \\ (6,1,1) \end{array}
\left\{
\begin{array}{rcl}
ds^2_{II} & = & H\left[H^{-1/2}\eta_{ij}dy^i dy^j - H^{1/2}dx^2\right] - H^{-3/2}dz^2 , \\
e^{\phi} & = & H^{1/8} , \\
k & = & H^{1/4} .
\end{array}
\right. \tag{3.239}
$$

This *not* a solution of our 9-dimensional massive type IIB theory. It would be a solution of another massive 9-dimensional type II theory with "Killing vectors" in its Lagrangian. Only after the elimination by reduction of the special isometric direction will we get a solution to some massive supergravity. Anyway, if we uplift this configuration to ten dimensions using the standard type IIB rules we get

$$
\begin{array}{c} Unknown \\ (6,2,1) \end{array}
\quad d\hat{s}^2_{IIB} = H\left[H^{-1/2}\eta_{ij}dy^i dy^j - H^{1/2}dx^2\right] - H^{-1/2}dy^2 - H^{-3/2}dz^2 . \tag{3.240}
$$

This purely gravitational configuration is similar to the KK-9M-brane but with 2 isometric directions instead of just one. Its presence as a 10-dimensional type IIB background will give an 8-dimensional fully covariant massive type II theory.

Objects of this kind can be useful in considering massive theories in lower dimensions, which are out of the scope of this paper and so we will not discuss them any further.

We have already checked the left hand side of Figure 3.3. It is convenient now to start from the KK-9M-brane, recently constructed and studied in Ref. [27][15]. This purely gravitational field configuration is not a solution of the standard 11-dimensional supergravity, but it is a solution of the massive 11-dimensional supergravity constructed in Ref. [25] which we have just generalized in a manifestly $SL(2,\mathbb{R})$-covariant way. Its defining property is that it has a special isometric direction $(z)$ and reduction in this direction gives the D-8-brane.

Choosing $\epsilon = -1$, the metric of the KK-9M-brane is

$$\begin{array}{c} KK9M \\ (8,1,1) \end{array} \left\{ \begin{array}{rcl} d\hat{\hat{s}}^2 & = & H^{1/3}\eta_{ij}dy^i dy^j - H^{-5/3}dz^2 - H^{4/3}dx^2 \,, \\ & & \\ H & = & c + Qx \,, \end{array} \right. \tag{3.241}$$

where now $i = 0, 1, \ldots, 8$.

If we reduce the KK-9M-brane in the isometry direction $(z)$ we get the D-8-brane

$$\begin{array}{c} D8 \\ (8,0,1) \end{array} \left\{ \begin{array}{rcl} d\hat{s}^2_{IIA} & = & H^{-1/2}\eta_{ij}dy^i dy^j - H^{1/2}dx^2 \,, \\ & & \\ e^{\hat{\phi}} & = & H^{-5/4} \,, \end{array} \right. \tag{3.242}$$

which is a solution of Romans' massive type IIA supergravity [98].

Reducing further in one of the spacelike worldvolume directions $(y^8)$ we get the 9-dimensional D-7-brane

$$\begin{array}{c} D7_9 \\ (7,0,2) \end{array} \left\{ \begin{array}{rcl} ds^2_{II} & = & H^{-1/2}\eta_{ij}dy^i dy^j - H^{1/2}dx^2 \,, \\ & & \\ e^{\phi} & = & H^{-9/8} \,, \\ & & \\ k & = & H^{-1/4} \,. \end{array} \right. \tag{3.243}$$

Uplifting to 10 dimensional using the type IIA rules we get the D-7-brane is also the solution we obtained by compactifying in a transverse dimension the D-7-brane. This establishes T duality between the D-8- and the D-7-brane [19].

If we reduce first the KK-9M-brane on a standard worldvolume direction we get the following field configuration

$$\begin{array}{c} KK8A \\ (7,1,1) \end{array} \left\{ \begin{array}{rcl} d\hat{s}^2_{IIA} & = & H\left[H^{-1/2}\eta_{ij}dy^i dy^j - H^{1/2}dx^2\right] - H^{-3/2}dz^2 \,, \\ & & \\ e^{\hat{\phi}} & = & H^{1/4} \,, \end{array} \right. \tag{3.244}$$

which we call KK-8A-brane. This is not a solution of any standard 10-dimensional supergravity. Instead, it is a solution of the massive type IIA supergravity that one finds by reduction of the massive 11-dimensional supergravity of Ref. [25] in a direction different from the isometric one. This theory is related by a rotation in internal space with Romans' massive supergravity [98].

Reducing further in the isometry direction $(z)$, we get the 9-dimensional Q-7-brane

$$\begin{array}{c} Q7_9 \\ (7,0,1) \end{array} \left\{ \begin{array}{rcl} ds^2_{II} & = & H\left[H^{-1/2}\eta_{ij}dy^i dy^j - H^{1/2}dx^2\right] \,, \\ & & \\ e^{\phi} & = & H^{5/8} \,, \\ & & \\ k & = & H^{-3/4} \,. \end{array} \right. \tag{3.245}$$

[15] In that reference it is called “M-9-brane”. We prefer the name KK-9M-brane because it stresses the fact that it has a special isometric direction as the usual KK monopole.

We observe again that we have obtained two different 9-dimensional results which must be related by a rotation in the 2-dimensional internal space and, therefore, by an S duality transformation in the T dual type IIB theory. Thus, not surprisingly, if we uplift the 9-dimensional Q-7-brane to ten dimensions using the standard type IIB rules we get

$$\begin{array}{c} Q7^{\text{bare}} \\ (7,0,2) \end{array} \quad \left\{ \begin{array}{rcl} d\hat{s}^{2\ \text{b}}_{IIB} & = & H\left[H^{-1/2}\left(\eta_{ij}dy^i dy^j - dy^2\right) - H^{1/2}d\omega d\overline{\omega}\right] \,, \\ \\ \hat{\lambda}^{\text{b}} & = & +iH^{-1}\,. \end{array} \right. \tag{3.246}$$

This is nothing but the *bare* field configuration of the Q-7-brane Eq. (3.237). Using the generalized rules for uplifting, the dependence on the internal coordinate is fully recovered. This establishes T duality between the Q-7-brane and the KK-8A-brane under the generalized Buscher T duality rules of Appendix B.3.

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# Appendix A

# Conventions *et. al.*

Our spactime conventions are such that the signature is $(+,-\ldots-)$, except for chapter (2) where we use the oposite signature. In any dimension curved indices are denoted by greek letters, and tangent space indices will be denoted by lowercase latin letters. When dealing with dimensional reduction, higher dimensional indices, and objects, will carry hats, whereas lower dimensional indices and objects will carry nothing.

Vielbeins are introduced by

$$g_{\mu\nu} \;=\; \eta_{ab}\, e_\mu{}^a\, e_\nu{}^b \;, \tag{A.1}$$

and we define the covariant derivative acting on a general object by

$$\nabla_\mu \mathcal{O}_\nu{}^\kappa{}_a{}^b \;=\; \partial_\mu \mathcal{O}_\nu{}^\kappa{}_a{}^b - \Gamma_{\mu\nu}{}^\lambda\, \mathcal{O}_\lambda{}^\kappa{}_a{}^b + \Gamma_{\mu\lambda}{}^\kappa\, \mathcal{O}_\nu{}^\lambda{}_a{}^b - \omega_{\mu a}{}^c\, \mathcal{O}_\nu{}^\kappa{}_c{}^b - \omega_\mu{}^b{}_c\, \mathcal{O}_\nu{}^\kappa{}_a{}^c \;. \tag{A.2}$$

The Levi-Cività- and the spin-connection are defined by

$$\begin{aligned} \Gamma_{\mu\nu}{}^\kappa &= \tfrac{1}{2} g^{\kappa\sigma}\left[\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right] \;, && (A.3)\\ \omega_\mu{}^{ab} &= e^{\nu b}\partial_\mu e_\nu{}^a - e_\nu{}^a\, e^{\rho b}\Gamma_{\mu\rho}{}^\nu \;. && (A.4) \end{aligned}$$

The various curvatures are then defined as

$$\begin{aligned} R_{\mu\nu\kappa}{}^\rho &= 2\partial_{[\mu}\Gamma_{\nu]\kappa}{}^\rho + 2\Gamma_{[\mu|\sigma}{}^\rho\Gamma_{\nu]\kappa}{}^\sigma \;,\\ R_{\mu\nu} &= R_{\mu\rho\nu}{}^\rho \;,\\ R &= R_\mu{}^\mu \;. \end{aligned} \tag{A.5}$$

The, d-dimensional, totally anti-symmetric Levi-Cività symbol is defined in the tangent-space as

$$\epsilon^{0\,1\ldots d} \;\equiv\; 1 \quad\rightarrow\quad \epsilon_{0\,1\ldots d} \;=\; (-1)^{d-1} \;, \tag{A.6}$$

and on curved space as

$$\epsilon^{\mu_1\cdots\mu_d} \;=\; \sqrt{|g|}\, e_{a_1}{}^{\mu_1}\ldots e_{a_d}{}^{\mu_d}\, \epsilon^{a_1\ldots a_d} \;. \tag{A.7}$$

All p-forms $\omega_{(p)}$, are defined to be of the form

$$\omega_{(p)} \;=\; \tfrac{1}{p!}\omega_{\mu_1\ldots\mu_p}\, dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p} \;. \tag{A.8}$$

With these definitions we can define the *Hodge dual* of a p-form, $\omega_{(p)}$ say, as a $(d-p)$-form ${}^\star\omega_{(p)}$ with components

$$\left({}^\star\omega_p\right)_{\nu_1\ldots\nu_{d-p}} \;=\; \frac{1}{p!\sqrt{|g|}}\,\epsilon_{\nu_1\ldots\nu_{d-p}\mu_1\ldots\mu_p}\,\omega^{\mu_1\ldots\mu_p} \;, \tag{A.9}$$

Using the Hodge duality twice, one can show that

$$
{}^\star\left({}^\star\omega_{(p)}\right) \;=\; (-)^{p(d-p)+d-1}\omega_{(p)} \tag{A.10}
$$

When not using form notation, we will be using a notation for which indices are not shown explicitly. In this notation we assume that all indices are completely antisymmetrized in the obvious order, *e.g.*

$$
H \;=\; 3\partial B\,, \tag{A.11}
$$

is shorthand for

$$
H_{\mu\nu\rho} \;=\; 3\partial_{[\mu}B_{\nu\rho]}\,. \tag{A.12}
$$

Note that all indices are antisymmetrized with unit weight.

# Appendix B

# KK Decompositions and T-Duality

## B.1. 9-Dimensional Einstein Fields Vs. 10-Dimensional Type IIA String Fields

In the main body of the paper we went directly from 11 to 9 dimensions and thus we need to repeat the reduction from 11 to 10 dimensions [115, 15] to be able to relate 9- with 10-dimensional fields.

As usual, we assume now that all fields are independent of the spacelike coordinate $z = x^{\underline{10}}$ and we rewrite the fields and action in a ten-dimensional form. The dimensional reduction of 11-dimensional supergravity Eq. (3.113) gives rise to the fields of the ten-dimensional $N = 2A, d = 10$ supergravity theory

$$\left\{\hat{g}_{\hat{\mu}\hat{\nu}}, \hat{B}_{\hat{\mu}\hat{\nu}}, \hat{\phi}, \hat{C}^{(3)}{}_{\hat{\mu}\hat{\nu}\hat{\rho}}, \hat{C}^{(1)}{}_{\hat{\mu}}, \right\} . \tag{B.1}$$

The metric, the two-form and the dilaton are NS-NS fields and the three-form and the vector are RR fields. We are going to use for RR forms the conventions proposed in Refs. [54, 22, 25].

The fields of the 11-dimensional theory can be expressed in terms of the 10-dimensional ones as follows:

$$\begin{array}{rclrcl}
\hat{\hat{g}}_{\hat{\mu}\hat{\nu}} & = & e^{-\frac{2}{3}\hat{\phi}}\hat{g}_{\hat{\mu}\hat{\nu}} - e^{\frac{4}{3}\hat{\phi}}\hat{C}^{(1)}{}_{\hat{\mu}}\hat{C}^{(1)}{}_{\hat{\nu}}\,, & \hat{\hat{C}}_{\hat{\mu}\hat{\nu}\hat{\rho}} & = & \hat{C}^{(3)}{}_{\hat{\mu}\hat{\nu}\hat{\rho}}\,, \\
\hat{\hat{g}}_{\hat{\mu}\underline{z}} & = & -e^{\frac{4}{3}\hat{\phi}}\hat{C}^{(1)}{}_{\hat{\mu}}\,, & \hat{\hat{C}}_{\hat{\mu}\hat{\nu}\underline{z}} & = & \hat{B}_{\hat{\mu}\hat{\nu}}\,, \\
\hat{\hat{g}}_{\underline{zz}} & = & -e^{\frac{4}{3}\hat{\phi}}\,. & & &
\end{array} \tag{B.2}$$

For the Elfbeins we have

$$\begin{array}{rcl}
\left(\hat{\hat{e}}_{\hat{\mu}}{}^{\hat{a}}\right) & = & \left(\begin{array}{cc} e^{-\frac{1}{3}\hat{\phi}}\hat{e}_{\hat{\mu}}{}^{\hat{a}} & e^{\frac{2}{3}\hat{\phi}}\hat{C}^{(1)}{}_{\hat{\mu}} \\ 0 & e^{\frac{2}{3}\hat{\phi}} \end{array}\right), \\
\left(\hat{\hat{e}}_{\hat{a}}{}^{\hat{\mu}}\right) & = & \left(\begin{array}{cc} e^{\frac{1}{3}\hat{\phi}}\hat{e}_{\hat{a}}{}^{\hat{\mu}} & -e^{\frac{1}{3}\hat{\phi}}\hat{C}^{(1)}{}_{\hat{a}} \\ 0 & e^{-\frac{2}{3}\hat{\phi}} \end{array}\right).
\end{array} \tag{B.3}$$

Conversely, the 10-dimensional fields can be expressed in terms of the 11-dimensional ones by:

$$
\begin{aligned}
\hat{g}_{\hat{\mu}\hat{\nu}} &= \left(-\hat{\hat{g}}_{\underline{zz}}\right)^{\frac{1}{2}} \left(\hat{\hat{g}}_{\hat{\mu}\hat{\nu}} - \hat{\hat{g}}_{\hat{\mu}\underline{z}}\hat{\hat{g}}_{\hat{\nu}\underline{z}}/\hat{\hat{g}}_{\underline{zz}}\right), & \hat{C}^{(3)}{}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= \hat{\hat{C}}_{\hat{\mu}\hat{\nu}\hat{\rho}}\,, \\
\hat{C}^{(1)}{}_{\hat{\mu}} &= \hat{\hat{g}}_{\hat{\mu}\underline{z}}/\hat{\hat{g}}_{\underline{zz}}\,, & \hat{B}_{\hat{\mu}\hat{\nu}} &= \hat{\hat{C}}_{\hat{\mu}\hat{\nu}\underline{z}}\,, \\
\hat{\phi} &= \tfrac{3}{4}\log\left(-\hat{\hat{g}}_{\underline{zz}}\right).
\end{aligned}
\tag{B.4}
$$

After some standard calculations that we omit we find the bosonic part of the $N=2A, d=10$ supergravity action in ten dimensions in the string frame:

$$
\begin{aligned}
\hat{S} = &\int d^{10}x\,\sqrt{|\hat{g}|}\left\{e^{-2\hat{\phi}}\left[\hat{R} - 4\left(\partial\hat{\phi}\right)^2 + \tfrac{1}{2\cdot 3!}\hat{H}^2\right]\right. \\
&\left. - \left[\tfrac{1}{4}\left(\hat{G}^{(2)}\right)^2 + \tfrac{1}{2\cdot 4!}\left(\hat{G}^{(4)}\right)^2\right] - \tfrac{1}{144}\tfrac{1}{\sqrt{|\hat{g}|}}\,\hat{\epsilon}\partial\hat{C}^{(3)}\partial\hat{C}^{(3)}\hat{B}\right\}.
\end{aligned}
\tag{B.5}
$$

where the fields strengths are defined as follows:

$$
\left\{
\begin{aligned}
\hat{H} &= 3\partial\hat{B}\,, \\
\hat{G}^{(2)} &= 2\partial\hat{C}^{(1)}\,, \\
\hat{G}^{(4)} &= 4\left(\partial\hat{C}^{(3)} - 3\partial\hat{B}\hat{C}^{(1)}\right),
\end{aligned}
\right.
\tag{B.6}
$$

and they are invariant under the gauge transformations

$$
\left\{
\begin{aligned}
\delta\hat{B} &= \partial\hat{\Lambda}\,, \\
\delta\hat{C}^{(1)} &= \partial\hat{\Lambda}^{(0)}\,, \\
\delta\hat{C}^{(3)} &= 3\partial\hat{\Lambda}^{(2)} + 3\hat{B}\partial\hat{\Lambda}^{(0)}\,.
\end{aligned}
\right.
\tag{B.7}
$$

Now, using these results together with the relation between 9- and 11-dimensional fields obtained in Section 3.3.1 we get

$$
\begin{aligned}
\mathcal{M} &= e^{\hat{\phi}}|\hat{g}_{\underline{xx}}|^{-1/2}\begin{pmatrix} e^{-2\hat{\phi}}|\hat{g}_{\underline{xx}}| + (\hat{C}^{(1)}{}_{\underline{x}})^2 & \hat{C}^{(1)}{}_{\underline{x}} \\ \hat{C}^{(1)}{}_{\underline{x}} & 1 \end{pmatrix}, \\
K &= e^{\hat{\phi}/3}|\hat{g}_{\underline{xx}}|^{1/2}\,, \\
A_{(1)\ \mu} &= \hat{B}_{\mu\underline{x}}\,,
\end{aligned}
\tag{B.8}
$$

$$
\begin{aligned}
\vec{A}_{(1)\ \mu} &= \begin{pmatrix} \hat{C}^{(1)}{}_{\mu} - \hat{C}^{(1)}{}_{\underline{x}}\hat{g}_{\mu\underline{x}}/\hat{g}_{\underline{xx}} \\ \\ -\hat{g}_{\mu\underline{x}}/\hat{g}_{\underline{xx}} \end{pmatrix}, \\
\vec{A}_{(2)\ \mu\nu} &= \begin{pmatrix} \hat{C}^{(3)}{}_{\mu\nu\underline{x}} - 2\hat{B}_{[\mu|\underline{x}|}\hat{C}^{(1)}{}_{\nu]} + 2\hat{C}^{(1)}{}_{\underline{x}}\hat{B}_{[\mu|\underline{x}|}\hat{g}_{\nu]\underline{x}}/\hat{g}_{\underline{xx}} \\ \\ \hat{B}_{\mu\nu} + 2\hat{B}_{[\mu|\underline{x}|}\hat{g}_{\nu]\underline{x}}/\hat{g}_{\underline{xx}} \end{pmatrix}, \\
A_{(3)\ \mu\nu\rho} &= \hat{C}^{(3)}{}_{\mu\nu\rho} - \tfrac{3}{2}\hat{g}_{[\mu|\underline{x}|}\hat{C}^{(3)}{}_{\nu\rho]\underline{x}}/\hat{g}_{\underline{xx}} - \tfrac{3}{2}\hat{C}^{(1)}{}_{\underline{x}}\hat{g}_{[\mu|\underline{x}|}\hat{B}_{\nu\rho}/\hat{g}_{\underline{xx}} \\
&\quad -\tfrac{3}{2}\hat{C}^{(1)}{}_{[\mu}\hat{B}_{\nu\rho]}, \\
g_{E\ \mu\nu} &= e^{-4\hat{\phi}/7}|\hat{g}_{\underline{xx}}|^{1/7}\left[\hat{g}_{\mu\nu} - \hat{g}_{\mu\underline{x}}\hat{g}_{\nu\underline{x}}/\hat{g}_{\underline{xx}}\right].
\end{aligned} \tag{B.9}
$$

## B.2. 9-Dimensional Einstein Fields Vs. 10-Dimensional Type IIB String Fields

Using the results of Section 3.2.2 we find

$$
\begin{aligned}
\mathcal{M} &= \Lambda^{-1}(y)\hat{\mathcal{M}}(\hat{x})(\Lambda^{-1})^{T}(y) = \hat{\mathcal{M}}^{\mathrm{b}} = e^{\hat{\varphi}^{\mathrm{b}}}\begin{pmatrix} |\hat{\lambda}^{\mathrm{b}}|^{2} & \hat{C}^{\mathrm{b}\ (0)} \\ \\ \hat{C}^{\mathrm{b}\ (0)} & 1 \end{pmatrix}, \\
K &= e^{\hat{\varphi}/3}|\hat{\jmath}_{\underline{yy}}|^{-2/3} = e^{\hat{\varphi}^{\mathrm{b}}/3}|\hat{\jmath}^{\mathrm{b}}{}_{\underline{yy}}|^{-2/3}, \\
A_{(1)\ \mu} &= \hat{\jmath}_{\mu\underline{y}}/\hat{\jmath}_{\underline{yy}} = \hat{\jmath}^{\mathrm{b}}{}_{\mu\underline{y}}/\hat{\jmath}^{\mathrm{b}}{}_{\underline{yy}}, \\
\vec{A}_{(1)\ \mu} &= -\Lambda^{-1}(y)\begin{pmatrix} \hat{C}^{(2)}{}_{\mu\underline{y}} \\ \\ \hat{\mathcal{B}}_{\mu\underline{y}} \end{pmatrix} = \begin{pmatrix} \hat{C}^{\mathrm{b}\ (2)}{}_{\mu\underline{y}} \\ \\ \hat{\mathcal{B}}^{\mathrm{b}}{}_{\mu\underline{y}} \end{pmatrix}, \\
\vec{A}_{(2)\ \mu\nu} &= \Lambda^{-1}(y)\begin{pmatrix} \hat{C}^{(2)}{}_{\mu\nu} \\ \\ \hat{\mathcal{B}}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \hat{C}^{\mathrm{b}\ (2)}{}_{\mu\nu} \\ \\ \hat{\mathcal{B}}^{\mathrm{b}}{}_{\mu\nu} \end{pmatrix}, \\
A_{(3)\ \mu\nu\rho} &= -\hat{C}^{(4)}{}_{\mu\nu\rho\underline{y}} - \tfrac{3}{2}\hat{\mathcal{B}}_{[\mu\nu}\hat{C}^{(2)}{}_{\rho]\underline{y}} - \tfrac{3}{2}\hat{\mathcal{B}}_{[\mu|\underline{y}|}\hat{C}^{(2)}{}_{\nu\rho]}, \\
g_{E\ \mu\nu} &= e^{-4\hat{\varphi}/7}|\hat{\jmath}_{\underline{yy}}|^{1/7}\left[\hat{\jmath}_{\mu\nu} - \hat{\jmath}_{\mu\underline{y}}\hat{\jmath}_{\nu\underline{y}}/\hat{\jmath}_{\underline{yy}}\right].
\end{aligned} \tag{B.10}
$$

## B.3. Generalized Buscher T-Duality Rules

Now we just have to compare the results of Appendix B.2 and Appendix B.1 to identify the 10-dimensional fields of the type IIA and IIB theories. This identification produces for us the searched for generalization of Buscher's T duality rules [30]. These rules generalize the standard type II T duality rules of Ref. [15] in the same way as those of Ref. [19]: The rules have exactly the same form as the massless ones if we replace the type IIB fields by the *bare type IIB* fields.

The only deficiency of these rules is with respect to the S duals of D-7-branes: It is necessary to dualize their 8-form potentials which transform independently of $\hat{\lambda}^{\mathrm{b}}$.

Thus, indicating by a superscript $b$ the bare type IIB fields the T duality rules take the form[1]:

**From IIA to IIB:**

$$\begin{aligned}
\hat{\jmath}^{\mathrm{b}}{}_{\mu\nu} &= \hat{g}_{\mu\nu} - \left(\hat{g}_{\mu\underline{x}}\hat{g}_{\nu\underline{x}} - \hat{B}_{\mu\underline{x}}\hat{B}_{\nu\underline{x}}\right)/\hat{g}_{\underline{xx}}\,, & \hat{\jmath}^{\mathrm{b}}{}_{\mu\underline{y}} &= \hat{B}_{\mu\underline{x}}/\hat{g}_{\underline{xx}}\,,\\
\hat{\mathcal{B}}^{\mathrm{b}}{}_{\mu\nu} &= \hat{B}_{\mu\nu} + 2\hat{g}_{[\mu|\underline{x}}\hat{B}_{\nu]\underline{x}}/\hat{g}_{\underline{xx}}\,, & \hat{\mathcal{B}}^{\mathrm{b}}{}_{\mu\underline{y}} &= \hat{g}_{\mu\underline{x}}/\hat{g}_{\underline{xx}}\,,\\
\hat{\varphi}^{\mathrm{b}} &= \hat{\phi} - \tfrac{1}{2}\log|\hat{g}_{\underline{xx}}|\,, & \hat{\jmath}^{\mathrm{b}}{}_{\underline{yy}} &= 1/\hat{g}_{\underline{xx}}\,,
\end{aligned}$$

$$\begin{aligned}
\hat{C}^{\mathrm{b}\ (2n)}{}_{\mu_1\ldots\mu_{2n}} &= \hat{C}^{(2n+1)}{}_{\mu_1\ldots\mu_{2n}\underline{x}} + 2n\hat{B}_{[\mu_1|\underline{x}|}\hat{C}^{(2n-1)}{}_{\mu_2\ldots\mu_{2n}]}\\
&\quad -2n(2n-1)\hat{B}_{[\mu_1|\underline{x}|}\hat{g}_{\mu_2|\underline{x}|}\hat{C}^{(2n-1)}{}_{\mu_3\ldots\mu_{2n}]\underline{x}}/\hat{g}_{\underline{xx}}\,,\\
\hat{C}^{\mathrm{b}\ (2n)}{}_{\mu_1\ldots\mu_{2n-1}\underline{y}} &= -\hat{C}^{(2n-1)}{}_{\mu_1\ldots\mu_{2n-1}}\\
&\quad +(2n-1)\hat{g}_{[\mu_1|\underline{x}|}\hat{C}^{(2n-1)}{}_{\mu_2\ldots\mu_{2n-1}]\underline{x}}/\hat{g}_{\underline{xx}}\,.
\end{aligned} \tag{B.11}$$

**From IIB to IIA:**

$$\begin{aligned}
\hat{g}_{\mu\nu} &= \hat{\jmath}^{\mathrm{b}}{}_{\mu\nu} - \left(\hat{\jmath}^{\mathrm{b}}{}_{\mu\underline{y}}\hat{\jmath}^{\mathrm{b}}{}_{\nu\underline{y}} - \hat{\mathcal{B}}^{\mathrm{b}}{}_{\mu\underline{y}}\hat{\mathcal{B}}^{\mathrm{b}}{}_{\nu\underline{y}}\right)/\hat{\jmath}^{\mathrm{b}}{}_{\underline{yy}}\,, & \hat{g}_{\mu\underline{x}} &= \hat{\mathcal{B}}^{\mathrm{b}}{}_{\mu\underline{y}}/\hat{\jmath}^{\mathrm{b}}{}_{\underline{yy}}\,,\\
\hat{B}_{\mu\nu} &= \hat{\mathcal{B}}^{\mathrm{b}}{}_{\mu\nu} + 2\hat{\jmath}^{\mathrm{b}}{}_{[\mu|\underline{y}|}\hat{\mathcal{B}}^{\mathrm{b}}{}_{\nu]\underline{y}}/\hat{\jmath}^{\mathrm{b}}{}_{\underline{yy}}\,, & \hat{B}_{\mu\underline{x}} &= \hat{\jmath}^{\mathrm{b}}{}_{\mu\underline{y}}/\hat{\jmath}^{\mathrm{b}}{}_{\underline{yy}}\,,\\
\hat{\phi} &= \hat{\varphi}^{\mathrm{b}} - \tfrac{1}{2}\log|\hat{\jmath}^{\mathrm{b}}{}_{\underline{yy}}|\,, & \hat{g}_{\underline{xx}} &= 1/\hat{\jmath}^{\mathrm{b}}{}_{\underline{yy}}\,,
\end{aligned}$$

$$\begin{aligned}
\hat{C}^{(2n+1)}{}_{\mu_1\ldots\mu_{2n+1}} &= -\hat{C}^{\mathrm{b}\ (2n+2)}{}_{\mu_1\ldots\mu_{2n+1}\underline{y}}\\
&\quad +(2n+1)\hat{\mathcal{B}}^{\mathrm{b}}{}_{[\mu_1|\underline{y}|}\hat{C}^{\mathrm{b}\ (2n)}{}_{\mu_2\ldots\mu_{2n+1}]}\\
&\quad -2n(2n+1)\hat{\mathcal{B}}^{\mathrm{b}}{}_{[\mu_1|\underline{y}|}\hat{\jmath}^{\mathrm{b}}{}_{\mu_2|\underline{y}}\hat{C}^{\mathrm{b}\ (2n)}{}_{\mu_3\ldots\mu_{2n+1}]\underline{y}}/\hat{\jmath}^{\mathrm{b}}{}_{\underline{yy}}\,,\\
\hat{C}^{(2n+1)}{}_{\mu_1\ldots\mu_{2n}\underline{x}} &= \hat{C}^{\mathrm{b}\ (2n)}{}_{\mu_1\ldots\mu_{2n}}\\
&\quad +2n\hat{\jmath}^{\mathrm{b}}{}_{[\mu_1|\underline{y}|}\hat{C}^{\mathrm{b}\ (2n)}{}_{\mu_2\ldots\mu_{2n}]\underline{y}}/\hat{\jmath}^{\mathrm{b}}{}_{\underline{yy}}\,.
\end{aligned} \tag{B.12}$$

The relation between the *bare* fields and the real fields is given in Section 3.2.2.

[1]These rules apply to RR $n$-forms for any $n$. For the values of $n$ that do not appear in the main body of this paper, one simply has to use the general expression for the RR field strengths and gauge transformations given in Ref. [25] inspired by those of Refs. [54, 22].

Figure B.1: Duality relations between classical solutions of 10- and 11-dimensional supergravity theories describing string/M-theory solitons: p-branes, M-branes, D-branes, gravitational waves, Kaluza-Klein monopoles and other KK-type solutions. Lines with two arrows denote T duality relations. Dashed lines denote S duality relations. Lines with a single arrow denote relations of dimensional reduction, either vertical (direct dimensional reduction) or diagonal (double dimensional reduction).

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