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Generalized spacetime symmetries with a Hopfian structure

P.A.A. Meessen

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Prof. Dr. C. Dullemond

Hecher, beser,
Di rod, di rod macht greser!
Grojs hot mir Got gemacht,
Glik hot er mir gebracht,
Huljet, Kinder a gantse nacht!
Di mesinke ojsgegebn!

Schtarker, frejlech,
Du di malke, ich - der mejlech.
Oj! oj! oj! - ich alejn
Hob mit majne ojgn gesen,
Wi Got hot mich matsliach gewen!
Di mesinke ojsgegebn!

Itsik - Schpitsik,
Wos schwajgstu mitn Schmitsik?
Ojf die klesmer gib a geschraj!
Tsi schpiln sej! Tsi schlofn sej?
Rajst di strunes ale ojf tswej!
Di mesinke ojs gegebn!

Motl - Schimen,
Di oreme lajt senen gekumen.
Schtelt far sej dem schensten tisch,
tajere wajnen, tajere fisch,
Oj, majne tochter gib mir a kusch!
Di mesinke ojsgegebn

Ajsik, Masik
Di bobe gejt a kosik,
on ein hore, seht nor seht,
Wi si tupet, wi si tret!
Oj, a simche, oj, a frejd!
Di mesinke ojsgegebn!

Mark Warschawski (1848-1907)

Contents

Most physical processes at high energies are described by means of Quantum Field Theories (QFT). The QFT are, necessarily, invariant under the Poincaré group so that its predictions are the same in every place and look the same in every inertial system, as is required by special relativity. Owing to the Noether theorem, this invariance leads to the conservation of energy, momentum and angular momentum. However, if one calculates physical processes, by means of, so called, Feynman integrals, one is plagued by diverging integrals. For this problem, there exists a successful programme in order to get rid of these infinities: renormalisation. This renormalisation-programme, however successfull it may be, is a bit artificial as it involves, for example, substractions of infinite quantities to yield finite quantities.

Seeing however the success of theories based on the Poincaré group in describing the low energy behaviour of physical processes [25], one might consider changing the Poincaré group in such a way that it only affects its high-energy behaviour. Since the divergencies of normal theories occur in this region, it follows that upon changing this behaviour one might obtain convergent predictions up to all orders in the perturbation expansion. And that is what one wants.

In the fifties Pauli noted that fermionic loops contribute with an relative minus sign, relative to bosonic loops, so that if fermions and bosons were to have the same mass and coupling to the propagators, their contribution to the Feynman integrals would vanish. A quick glance at the masses of elementary particles however tells us that fermions and bosons don't have the same mass.

In the seventies however, it was found that under certain circumstances one could break up particles with the same mass into sets of particles with different masses by means of the so-called spontaneous breakdown of symmetry (SBS). It was also proven that if a theory were renormalizable before the SBS, it would also be renormalizable after the SBS. This clearly opened up one possibility of obtaining renormalizable theories which contain particles of different masses. The trouble with this procedure is that it can only change the masses of particles with the same spin, whereas one needs to break up the mass-equivalence between particles with different spins. In the seventies Wess and Zumino found that some Lagrangians admit a symmetry between fermions and bosons which restricted all fermions and bosons to have the same mass and coupling to whatever [46]. Since this was a symmetry one can apply SBS to it, in order to obtain a renormalizable theory. Since everybody thought of this symmetry as being a super idea, it is nowadays referred to as supersymmetry. Supersymmetry has a disadvantage however because for every particle one needs to introduce a partner particle for the symmetry to work. These partner particles are vastly looked for, but haven't been found to date, a fact which makes supersymmetry a bit unattractive.

Therefore the search for renormalizable QFT continues. As was mentioned before, renormalizability depends on the behaviour of propagators in the high momenta limit. Therefore, changing this behaviour might result in superrenormalizable QFT, which is what one wants.

In the last few years physicist have been looking at deformations of Lie algebra (or Lie-groups) in order to find new symmetries in physics. One requirement on these deformations is that representations should exist. If no representations exist, there can be no such symmetry between particles or states in the, needed, Hilbert spaces that discribe nature in the eyes of QM. Now a Hopf algebra is a natural extention of Lie-algebras, every Lie-algebra being one, and has the nice property that it has representations.

For semisimple Lie-algebras there was developped a unique procedure to deform them, and turn them ino a family of Hopf algebras: The Drinfel'd - Jimbo method [19, 27]. However the Poincaré algebra isn't a semisimple algebra since it is a semi-direct-sum Lie algebra. Therefore one has to find other methods to deform the Poincaré algebra and to find lagrangians invariant under it.

There exist a few different methods to arrive at a deformed Poincaré algebra. One of the first attempts [12, 33] were based on the well-known fact that the Poincaré algebra can be obtained by making a so-called Wigner-Inönü contraction on the algebra $so(2,3)$ [24]. First of all one turns this group into the Hopf family $so_q(2,3)$ using the Drinfel'd-Jimbo method and then contracts the whole thing using some kind of Wigner-Inönü contraction. This procedure enables us to introduce a fundamental length into physical theories, which will improve their renormalization behaviour [53].

Another scheme is based on the equivalence between the Lorentz group and the group $SL(2,\mathbb{C})$. For this group, $SL(2,\mathbb{C})$, Manin [43] devised a quantized version which consists of four non-commuting numbers. Then using some generalizations of the usual construction, one can define a consistent deformed Lorentz algebra, which by construction is defined on spinors [49, 50]. From these spinors it's easy to define vectors under the Lorentz group, which then can be looked upon as generators of a quantized Minkowski space. On this Minkowski space one can define differential operators, interpreted as the translations, and the action of the deformed Lorentz algebra on the translations.

A recent idea is to change the (linear) action of the Lorentz group onto the translation algebra, into a non-linear one [31]. This construction has the advantage that one keeps the Lorentz algebra complete, so that one knows that fields will have the old spins and that the light-cone is invariant under increases in energies of the photons. This construction is plagued, however, by a structure function which can be chosen to ones (dis)liking. Physics however, puts some constraints on this structure function. One of these constraints is that in the low-energy-region, the algebra has to act like the Poincaré algebra and thus ensuring that theories based on this invariance act as a QFT in the low energies. Another constraint is the existence of a well-defined composition of generators describing different physical particles into one system. This constraint clearly puts one in a position to define polyparticle states and polyparticle theories.

The author would like to mention that this review is somehow incomplete. This is due to the fact that the subject is, at the moment this line is being written, very young and needs a lot of investigation to complete. Therefore this thesis ought to be looked upon, not as a full review as could be written on a subject like classification of semisimple Lie algebras, but rather as an introduction to the world of possibilities which occurs when applying new ideas to spacetime symmetries.

This thesis wouldn't be a thesis without the joyfull 'thank you's' so that I'll start by thanking my family and girlfriend for their (financial) support and goodwill. It's also a pleasure to thank the inhabitants of the department, for their help during my stay at the department, and my friends in Nuth, Nijmegen and Athens for loads of fun and helping to spend the above mentioned financial support. The biggest '**THANK YOU**' goes to A.A. Kehagias and G. Zoupanos for showing me what research is all about, teaching me loads of physics, a lot of fun and some extremely nice diners.

Chapter 1

What you always wanted to know about the Poincaré algebra (but were afraid to ask)

This chapter is intended to give a short introduction to the Poincaré algebra rather than to give a thorough mathematical description. One might state that it only intends to define the conventions and to explain some things that normally are not dealt with, but will be used in this thesis.

In the first section we'll derive the D-dimensional Poincaré algebra, after which the four dimensional Poincaré algebra is studied in the second section. In the third section we'll show how to obtain the four dimensional Poincaré algebra by a so called Wigner-Inönü contraction on the Anti-de Sitter algebra $so(2,3)$. In the last section we'll have a go at the Newton-Wigner position operator describing a spin-0 particle on mass-shell.

1.1 The Lie algebra $\pi_{p,q}$

The Poincaré group[1] Π is defined as the group of transformations that leave the Minkowski distance invariant. Normally, this distance would be defined on a four dimensional pseudo-euclidean space $\mathcal{M}_{1,3}$, i.e. its metric $\eta_{\mu\nu}$ would have as only non-zero components $\eta_{00} = -\eta_{ii} = 1$, (i=1,2,3). For future convenience, however, it would be handy to look at a class of pseudo-euclidean spaces which are denoted by $\mathcal{M}_{p,q}$. The distance on these spaces is defined by

$$d(x,y) \;\equiv\; \eta_{\mu\nu}\,(x-y)^{\mu}(x-y)^{\nu} \tag{1.1}$$

where the non-zero elements of the metric η are given by: $\eta_{00} = \ldots = \eta_{pp} = -\eta_{p+1\,p+1} = \ldots = -\eta_{p+q\,p+q} = 1$. The set of all transformations that leave the distance (1.1) invariant will be denoted by $\Pi_{p,q}$. In complete analogy with the normal case one sees that the most general invariance transformation, written as (Λ, a), is given by

$$x^{\mu} \to (\Lambda, a)\circ x^{\mu} \;\equiv\; \Lambda^{\mu\cdot}_{\cdot\nu}x^{\nu} + a^{\mu}\,. \tag{1.2}$$

[1]Throughout this paper groups shall be denoted by capital letters, whereas Lie algebras shall be denoted by small letters.

Here a^μ is a translation on $\mathcal{M}_{p,q}$ and Λ is a matrix whose elements satisfy

$$\Lambda^{\mu\cdot}_{\cdot\nu}\,\eta_{\mu\alpha}\,\Lambda^{\alpha\cdot}_{\cdot\beta} \;=\; \eta_{\nu\beta}\,. \tag{1.3}$$

As will be readily acknowledged the set of all Λ's constitutes the group $O(p,q)$ [16] and the set of all translations forms an abelian group which will be denoted by $T_{p,q}$. From eq.(1.3) it follows , by multiplying eq.(1.3) from the right with $(\Lambda^{-1})^{\beta\cdot}_{\cdot\gamma}$ and renaming the indices, that

$$(\Lambda^{-1})^{\mu\cdot}_{\cdot\nu} \;=\; \Lambda^{\cdot\mu}_{\nu\cdot}\,, \tag{1.4}$$

where we used $\Lambda^{\cdot\beta}_{\alpha\cdot} \equiv \eta^{\beta\nu}\Lambda^{\mu\cdot}_{\cdot\nu}\eta_{\mu\alpha}$.

Having eq.(1.2) it's very easy to see that $\Pi_{p,q}$ indeed forms a Lie group. First of all there is an identity, namely $(e,0)$ with e the identity element of $O(p,q)$. The product, which of course is associative, of two elements is again an element of $\Pi_{p,q}$

$$(\Lambda_1,a_1)\circ(\Lambda_2,a_2) \;=\; (\Lambda_1\Lambda_2,\Lambda_1\cdot a_2+a_1)\,, \tag{1.5}$$

as one verifies by using eq.(1.2). Furthermore, an inverse element can be defined for any (Λ,a):

$$(\Lambda,a)^{-1} \;=\; (\Lambda^{-1},-(\Lambda^{-1})a)\,. \tag{1.6}$$

From this it is clear that $\Pi_{p,q}$ indeed forms a group, moreover, from eq.(1.5) one sees that it actually forms a semidirectproduct group [8], which will be written as:

$$\Pi_{p,q} \simeq O(p,q) \ni\!\!\!\!\supset T_{p,q}\,. \tag{1.7}$$

The foregoing relations are sufficient to derive the Lie algebra completely. In the derivation use will be made of infinitesimal transformations [46], in stead of the one parameter-subgroup method [8], because it's easier and more straightforward.

In order to derive the Lie algebra of the group $O(p,q)$, which is isomorphic to $so(p,q)$, make an infinitesimal transformation $(\Lambda,0)$ on the x^μ as defined by eq.(1.2). Writing the transformation as $\Lambda^{\mu\cdot}_{\cdot\nu} = \delta^{\mu\cdot}_{\cdot\nu} + \omega^{\mu\cdot}_{\cdot\nu}$, using this in eq.(1.3) and disregarding the, very small, ω^2 term one arrives at

$$\omega_{\mu\nu} \;=\; -\omega_{\nu\mu}\,. \tag{1.8}$$

It is a well-known fact [16] that any element in the connected part of a Lie group, containing the identity, can be written as the exponential of a linear combination of the generators of the corresponding Lie algebra. If we call the generators of $so(p,q)$ generically $M_{\mu\nu}$, and its representation $\Gamma(M_{\mu\nu})$, such an element of the (matrix)group can be written as

$$\Lambda^{\alpha\cdot}_{\cdot\beta} \;=\; \exp(-\frac{i}{2}\omega^{\mu\nu}\Gamma(M_{\mu\nu}))^{\alpha\cdot}_{\cdot\beta}\,. \tag{1.9}$$

By making, again, an infinitesimal transformation on the coordinates, but this time using eq.(1.9), one finds

$$\omega^{\alpha\cdot}_{\cdot\beta} \;=\; -\frac{i}{2}\omega^{\mu\nu}\Gamma(M_{\mu\nu})^{\alpha\cdot}_{\cdot\beta} + O(\omega^2)\,. \tag{1.10}$$

Neglecting the $O(\omega^2)$ term, a $p+q$-dimensional defining representation of the Lie algebra $so(p,q)$ is found to be

$$\Gamma(M_{\mu\nu})^{\alpha\cdot}_{\cdot\beta} \;=\; i(\eta_{\mu\beta}\delta^{\alpha\cdot}_{\cdot\nu} \;-\; \eta_{\nu\beta}\delta^{\alpha\cdot}_{\cdot\mu})\,. \tag{1.11}$$

Having found an explicit form for the $\Gamma(M)$ matrices, which form a matrix-representations of the $so(p,q)$ generators $M_{\mu\nu}$, one can make a straightforward calculation of the commutator of two M's, by virtue of the representation, and thus defining the Lie algebra $so(p,q)$ completely since the $\Gamma(M)$ matrices form a defining representation of $so(p,q)$. The result of such a calculation is

$$[M_{\mu\nu}, M_{\alpha\beta}] \;=\; i(\eta_{\mu\beta}M_{\nu\alpha} - \eta_{\mu\alpha}M_{\nu\beta} + \eta_{\nu\alpha}M_{\mu\beta} - \eta_{\nu\beta}M_{\mu\alpha})\,, \tag{1.12}$$

which gives us, as was mentioned above, the explicit form of the algebra $so(p,q)$ in terms of its generators $M_{\mu\nu}$.

From eq.(1.5) it follows that, since $T_{p,q}$ is an abelian group, all the generators P_μ of the Lie algebra $t_{p,q}$ commute, i.e.

$$[P_\mu, P_\nu] \;=\; 0\,. \tag{1.13}$$

To characterize the Lie algebra completely it is sufficient to define the remaining commutators between the M's and the P's. In order to do this write a general element of the group in the neighbourhood of unity as

$$(\Lambda, a) \;=\; \exp(-\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu} \;+\; ia^\mu P_\mu)\,. \tag{1.14}$$

Then it is found from eq.(1.5) that

$$\begin{aligned}(\Lambda^{-1},0)\circ(e,c)\circ(\Lambda,0) \;&=\; (\Lambda^{-1},0)\circ(\Lambda,c)\\ &=\; (e,\Lambda^{-1}c)\,.\end{aligned} \tag{1.15}$$

Write for an infinitesimal translation $(e,a) = 1 + ia^\mu P_\mu$, then it follows from eq.(1.15) and eq.(1.4) that P_μ transforms as a vector under $O(p,q)$, i.e.

$$(\Lambda^{-1},0)\circ P_\mu \circ (\Lambda,0) \;=\; \Lambda_{\mu\cdot}^{\cdot\nu}P_\nu\,. \tag{1.16}$$

Making an infinitesimal $(\Lambda,0)$ transformation, using eq.(1.9), on the left hand side of eq.(1.16), one finds that

$$(\Lambda^{-1},0)\circ P_\mu \circ (\Lambda,0) \;=\; P_\mu \;+\; \frac{i}{2}\omega^{\alpha\beta}[M_{\alpha\beta}, P_\mu]\,. \tag{1.17}$$

From the right hand side of eq.(1.16) it then follows that

$$=\; P_\mu + \eta_{\mu\nu}\omega^{\nu\kappa}P_\kappa \;=\; P_\mu + \frac{1}{2}\omega^{\alpha\beta}\{\eta_{\mu\alpha}P_\beta - \eta_{\mu\beta}P_\alpha\}\,. \tag{1.18}$$

Upon equating both sides of eq.(1.16) the remaining commutator is found:

$$[M_{\alpha\beta}, P_\mu] \;=\; i\{\eta_{\mu\beta}P_\alpha - \eta_{\mu\alpha}P_\beta\}\,. \tag{1.19}$$

From eqs.(1.12,1.13,1.19) it is, again, clear that the Lie algebra is a semidirect sum of two subalgebras:

$$\pi_{p,q} \simeq so(p,q) \ni t_{p,q}\,. \tag{1.20}$$

1.2 The Lie algebra $\pi_{1,3}$

1.2.1 The Cartan decomposition

Having found the algebra for $\pi_{p,q}$, it's easy to write down the algebra for $\pi_{1,3}$. It is given by eqs.(1.12,1.13,1.19)

$$\begin{aligned} [M_{\mu\nu}, M_{\alpha\beta}] &= i(\eta_{\mu\beta}M_{\nu\alpha} - \eta_{\mu\alpha}M_{\nu\beta} + \eta_{\nu\alpha}M_{\mu\beta} - \eta_{\nu\beta}M_{\mu\alpha}) , \\ [M_{\alpha\beta}, P_\mu] &= i(\eta_{\mu\beta}P_\alpha - \eta_{\mu\alpha}P_\beta) , \\ [P_\mu, P_\nu] &= 0 , \end{aligned} \tag{1.21}$$

where the indices run from 0 to 3. In some cases it may come in handy to have another form of this algebra. A form which is often used is the Cartan form [8, 11], which can be obtained as follows: Define the following generators:

$$\begin{aligned} J_i &\equiv \frac{1}{2}\epsilon_{ijk}M_{jk} , \\ K_i &\equiv -M_{0i} , \end{aligned} \tag{1.22}$$

where i=1..3. From eq.(1.12) it then follows that

$$\begin{aligned} [K_i, K_j] &= [M_{0i}, M_{0j}] \\ &= i(\eta_{0j}M_{i0} - \eta_{00}M_{ij} + \eta_{i0}M_{0j} - \eta_{ij}M_{00}) \\ &= -iM_{ij} , \end{aligned} \tag{1.23}$$

where the last term vanishes on behalf of the fact that the generators are antisymmetric in their indices, and the others drop out because of the diagonality of the metric. Upon using the next identity

$$\begin{aligned} M_{ij} &= \frac{1}{2}(M_{ij} - M_{ji}) \\ &= \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il})M_{kl} \\ &= \frac{1}{2}\epsilon_{mij}\epsilon_{mkl}M_{kl} \\ &\equiv \epsilon_{ijm}J_m , \end{aligned}$$

in eq.(1.23), one arrives at

$$[K_i, K_j] = -i\epsilon_{ijk}J_k . \tag{1.24}$$

In the same way the next commutators can be derived

$$[J_i, J_j] = i\epsilon_{ijk}J_k , \tag{1.25}$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k . \tag{1.26}$$

Splitting up the four-vector P_μ into a vector part (P_i) and a scalar part (P_0) it follows from eq.(1.13) that

$$[P_i, P_j] = [P_i, P_0] = 0 , \tag{1.27}$$

whereas from eq.(1.19) it follows that

$$\begin{aligned} [J_i, P_j] &= i\epsilon_{ijk} P_k \;, & (1.28)\\ [J_i, P_0] &= 0 \;, & (1.29)\\ [K_i, P_j] &= iP_0 \delta_{ij} \;, & (1.30)\\ [K_i, P_0] &= iP_i \;. & (1.31) \end{aligned}$$

From the eqs.(1.24-1.31) it is obvious that the J's generate the rotations and that the K's generate the Lorentz boosts.

1.2.2 The $su(2)$ decomposition

A decomposition which is especially useful for the construction of irreducible representations (irreps) of the Lorentz algebra is the following. Consider the following linear combinations

$$\begin{aligned} J_i^1 &= \frac{1}{2}(J_i + iK_i) \quad , \quad J_i = J_i^1 + J_i^2 \,, \\ J_i^2 &= \frac{1}{2}(J_i - iK_i) \quad , \quad K_i = i(J_i^2 - J_i^1)\,. \end{aligned} \tag{1.32}$$

These combinations satisfy, as one can readily convince oneself,

$$[J_i^\alpha, J_j^\beta] = i\epsilon_{ijk}\delta^{\alpha\beta} J_k^\beta \;, \tag{1.33}$$

where $\alpha, \beta = 1, 2$. This decomposition shows that the Lie algebra $so(1,3)$ is isomorphic to $su(2) \oplus su(2)$. Since every irrep of $su(2)$ is labeled by a quantum number j, which is integer or half-integer, every irrep of $so(1,3)$ can be labeled by two quantum numbers j_1, j_2 that are integer or half-integer. Let us write $D^{(j_1,j_2)}$ for an arbitrary irrep of $so(1,3)$, then this irrep is $(2j_1+1)(2j_2+1)$ dimensional, owing to the $su(2)$-structure.

In the usual case, the construction of the Clebsch-Gordon series, or coefficients, is not a trivial task. In this case, however, it is, since everything is known for $su(2)$. The Clebsch-Gordon series for example are found to be

$$\begin{aligned} D^{(j_1,j_2)} \otimes D^{(k_1,k_2)} &= D^{(j_1+k_1,j_2+k_2)} \oplus D^{(j_1+k_1-1,j_2+k_2)} \oplus D^{(j_1+k_1,j_2+k_2-1)} \\ &\quad \oplus \ldots \oplus D^{(|j_1-k_1|,|j_2-k_2|)} \,. \end{aligned} \tag{1.34}$$

So every irrep occurs only once in the decomposition (1.34), as it is also the case for $su(2)$.

The force of this decomposition lies in the fact that for all finite dimensional, linear representations the momenta, P_μ, are represented trivial.

1.2.3 The Casimir invariants

Casimir invariants for a certain Lie algebra are operators which commute with every element of that algebra, but needn't be part of the Lie algebra. In fact the Casimir operators lie in the center of the enveloping algebra of the Lie algebra (see chapter 2). Their force lies in the second lemma of Schur which states that if a matrix commutes with every element of a matrix representation, the matrix has to be a multiple of the unity matrix. So it is clear that the Casimir operators have to be in the complete set of operators with which a physical state is described. However, because of the fact that the Casimir invariants commute with everything, they not only label states but a whole set of states that transform according to a given representation, i.e. they can be used to label the representations. This is the equivalence between representations of the Poincaré group and physical particles, which was first noted by Wigner [63].

The first Casimir invariant of the Poincaré algebra is readily found by remembering that the distance (1.1) is invariant. Because this invariance also holds in momentum space one tries

$$P_\mu P^\mu = P_0^2 - \vec{P}^2 = m^2 , \tag{1.35}$$

as a Casimir for the Poincaré algebra. As it happens this actually is a Casimir invariant for the Poincaré algebra (as if you didn't know). Another invariant is found through the Pauli-Lubanski four-vector [11, 25]

$$W_\alpha = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta} M^{\beta\gamma} P^\delta . \tag{1.36}$$

With this definition it can be shown [5] that

$$\begin{aligned} [P_\alpha, W_\beta] &= 0 , & (1.37)\\ [M_{\alpha\beta}, W_\gamma] &= i(\eta_{\gamma\beta} W_\alpha - \eta_{\gamma\alpha} W_\beta) , & (1.38)\\ [W_\alpha, W_\beta] &= -i\epsilon_{\alpha\beta\gamma\delta} W^\gamma P^\delta . & (1.39) \end{aligned}$$

From the above relations it's straightforward to deduce that the length of the Pauli-Lubanski four-vector is a Casimir invariant for the Poincaré algebra.

Eq.(1.36) can most easily be expressed in terms of the generators J_i and K_i just by using the definition (1.22). A quick calculation [11] then shows that:

$$\begin{aligned} W_0 &= \vec{J} \cdot \vec{P} , \\ W_i &= P_0 J_i + \epsilon_{ijk} P_j K_k . \end{aligned} \tag{1.40}$$

Because W^2 is a Lorentz invariant, one can use eqs.(1.40) to deduce its value by evaluating it in the restframe. In this frame one has, owing to eqs.(1.35,1.40)

$$\begin{aligned} p_\mu &= (m, \vec{0}) , \\ W_0 &= 0 , \\ W_i &= m J_i , \end{aligned} \tag{1.41}$$

so that W^2 is given by

$$W^2 = -m^2 \vec{J} \cdot \vec{J} . \tag{1.42}$$

From eq.(1.25) one sees that the J_i are the generators of the $su(2)$ subalgebra for which the value of the Casimir, $\vec{J} \cdot \vec{J}$, is known. Hence we arrive at

$$W^2 = -m^2 s(s+1) , \tag{1.43}$$

where s is integer or half-integer and is called the spin of the particle which is described by the representation.

1.3 The Wigner-Inönü contraction

A new Lie algebra can be obtained, with some restrictions, by contracting another Lie algebra [8]. Take, for example, a Lie algebra spanned by generators X_i, $i = 1 \dots r$, and structure constants ${f_{ij}}^k$, which satisfy

$$[X_i, X_j] \;=\; {f_{ij}}^n X_n \; . \tag{1.44}$$

By redefining a subset of these generators like

$$Y_i \;\equiv\; \lambda X_i \; , \tag{1.45}$$

where the index i runs from 1 to $s < r$. The commutators (1.44) can now be written as

$$\begin{aligned} [X_i, X_j] &= \lambda {f_{ij}}^m Y_m \,+\, {f_{ij}}^k X_k \; , & (1.46)\\ [X_i, Y_m] &= {f_{im}}^n Y_n \,+\, \lambda^{-1} {f_{im}}^j X_j \; , & (1.47)\\ [Y_m, Y_n] &= \lambda^{-1} {f_{mn}}^p Y_p \,+\, \lambda^{-2} {f_{mn}}^i X_i \; , & (1.48) \end{aligned}$$

where $i, j, k = s+1 \dots r$ and $m, n, p = 1 \dots s$. From the equations stated above it follows that, by taking $\lambda \to \infty$, the system will again form a Lie algebra if ${f_{ij}}^m$ in eq.(1.46) be zero. The resulting algebra then forms a semidirect-sum algebra:

$$\begin{aligned} [X_i, X_j] &= {f_{ij}}^k X_k \; ,\\ [X_i, Y_m] &= {f_{im}}^n Y_n \; ,\\ [Y_m, Y_n] &= 0 \; . \end{aligned} \tag{1.49}$$

So now it is natural to look for a Lie Algebra whose contraction results in the Poincaré algebra.

It is possible to contract the $so(2,3)$, or the $so(1,4)$, algebra to the Poincaré algebra by means of a so-called Wigner-Inönü contraction [24]. The algebra $so(2,3)$ is completely specified by eq.(1.12) where the (capital) latin indices run from 0 to 4, and the greek indices will run from 0 to 3. For simplicity take the metric in eq.(1.1) to be diagonal and have signature $(+---+)$. By rescaling some of the generators M_{MN} by the anti-de Sitter radius R

$$M_{4\mu} \;\equiv\; R P_\mu \; , \tag{1.50}$$

then it can be found from eqs.(1.12), by taking $R \to \infty$, that

$$\begin{aligned} [P_\mu, P_\nu] &= R^{-2} [M_{4\mu}, M_{4\nu}]\\ &= \frac{-i}{R^2} M_{\mu\nu} \;=\; 0 \; , \end{aligned} \tag{1.51}$$

and that

$$\begin{aligned} [M_{\mu\nu}, P_\beta] &= R^{-1} [M_{\mu\nu}, M_{4\beta}]\\ &= \frac{i}{R} (\eta_{\mu\beta} M_{\nu 4} - \eta_{\mu 4} M_{\nu\beta} + \eta_{\nu 4} M_{\mu\beta} - \eta_{\nu\beta} M_{\mu 4})\\ &= i(\eta_{\nu\beta} P_\mu - \eta_{\mu\beta} P_\nu) \; . \end{aligned} \tag{1.52}$$

Upon comparing eq.(1.51) with eq.(1.13) and eq.(1.52) with eq.(1.19) it follows that by contracting the $so(2,3)$ algebra one has obtained the Poincaré algebra $\pi_{1,3}$.

The Casimir operators for $so(2,3)$ are given by [26]:

$$C_1 = \frac{1}{2}M_{AB}M^{AB} , \tag{1.53}$$
$$C_2 = W_A W^A , \tag{1.54}$$

where the W_A are defined by

$$W_A = -\frac{1}{8}\epsilon_{ABCDE}M^{BC}M^{DE} . \tag{1.55}$$

In order to do the rescaling (1.50) write the eqs.(1.53,1.55) as

$$\begin{aligned} C_1 &= \frac{1}{2}M_{\mu\nu}M^{\mu\nu} + M_{4\mu}M^{4\mu} , \\ W_\mu &= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma 4}M^{\nu\rho}M^{\sigma 4} \\ &\equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}M^{\nu\rho}M^{4\sigma} \\ &= \frac{R}{2}\epsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^{\sigma} , \end{aligned} \tag{1.56}$$
$$W_4 = -\frac{1}{8}\epsilon_{\mu\nu\rho\sigma}M^{\mu\nu}M^{\rho\sigma} . \tag{1.57}$$

Upon actually doing the rescaling (1.50) one can define new Casimir invariants that look like

$$\begin{aligned} D_1 = \frac{C_1}{R^2} &= \frac{1}{2R^2}M_{\mu\nu}M^{\mu\nu} + P_\mu P^\mu , \\ D_2 = \frac{C_2}{R^2} &= W_\mu W^\mu + (\frac{W_4}{R})^2 . \end{aligned} \tag{1.58}$$

Seeing the equivalences between the equations (1.56,1.58) and eqs.(1.36,1.35), it is clear that, upon taking the limit $R \to \infty$ one recovers from eqs.(1.58) the Casimir invariants for the Poincaré algebra $\pi_{1,3}$.

1.4 A relativistic position operator

The notion of a position operator as is used in non-relativistic quantum mechanics is well-known, and the intention of this subsection is to generalize this idea to relativistic particles. In the Schrödinger picture one can obtain this operator in momentum-space by taking the Fourier transformation of this operator, i.e. $q_k \to i\frac{\partial}{\partial p_k}$. Furthermore one knows that this operator is Hermitean with respect to the inproduct on the Hilbert space, which is used to describe the processes in the Schrödinger picture. In the relativistic case, this inproduct is different from the classical one since one has a mass-shell condition. The inproduct on momentum-space is well-known and has the form [9]

$$\begin{aligned} (\Phi,\Psi) &= \int \frac{d^4p}{(2\pi)^3}\,\theta p_0 \delta(p_\mu p^\mu - m^2)\overline{\Phi}(p)\Psi(p) \\ &= \int \frac{d^3p}{(2\pi)^3 2p_0}\,\overline{\Phi}(p)\Psi(p) , \end{aligned} \tag{1.59}$$

and one can easily convince oneselve that the old position operator isn't Hermitean to this inproduct. On this inproduct one can define something like a generalized position operator. Actually, one can define more such operators [8], but here we'll follow the account of Newton and Wigner [47]. They introduced a Hermitean operator on the form (1.59) which has as the non-relativistic limit the old position operator in the Schrödinger picture. This operator takes the form[2]

$$Q_i \;=\; i\left(\frac{\partial}{\partial p_i} - \frac{p_i}{2p_0^2}\right) \;=\; {Q_i}^\dagger \,, \tag{1.60}$$

where one should note that, since this is a representation on mass-shell[3], p_0 is just a short-hand for $\sqrt{\vec{p}^2 + m^2}$.

In order to investigate whether this position operator satisfies some obvious physical requirements, one has to calculate the action of the Poincaré algebra on this operator. This action can be found by making use of a defining representation on the mass-shell. This representation can be found to be

$$P_\mu \;=\; p_\mu \,, \qquad J_i = -i\epsilon_{ijk} p_j \frac{\partial}{\partial p_k} \,, \qquad K_i = i p_0 \frac{\partial}{\partial p_i} \,, \tag{1.61}$$

where one has to remind oneself that one is looking for transformations along the mass-shell without a spin part in the representation. By using this representation together with eq.(1.60) one can show that it is a vector under $so(3)$, that all Q's commute with each other and that together with the momenta they form a Dirac algebra, i.e. $[P_i, Q_j] = i\delta_{ij}$. So, upto now we can say that Q forms a legitimate quantum position operator.

The question whether it really forms a relativistic position operator is still not clear. Let's see whether the velocity of the particle can be calculated. Clearly, if the operator is a position operator its time evolution has to be a velocity. The time evolution of the position operator is as usual, in the Heisenberg picture, given by

$$\frac{dQ_i}{dt} \;=\; i[P_0, Q_i] \;=\; \frac{p_i}{p_0} \,. \tag{1.62}$$

Now it is widely known that in special relativity there exists a relationship between velocity and momenta, which reads

$$p_i \;=\; p_0 v_i \,, \tag{1.63}$$

where v is the velocity of the particle. Seeing these two equations, eqs.(1.62,1.63), one sees the consistency in our definitions. Therefore we can say that we have found an acceptable relativistic position operator, which is given by eq.(1.60).

At the end of this chapter let us state that eq.(1.60) is equivalent to

$$Q_i \;=\; \frac{1}{2P_0} K_i + K_i \frac{1}{2P_0} \;=\; \frac{1}{p_0} K_i - \frac{iP_i}{2P_0^2} \,, \tag{1.64}$$

if we use the spinless mass-shell representation (1.61). If we allow for a spin part in our representation and take eq.(1.64) as the definition for our position operator, the commutator of two Q's is not zero but reads

$$[Q_i, Q_j] \;=\; -i\epsilon_{ijk} \frac{1}{P_0^3} W_k \,, \tag{1.65}$$

[2]Factors like $\hbar$ and c have been put to one for sake of convenience. Upon reinstating these factors one can see that in the limit $c \to \infty$ one recovers the above stated result

[3]This means that we can use $\frac{\partial p_0}{\partial p_i} = \frac{p_i}{p_0}$ in all the calculations.

and describes a so-called spinning particle [8]. Note that the position operator defined in eq.(1.64) still satisfies eq.(1.62) and is thus, from a relativistic point of view, an acceptable position operator.

Chapter 2

Hopf algebras for pedestrians

In this chapter the notion of a Hopf algebra is introduced. People that already know something about Hopf algebras can safely skip the first section and go straight on to the second chapter, where we give the definition of a Hopf algebra. If one wants to know what it is all about, one'd better start with the first section where we intend to give a complete, however not too mathematical, introduction to Hopf algebras. The idea of Hopf algebras will then be visualized by means of some examples, the quaternions and the enveloping algebra of $su(2)$, in the third section.

2.1 Formal structure of Hopf Algebras

An algebra A over $\mathbb{C}$ is called a $\mathbb{C}$-algebra[1], or "an associative algebra over $\mathbb{C}$ with unity", if:

- The algebra A forms a vector space over $\mathbb{C}$. This means that for elements Φ, Ψ, Υ in A and a, b in $\mathbb{C}$, a scalar multiplication $a\Phi$ and an addition $\Phi + \Psi$ has to be defined whose result lies in A. Furthermore, these actions have to satisfy [16]: $a(b\Phi) = (ab)\Phi$, $\Phi + \Psi = \Psi + \Phi$, $\Phi + (\Psi + \Upsilon) = (\Phi + \Psi) + \Upsilon$, $a(\Phi + \Psi) = a\Phi + a\Psi$ and $(a + b)\Phi = a\Phi + b\Phi$. The algebra also needs to contain an element 0, so that for every element Φ in A we have $\Phi + 0 = \Phi$.

- On it one can define a multiplication map $m : A \otimes A \to A$, that is associative. This associativity is expressed by

$$m \circ (id \otimes m) \;=\; m \circ (m \otimes id) \,, \tag{2.1}$$

where 'id' means the identity map on A and '$\circ$' defines the action over something, like in $(f \circ g)(x) = f(g(x))$. Note that the above equation is a mapping working on an element of $A \otimes A \otimes A$, $a \otimes b \otimes c$ say, as $m(m(a \otimes b) \otimes c) = m(a \otimes m(b \otimes c))$. If one defines the usual multiplication $m(a \otimes b) = a \cdot b$, one can see that (2.1), working on $a \otimes b \otimes c$, results in the well-known associativity rule

$$a \cdot (b \cdot c) \;=\; (a \cdot b) \cdot c\,. \tag{2.2}$$

Hence the name of (2.1).

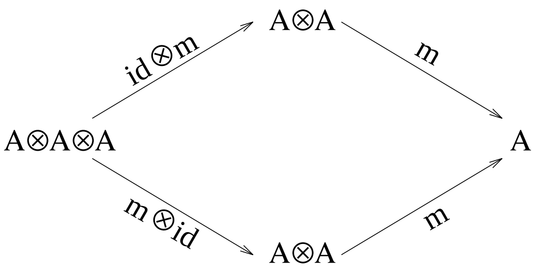
[1] $\mathbb{C}$ is the ring of complex numbers. For a generalization of $\mathbb{C}$-algebras to K-algebras, where K is an arbitrary ring, one is kindly referred to [1, 38, 43].

- The algebra contains a unit element, denoted by e, which, for all elements a of A, satisfies

$$m(a \otimes e) \;=\; m(e \otimes a) \;=\; a\,. \tag{2.3}$$

Note that the multiplication, m, may have nothing to do with the defining relations between the generators of the $\mathbb{C}$-algebra.

There exists a way of putting axioms and lemmas in terms of commuting diagrams. A diagram is called commutative if any two paths, with the same beginning and ending, along directed arrows result in the same mapping. An example of such diagrams is the following, which is equivalent to eq.(2.1),



Let's give some examples: Of course $\mathbb{C}$ is an algebra with the standard multiplication and addition rules. Also the set of all polynomials in one unknown over $\mathbb{C}$ forms an $\mathbb{C}$-algebra under the standard addition and multiplication rules.

It should be clear that a group can never be a $\mathbb{C}$-algebra because it isn't a vector space. A Lie algebra however, is a vectorspace but one runs into trouble with the second and the third point (2.1,2.3): If we take m to be the usual multiplication we don't end up with an element of the Lie algebra, whereas if we define m to be the Lie product it isn't associative due to the Jacobi identities. This fact will lead to the enveloping algebra as we'll see in a few inches.

Let's also introduce a mapping from $\mathbb{C}$ onto A, call it i for inclusion and define it by

$$i : \mathbb{C} \to A : i(\alpha) = \alpha e\,. \tag{2.4}$$

This inclusion map obviously satisfies the following commutative diagram

where the isomorphism $\mathbb{C} \otimes A \cong A$ is found by $\alpha \otimes a \equiv \alpha a$. One can show that the existence of an inclusion map which satisfies the above diagram is equivalent to the requirement of a unit element in A [1, 38].

Let's look at the space of all functionals on A, which shall be called A^* and is equal to the space of all homomorphims from A onto $\mathbb{C}$, denoted $Hom_{\mathbb{C}}(A, \mathbb{C})$ [38][2]. Since on A we

[2] A mapping α is called a $\mathbb{C}$-algebra homomorphism from a $\mathbb{C}$-algebra A to a $\mathbb{C}$-algebra B if for all a, b in A and r in $\mathbb{C}$ we have $\alpha(a+b) = \alpha(a) + \alpha(b)$ and $\alpha(ra) = r\alpha(a)$. Furthermore, a homomorphism has to satisfy $m_B \circ (\alpha \otimes \alpha) = \alpha \circ m_A$ and $\alpha(e_A) = e_B$, where m_X (e_X) is the multiplication (resp. the unit) on the $\mathbb{C}$-algebra X.

have the multiplication map, we would like to know how an element ℓ of A^* can be defined to work on elements of $A \otimes A$. Therefore we introduce a mapping $\Delta^* : A^* \to A^* \otimes A^*$ which will be defined by

$$\Delta^*(\ell) \circ (a \otimes b) \;=\; \ell \circ m(a \otimes b)\,, \tag{2.5}$$

where one should note that, formally, the left-hand-side results in an element of $\mathbb{C} \otimes \mathbb{C}$, whereas the right-hand-side is an element of $\mathbb{C}$. The fact that both sides are indeed equal follows from the (trivial?) identification $\alpha \otimes \beta \equiv \alpha\beta$. This mapping, Δ^*, is called the coproduct on A^*. Seeing the fact that m satisfies associativity, we expect Δ^*, since it is the dual of m, to satisfy some kind of associativity, which we'll call coassociativity:

$$(\Delta^* \otimes id) \circ \Delta^* \;=\; (id \otimes \Delta^*) \circ \Delta^*\,. \tag{2.6}$$

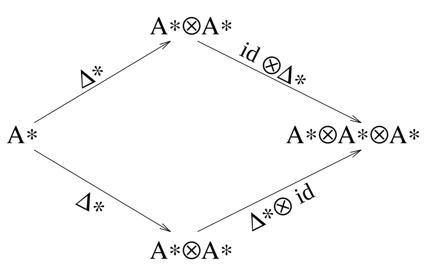
The inclusion map, i, also induces a map on A^*! We define a map $\epsilon^* : A^* \to \mathbb{C}$, called the counit, by

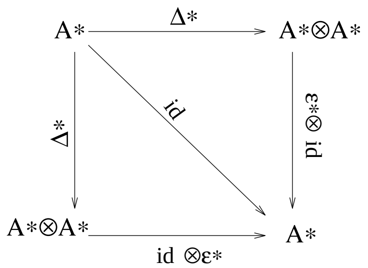
$$\epsilon^*(\ell) \;=\; \ell(e)\,, \tag{2.7}$$

which we take to satisfy

$$(id \otimes \epsilon^*) \circ \Delta^* \;=\; (\epsilon^* \otimes id) \circ \Delta^* \;=\; id\,. \tag{2.8}$$

We take this last equation because $\mathbb{C} \otimes A \simeq A$, i.e. the algebra multiplied by $\mathbb{C}$ is isomorphic to the algebra. These axioms for the coproduct and the counit on A^* are equivalent to the following commutative diagrams





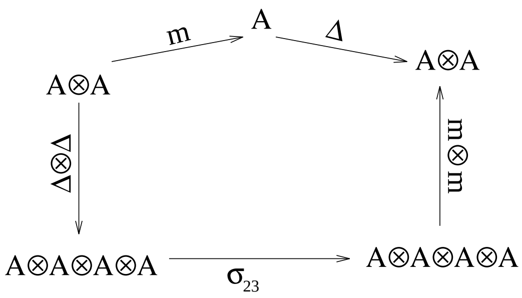
One can see that by dualizing the diagrams for m and i, i.e. by replacing $m \to \Delta^*$, $i \to \epsilon^*$, $A \to A^*$ and reversing the direction of the arrows, that coassociativity follows naturally from associativity and that the axiom for the counit follows from the fact that A is a $\mathbb{C}$-algebra.

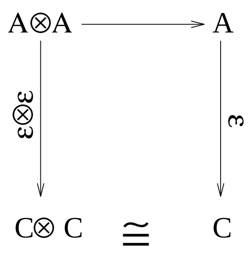
A $\mathbb{C}$-algebra is called a coalgebra if there exists a coproduct Δ and counit ϵ, on the algebra A itself, that satisfy eqs.(2.6,2.8), but then without the *'s. Note that these maps induce an associative multiplication and an inclusion map on A^*. It is clear that if A is a coalgebra, then A^* is a $\mathbb{C}$-algebra and the converse is also true.

Let's define more types of algebras: A $\mathbb{C}$-algebra is called a bialgebra if it's also a coalgebra and its coproduct, Δ, and counit, ϵ, satisfy

$$\Delta(a \cdot b) \;=\; \Delta(a) \cdot \Delta(b)\;,\; \epsilon(a \cdot b) \;=\; \epsilon(a) \cdot \epsilon(b)\,, \tag{2.9}$$

i.e. are elements of $Hom_{\mathbb{C}}(A, A \otimes A)$, resp. $Hom_{\mathbb{C}}(A, \mathbb{C})$, which means that they constitute a $\mathbb{C}$-algebra homomorphism between A and $A \otimes A$, resp. $\mathbb{C}$. The fact that they constitute $\mathbb{C}$-algebra homomorphisms can be stated elegantly by the following commutative diagram





which explicitly read

$$\Delta \circ m = (m \otimes m) \circ \sigma_{23} \circ (\Delta \otimes \Delta) \quad , \quad \epsilon \circ m = \epsilon \otimes \epsilon \,, \tag{2.10}$$

where σ_{23} permutates the second and third term in the element on which it works. Let's elucidate a bit on this equivalence. The meaning of the second diagram is clearly the same as the homomorphic character of the counit, so that we'll focus on the first diagram. Take a general coproduct for an element of A to be $\Delta(a) = \sum a_l^i \otimes a_r^i = a_l \otimes a_r$ so that we imply that the coproduct is a sum of elements. From the left-hand-side we immediatly obtain $\Delta \circ m(a \otimes b) = \Delta(ab)$. The right-hand-side can be calculated to be

$$\begin{aligned} (m \otimes m) \circ \sigma_{23} \circ (\Delta \otimes \Delta)(a \otimes b) &= (m \otimes m) \circ \sigma_{23}(a_l \otimes a_r \otimes b_l \otimes b_r) && (2.11) \\ &= (m \otimes m)(a_l \otimes b_l \otimes a_r \otimes b_r) && (2.12) \\ &= a_l b_l \otimes a_r b_r \;=\; (a_l \otimes a_r) \cdot (b_l \otimes b_r) && (2.13) \\ &\equiv \Delta(a) \cdot \Delta(b) \,, && (2.14) \end{aligned}$$

so that the diagram amounts up to saying that Δ is a homomorphism on A. It ought to be clear that the σ_{23} occurs due to the multiplication structure on $A \otimes A$.

What about the physical significance of this stuff?

Let's look at a bialgebra with a set of defining relations given, in general, by $\mathcal{F}(X) = 0$, where X are the generators of the algebra. Since the coproduct Δ is a homomorphism on the algebra it maintaines the defining relations, i.e. $\mathcal{F}(\Delta(X)) = 0$, and thus behaves as an element of the algebra. However, Δ is an element of $A \otimes A$ and can thus work on different Hilbert spaces, which may describe different particles. This means that we can use the coproduct to define the action of our algebra on composite systems, i.e. polyparticle states, and **thus**: The coproduct tells us how to add the quantum numbers for the observables! By the same reasoning we can obtain $\mathcal{F}(\epsilon(X)) = 0$, which clearly states that ϵ corresponds to a possible one dimensional representation over $\mathbb{C}$. This last point may come in handy whilst defining the counit on an algebra.

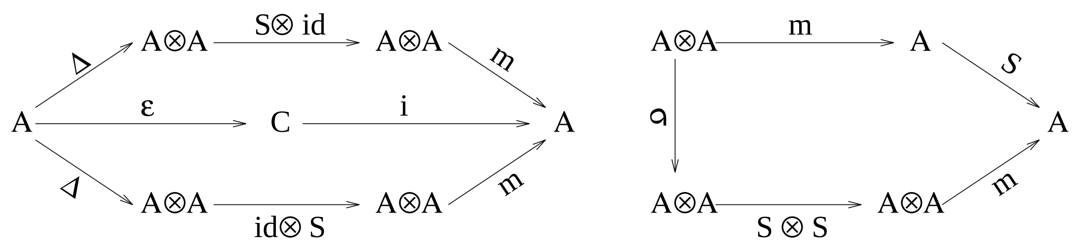
And finally: A bialgebra is called a Hopf algebra if there exists a bijective map $S : A \to A$, called the antipode, which is an anti-homomorphism on A:

$$S(a \cdot b) \;=\; S(b) \cdot S(a) \,, \tag{2.15}$$

and satisfies

$$m \circ (S \otimes id) \circ \Delta \;=\; m \circ (id \otimes S) \circ \Delta \;=\; i \circ \epsilon \,. \tag{2.16}$$

In the language of diagrams eqs.(2.16,2.15) read



Note that this antipode also induces an antipode* for A^* by the duality $S^*(\ell) \circ a = \ell \circ S(a)$. By dualizing eq.(2.16) we see that S^* has to satisfy

$$m^* \circ (S^* \otimes id) \circ \Delta^* \;=\; m^* \circ (id \otimes S^*) \circ \Delta^* \;=\; i^* \circ \epsilon^* \,. \tag{2.17}$$

This equation tells us that S^* looks like an antipode, but needn't be an antihomomorphism on A^*. There exist a theorem [1] which states that if it is possible to define on a bialgebra a mapping S that satisfies (2.16), then this mapping S is unique and is an antihomomorphism. This means that if A is a Hopf algebra then A^* is a Hopf algebra, i.e. there is a complete dualization between A and A^*!

By looking at the dual-type diagram of eq.(2.15), i.e. $S^* \circ m^* = m^* \circ (S^* \otimes S^*) \circ \sigma$, we find by dualization that

$$\Delta \circ S \;=\; \sigma \circ (S \otimes S) \circ \Delta \,, \tag{2.18}$$

which tells us how to compose the antipode and the coproduct.

2.2 Hopf algebras: The fast way

A Hopf algebra is a set of six $(A, m, i, \Delta, \epsilon, S)$ where A is a $\mathbb{C}$-algebra, with unit element e, on which we are able to define the 5 maps $m : A \to A \otimes A$, $i : \mathbb{C} \to A$, $\Delta : A \to A \otimes A$, $\epsilon : A \to \mathbb{C}$ and $S : A \to A$. These maps have to satisfy some equations:

- The multiplication on A, m, has to be associative, i.e.

$$m \circ (id \otimes m) = m \circ (m \otimes id). \tag{2.19}$$

- The inclusion map, i, is defined by $i(\alpha) = \alpha e$, for any α in $\mathbb{C}$.

- The coproduct, Δ, has to be a homomorphism on A and has to satisfy

$$(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta, \tag{2.20}$$

  which is called coassociativity.

- The counit, ϵ, has to be a homomorphism on A and has to satisfy

$$(id \otimes \epsilon) \circ \Delta = (\epsilon \otimes id) \circ \Delta = id. \tag{2.21}$$

- The antipode, S, has to be an antihomomorphism on A and needs to satisfy

$$m \circ (S \otimes id) \circ \Delta = m \circ (id \otimes S) \circ \Delta = i \circ \epsilon \,. \tag{2.22}$$

On a Hopf algebra it is possible to construct the space of all functionals, which turns out to be a Hopf algebra as well [1, 61].

The physical interest in Hopf algebras arises from the fact that the coproduct Δ tells us how to add the operators working on different vectorspaces, and thus tells us how the algebra acts on polyparticle states. In this context it is worth remarking that the counit ϵ is equal to one of the possible one-dimensional representations of A.

2.3 Two examples of Hopf algebras

The two examples will be quaternions, which won't work, and the enveloping algebra of $su(2)$ [61], which fortunately will work.

Let's look at the quaternions, generated by elements e, i, j, k that satisfy

$$\begin{array}{ccccc} i \cdot j = k & , & j \cdot i = -k & , & i \cdot i = -e \; , \\ j \cdot k = i & , & k \cdot j = -i & , & j \cdot j = -e \; , \\ k \cdot i = j & , & i \cdot k = -j & , & k \cdot k = -e \; , \end{array} \tag{2.23}$$

where e is the unit element of the quaternions. On the quaternions we define the usual multiplication of a quaternion with a $\mathbb{C}$-number and the usual definition for the addition between quaternions. It's clear that with these definition the quaternions form a vectorspace over $\mathbb{C}$.

If we want to turn the quaternions into a $\mathbb{C}$-algebra we need to define an inclusion, i, and a multiplication m. This can be done by defining

$$i(\alpha) \;=\; \alpha e \quad , \quad m(a \otimes b) \;=\; a \cdot b \; , \tag{2.24}$$

where α is a $\mathbb{C}$-number and a, b are elements of the quaternion-vectorspace. Since the multiplication on the quaternions is associative we can say that we've constructed a $\mathbb{C}$-algebra.

I already said that this doesn't form a Hopf algebra and here is the reason why: The counit, ϵ, is supposed to be a homomorphism on the quaternions and thus also on their defining relations, e.g. $\epsilon(i)\epsilon(j) = \epsilon(k)$ and $\epsilon(j)\epsilon(i) = -\epsilon(k)$. From these relations we have to conclude that $\epsilon(i, j, k) = 0$ which due to eq.(2.23) leads to $\epsilon(e) = 0$ and thus to $\epsilon(a) = 0$ for all a in the quaternions. Applying this result to the axiom for the counit, eq.(2.21), we find that '$0 = id$', which points out the impossibility of a Hopf structure on the quaternions.

How do normal Lie algebras fit into the Hopf algebras? As was said before, a Lie algebra g can never be a $\mathbb{C}$-algebra, let alone a Hopf algebra. This forces us to look for another algebra, which allows for a $\mathbb{C}$-algebra structure whilst embedding the Lie algebra. This structure is readily found in the enveloping algebra of g, denoted $U(g)$. $U(g)$ contains all the polynomials consisting of powers of the generators of g, e.g. $U(su(2))$ can contain $J_1 J_2 J_3$ whereas the Lie algebra cannot. One can easily convince oneselve that $U(g)$ indeed forms a vector space under the usual addition, and indeed is a $\mathbb{C}$-algebra with respect to the usual multiplication (like in eq.(2.2)).

As an example we're going to look at the enveloping algebra of $su(2)$, from which one can deduce the general case with great ease [61]. Take the generators of $su(2)$, (J_i), to satisfy

$$[J_i, J_j] \;=\; i\epsilon_{ijk} J_k \, . \tag{2.25}$$

To these three generators we add a unit element, e, which we define by $eJ_i = J_i e = J_i$. This enables us to look at the enveloping algebra of $su(2)$, denoted $U(su(2))$, which then defines

$U(su(2))$ as a $\mathbb{C}$-algebra with the usual addition and multiplication defined on it. Due to the defining relations, i.e. the Lie algebra structure, we can order any element of $U(su(2))$ to the form

$$\alpha e + \sum_{n,m,p=0}^{\infty} \alpha_{n,m,p} J_1^n J_2^m J_3^p \;\equiv\; \alpha e + \alpha_{\{n\}} J^{\{n\}} \;, \tag{2.26}$$

where the α's are just plain $\mathbb{C}$-numbers.

We'll not bother ourselves with $U^*(su(2))$ since if we satisfy (2.19-2.22) we get the Hopf structure of $U^*(su(2))$ for free. So we need to seek a coproduct and a counit which have to satisfy (2.20,2.21) in order to define a bialgebra on $U(su(2))$. Since these maps are homomorphisms we know that they have to satisfy

$$[\epsilon(J_i), \epsilon(J_j)] = i\epsilon_{ijk}\epsilon(J_k) \quad , \quad [\Delta(J_i), \Delta(J_j)] = i\epsilon_{ijk}\Delta(J_k) \;. \tag{2.27}$$

Because $\epsilon(J_i)$ is a number we have to conclude that $\epsilon(J_i) = 0$. The bialgebra structure of the unit element is readliy found by defining

$$\Delta(e) \;=\; e \otimes e \quad , \quad \epsilon(e) \;=\; 1 \;. \tag{2.28}$$

Note that it was due to the zero-ness of the counit on the quaternions that the quaternions couldn't form a bialgebra; A problem which doesn't occur here.

Now we are in a position to deduce the coproduct. By using eq.(2.21) on J_i we find that $(\epsilon \otimes id) \circ \Delta(J_i) = J_i$ and $(id \otimes \epsilon) \circ \Delta(J_i) = J_i$, from which we have to conclude that

$$\Delta(J_i) \;=\; J_i \otimes e + e \otimes J_i \;\equiv\; J_i \oplus J_i \;. \tag{2.29}$$

Because we said that the coproduct is a homomorphism on $U(su(2))$, we need to check whether the coproduct satisfies eq.(2.27). This is done by

$$\begin{aligned} [\Delta(J_i), \Delta(J_j)] &= [J_i \oplus J_i, J_j \oplus J_j] \\ &= [J_i \otimes e, J_j \otimes e] + [e \otimes J_i, e \otimes J_j] \\ &= [J_i, J_j] \otimes e + e \otimes [J_i, J_j] \\ &= i\epsilon_{ijk}(J_k \oplus J_k) \;\equiv\; i\epsilon_{ijk}\Delta(J_k) \;, \end{aligned} \tag{2.30}$$

where use has been made of the relation $[A \otimes B, C \otimes D] = [A, C] \otimes BD + CA \otimes [B, D]$. Also we need to check whether the coproduct is coassociative, i.e. satisfies eq.(2.20). This is left to the reader, however, since the proof of coassociativity of the coproduct (2.29) needs some very extensive use of Galois-theory [61].

Now that we know that the Δ we've found is a homomorphism on the defining relations we extend this structure to the whole enveloping algebra by definig it to be a homomorphism on $U(su(2))$. Let's show that this imposing of homomorpic character is consistent. Consistency means that the definition of homomorphicness has to be compatible with eq.(2.10). It is clear that if we apply the rule (2.10) on $e \otimes e$ and on $e \otimes J^{\{n\}}$ it is satisfied trivially. Therefore look at

$$\Delta \circ m(J^{\{n\}} \otimes J^{\{m\}}) \;=\; \Delta(J^{\{n\}} J^{\{m\}}) \;\equiv\; \Delta(J)^{\{n\}} \Delta(J)^{\{m\}} \;. \tag{2.31}$$

This can be rewritten by use of

$$
\begin{aligned}
\Delta(J)^{\{n\}} &= \Delta(J_1)^n \Delta(J_2)^m \Delta(J_3)^p \;=\; (J_1 \oplus J_1)^n (J_2 \oplus J_2)^m (J_3 \oplus J_3)^p \\
&= \left[\sum_{\nu=0}^{n} \begin{pmatrix} n \\ \nu \end{pmatrix} J_1^{n-\nu} \otimes J_1^{\nu}) \right] * \text{analogue terms for } J_2 \text{ and } J_3 \\
&= \sum_{\nu,\mu,\eta=0}^{n,m,p} \begin{pmatrix} n \\ \nu \end{pmatrix} \begin{pmatrix} m \\ \mu \end{pmatrix} \begin{pmatrix} p \\ \eta \end{pmatrix} J_1^{n-\nu} J_2^{m-\mu} J_3^{p-\eta} \otimes J_1^{\nu} J_2^{\mu} J_3^{\eta} \\
&\equiv \sum_{\{\nu\}=0}^{\{n\}} \begin{pmatrix} \{n\} \\ \{\nu\} \end{pmatrix} J^{\{n-\nu\}} \otimes J^{\{\nu\}} \,,
\end{aligned}
\tag{2.32}
$$

to the form

$$
(2.31) \;=\; \sum_{\{\nu\},\{\mu\}=0}^{\{n\},\{m\}} \begin{pmatrix} \{n\} \\ \{\nu\} \end{pmatrix} \begin{pmatrix} \{m\} \\ \{\mu\} \end{pmatrix} J^{\{n-\nu\}} J^{\{m-\mu\}} \otimes J^{\{\nu\}} J^{\{\mu\}} \,. \tag{2.33}
$$

The other side can now be calculated with great ease

$$
\begin{aligned}
m \otimes m \circ \sigma_{23} \circ \Delta \otimes \Delta(J^{\{n\}} \otimes J^{\{m\}}) \;&=\; \sum_{\{\nu\},\{\mu\}=0}^{\{n\},\{m\}} \begin{pmatrix} \{n\} \\ \{\nu\} \end{pmatrix} \begin{pmatrix} \{m\} \\ \{\mu\} \end{pmatrix} \\
&\quad \times m \otimes m \circ \sigma_{23} (J^{\{n-\nu\}} \otimes J^{\{\nu\}} \otimes J^{\{m-\mu\}} \otimes J^{\{\mu\}}) \\
&= \sum_{\{\nu\},\{\mu\}=0}^{\{n\},\{m\}} \begin{pmatrix} \{n\} \\ \{\nu\} \end{pmatrix} \begin{pmatrix} \{m\} \\ \{\mu\} \end{pmatrix} J^{\{n-\nu\}} J^{\{m-\mu\}} \otimes J^{\{\nu\}} J^{\{\mu\}} .
\end{aligned}
\tag{2.34}
$$

Comparing these two results, we have to conclude that our definition is consistent, i.e. our coproduct is well-defined.

The most economic way of defining the antipode is by using eq.(2.22) as the guiding principle. If we use eq.(2.22) on the unit e and the J's we find

$$
\begin{aligned}
& m \circ (S(e) \otimes e) \;=\; m \circ (e \otimes S(e)) \;=\; i \circ \epsilon(e) \;=\; e \\
& S(J_i) \,+\, S(e) J_i \;=\; J_i S(e) \,+\, S(J_i) \;=\; i \circ \epsilon(J_i) = 0 \,.
\end{aligned}
\tag{2.35}
$$

From the first equation we have to conclude that $S(e) = e$, whereupon we find from the second that $S(J_i) = -J_i$. This structure is then enlarged to the whole $U(su(2))$ by imposing that it is an antihomomorphism. By methods analogous to the ones used for the coproduct one can see that this definition is consistent with the antipode diagrams and eq.(2.18).

From these examples it is clear that the defining relations play a major role in the construction of a Hopfian structure. From the quaternion example we know that the defining relations forced the counit to be zero for the whole algebra, which made it impossible to define a bialgebra structure. In the second example we constructed the coproduct by virtue of eq.(2.21), and then showed that this coproduct satisfies eq.(2.27). This construction also works the other way: First we find a coproduct on the generators which satisfies eq.(2.27) and then we can show that it satisfies the axioms.

These remarks will be the key idea for finding the Hopf structure on the algebras we're going to construct: First we define the coproduct and counit compatible with the defining relations. Then we define the antipode by virtue of eq.(2.22). Afterwards one can check that the then defined Hopfian structure is consistent with the diagrams for homomorphisms and antihomomorphisms.

Chapter 3

The Lukierski-Nowicki-Ruegg algebra

The Lukierski-Nowicki-Ruegg (LNR) method to a deformed Poincaré algebra is based on a family of Hopf algebras , which contains, as a special case, the Lie-algebra. This family can be obtained by the Drinfel'd-Jimbo method, which is applicable to any simple Lie algebra. As was mentioned before the Poincaré algebra isn't (semi)simple[1]. The Poincaré algebra can be obtained by a Wigner-Inönü contraction of the simple Lie algebra $so(2,3)$. The LNR-method consists then of a Wigner-Inönü contraction of the Drinfel'd-Jimbo Hopf algebra, normally denoted $so_q(2,3)$ or $U_q(so(2,3))$, containing the $so(2,3)$ algebra. This method has the advantage that, owing to the procedure, we can form out of two degrees of freedom, by the contraction, one degree of freedom, which will be interpreted as a mass-scale, or equivalently a minimal length.

3.1 Cartan classification and $so(2,3)$

The Cartan-classification of simple Lie-algebras is well-known [16]. In terms of this classification $so(2,3)$, the so-called Anti-de Sitter algebra, is isomorphic to C_2 and B_2. The fact that C_2 and B_2 are isomorphic follows immediately from their Cartan matrices A_{ij}, i.e.

$$A_{C_2} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} , \; A_{B_2} = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} . \tag{3.1}$$

Seeing this isomorphism we'll drop B_2 as an object of our interest, and we'll focus on C_2 in the rest of this construction.

Out of the knowledge of the Cartan matrix one can construct the whole Lie-algebra. At least, we were taught that it was possible. As an example the whole construction will be done for C_2, which will allow us to identify the elements of C_2 with those of $so(2,3)$.

Since the Cartan matrix is two-dimensional, there are two principle roots, which will be called α_1 and α_2. In this case the principle roots can be deduced very easily, in fact trivially,

[1] Remember that a semisimple Lie algebra can always be written as a direct sum of simple Lie algebras.

by noting that the Cartan matrix can be 'symmetrized'[2] by

$$A_{ij} \equiv \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = \frac{2}{(\alpha_i, \alpha_i)} \cdot (\alpha_i, \alpha_j)\,, \tag{3.2}$$

where $(,)$ means a bilinear form on the root-space. Using this diagonalization one easily finds $(\alpha_1, \alpha_1) = 1$, $(\alpha_1, \alpha_2) = -1$, $(\alpha_2, \alpha_2) = 2$, which leads to $\alpha_1 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and $\alpha_2 = (0, \sqrt{2})$. It is common-knowledge that the operators associated with every simple root α_i,here denoted h_i and $e_{\pm i}$, enjoy

$$\begin{aligned} [h_i, h_j] = 0 \quad &, \quad [e_{+i}, e_{-j}] = -\delta_{ij} h_j\,, \\ [h_i, e_{\pm j}] &= \pm A_{ij} e_{\pm j}\,, \end{aligned} \tag{3.3}$$

as some of their commutation relations. A very useful base is found by defining $H_i = \frac{2}{\alpha_{ii}} h_i$, $E_{\pm i} = \pm\sqrt{\frac{2}{\alpha_{ii}}} e_{\pm i}$, where we have defined $\alpha_{ij} = (\alpha_i, \alpha_j)$. In this base, often called the (Cartan-)Chevalley base, the commutation relations take the form

$$\begin{aligned} [H_i, H_j] = 0 \quad &, \quad [E_i, E_{-j}] = \delta_{ij} H_i\,, \\ [H_i, E_{\pm j}] &= \pm \alpha_{ij} E_{\pm j}\,. \end{aligned} \tag{3.4}$$

Note however, that these commutation relations do not define the commutation relations between E_1 and E_2. These commutation relations will have to be defined in a consistent way as to support the general theory of simple Lie algebras. This definition is found out of a theorem, which states that if one denotes the subalgebras corresponding to one root α, $\mathcal{L}_\alpha$ then

$$[\mathcal{L}_\alpha, \mathcal{L}_\beta] \subseteq \mathcal{L}_{\alpha+\beta}\,, \tag{3.5}$$

where we look upon the Cartan subalgebra as being $\mathcal{L}_0$. This means that we can identify, up to some scaling,

$$E_3 = [E_1, E_2]\,, \ \text{etc...} \tag{3.6}$$

In order to determine the exact form of the root-system, i.e. how many non-simple roots exist, we are helped by the Serre relations

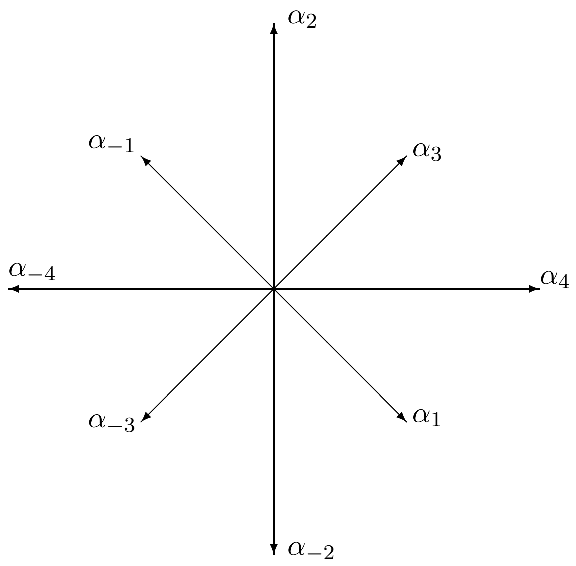
$$i \neq j: \quad (ad\, E_{\pm i})^{1-A_{ij}}\, E_{\pm j} = 0\,. \tag{3.7}$$

A more geometrical interpretation of the construction of root-systems can be found by using the notion of an α-string (of roots) containing β. The vastness of this string, defined as $\beta + k\alpha$ with $-p \leq k \leq q$, can be found by using

$$p - q \leq \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\,. \tag{3.8}$$

Note that this string-relation is equivalent to the Serre relations, as can be seen by the homomorphism $E_{\pm i} \to \alpha_{\pm i}$, $[E_\alpha, E_\beta] \to \alpha + \beta$. Equipped with this knowledge one can construct the whole root-system, see fig.(3.1), from which it follows that the thus constructed algebra contains 10 independent generators.

[2] A Cartan matrix, A, is called symmetrizable if an invertible matrix D exists so that the composition DA is symmetric.



$$\alpha_1 = (\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}})$$
$$\alpha_2 = (0, \sqrt{2})$$
$$\alpha_3 = (\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}})$$
$$\alpha_4 = (\sqrt{2}, 0)$$

$$\alpha_{-1} = (\tfrac{1}{-\sqrt{2}}, \tfrac{1}{\sqrt{2}})$$
$$\alpha_{-2} = (0, -\sqrt{2})$$
$$\alpha_{-3} = (-\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}})$$
$$\alpha_{-4} = (-\sqrt{2}, 0)$$

Figure 3.1: The rootsystem for C_2.

The generators corresponding to the non-simple roots can thus be defined by[3]

$$\begin{aligned} E_3 &= [E_1, E_2] \quad , \quad E_{-3} = [E_{-2}, E_{-1}] , \\ E_4 &= [E_1, E_3] \quad , \quad E_{-4} = [E_{-3}, E_{-1}] , \end{aligned} \tag{3.9}$$

Now, all the knowledge needed to calculate the rest of the commutation relations is available. Let's see what is meant by this.

First of all the action of the H's on the E's can be extended to the whole algebra, remember that this action is defined like a linear functional [8, 16], by putting

$$[H_i, E_{\pm j}] = \pm\alpha_{ij} E_{\pm j} , \tag{3.10}$$

where $i, j = 1 \ldots 4$ and we have defined, in correspondence with the rootsystem

$$H_3 = H_1 + H_2 , \ H_4 = H_2 + 2H_1 . \tag{3.11}$$

Secondly, the rest of the commutation relations can be calculated by using eqs(3.4,3.7,3.9) and the Jacobi identities, e.g.

$$\begin{aligned} [E_3, E_{-1}] &= [[E_1, E_2], E_{-1}] \\ &= [E_1, [E_2, E_{-1}]] - [E_2, [E_1, E_{-1}]] \\ &= [H_1, E_2] = -E_2 . \end{aligned} \tag{3.12}$$

Doing this type of calculation a lot of times, one is bound to end up with $(i, j = 1, 2; k, l = 1 \ldots 4)$

$$\begin{array}{lll} [E_i, E_{-j}] = \delta_{ij} H_i & , & [H_k, H_l] = 0 , \\ [H_k, E_{\pm l}] = \pm\alpha_{kl} E_{\pm l} & , & \\ [E_3, E_{-3}] = H_3 & , & [E_4, E_{-4}] = H_4 , \\ [E_1, E_4] = 0 & , & [E_4, E_3] = 0 , \\ [E_3, E_2] = 0 & , & [E_2, E_4] = 0 , \\ [E_{-1}, E_{-4}] = 0 & , & [E_{-2}, E_{-4}] = 0 , \\ [E_{-2}, E_{-3}] = 0 & , & [E_{-3}, E_{-4}] = 0 , \\ [E_1, E_{-3}] = -E_{-2} & , & [E_3, E_{-1}] = -E_2 , \\ [E_1, E_{-4}] = -E_{-3} & , & [E_4, E_{-1}] = -E_3 , \\ [E_2, E_{-3}] = E_{-1} & , & [E_3, E_{-2}] = E_1 , \\ [E_2, E_{-4}] = 0 & , & [E_4, E_{-2}] = 0 , \\ [E_3, E_{-4}] = E_{-1} & , & [E_4, E_{-3}] = E_1 . \end{array} \tag{3.13}$$

It is important to identify this algebra with the $so(2, 3)$ algebra at this point, because after the Drinfel'd-Jimbo quantization the same identification, for an obvious reason, will be applied. And thus one looks for an identification between eq.(3.13) and the algebra $so(2, 3)$, given by eq.(1.12), which can be found by identifying

$$\begin{array}{lll} M_{12} = H_1 & , & M_{23} = \frac{1}{\sqrt{2}}(E_1 + E_{-1}) , \\ M_{31} = \frac{1}{i\sqrt{2}}(E_1 - E_{-1}) & , & M_{04} = H_3 , \\ M_{34} = \frac{1}{\sqrt{2}}(E_{-3} - E_3) & , & M_{03} = \frac{1}{i\sqrt{2}}(E_3 + E_{-3}) , \\ M_{02} = \frac{1}{2}(E_4 - E_{-4} + E_2 - E_{-2}) & , & M_{01} = \frac{i}{2}(E_4 + E_{-4} - E_2 - E_{-2}) , \\ M_{24} = \frac{1}{2i}(E_4 + E_{-4} + E_2 + E_{-2}) & , & M_{14} = \frac{1}{2}(E_4 - E_{-4} - E_2 + E_{-2}) , \end{array} \tag{3.14}$$

[3]The fact that for positive roots we use an anti-clockwise, and for negative roots a clockwise, orientation of the Lie-product, is a mere conventional feat. One might as well take everything (anti-)clockwise and the two algebras would be isomorphic.

where the use of the same metric as was used in section (1.3) is highly recommended.

Now that we know the identification between C_2 and $so(2,3)$, and we've refreshen our memory about the Cartan classification of simple Lie algebras, we can go about our business and try to explain the Drinfel'd-Jimbo quantization of simple Lie algebras.

3.2 The Drinfel'd-Jimbo method

The Drinfel'd-Jimbo method [19, 27] is a way to construct a family of non-cocommuting Hopf algebras, which embeds (semi)simple Lie algebras. The principle idea in the Drinfel'd-Jimbo method is to look at a deformation of the Serre relations, which allows for a Hopfian structure. Since the resulting algebra forms a Hopf algebra, we are sure that one can define the action of the algebra on *poly*particle states, which is just what one needs when talking about Field theories.

Drinfel'd and Jimbo formulated their algebra, by making use of a symmetrizable Cartan matrix and a set of simple roots α_i ($i = 1 \dots N$). Conjugated to these simple roots they introduced, as in the foregoing section, operators $h_i, e_{\pm i}$ but now subjected to different commutation relations[4]

$$\begin{array}{ll} [h_i, h_j] = 0 \quad , & [h_i, e_{\pm j}] = \pm \alpha_{ij} e_{\pm j} \;, \\ [e_i, e_{-j}] = \delta_{ij} [h_i]_q \quad , & [x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}} \;, \end{array} \tag{3.15}$$

$$i \neq j \; : \quad \sum_{\nu=0}^{1-A_{ij}} (-1)^\nu \left[\begin{array}{c} 1 - A_{ij} \\ \nu \end{array} \right]_{q^{d_i}} e_{\pm i}^{1-A_{ij}-\nu} e_{\pm j} e_{\pm i}^\nu = 0 \,, \tag{3.16}$$

where d_i is defined as $d_i = \frac{\alpha_{ii}}{2}$. In eq.(3.16) use has been made of the so-called q-binomial [20], which is a generalisation of the normal binomial. It is defined as

$$\left[\begin{array}{c} m \\ n \end{array} \right]_t = \left\{ \begin{array}{ll} \prod_{i=1}^n \frac{t^{m-i+1} - t^{-m+i-1}}{t^i - t^{-i}} & \text{if } m > n > 0 \\ 1 & \text{if } n = 0 \text{ or } m = n \end{array} \right. \tag{3.17}$$

It is easy to see that eq.(3.16) is a straightforward q-generalization of the Serre relations (3.7). Using the principle roots for the case C_2, we can calculate the q-Serre relations to be

$$\begin{array}{rcl} e_1^3 e_2 - (q + 1 + q^{-1}) e_1^2 e_2 e_1 + (q + 1 + q^{-1}) e_1 e_2 e_1^2 - e_2 e_1^3 & = & 0 \,, \\ e_2^2 e_1 - (q + q^{-1}) e_2 e_1 e_2 + e_1 e_2^2 & = & 0 \,, \end{array} \tag{3.18}$$

and analogous relations for the e_{-j}'s. Note that this algebra, in the form it was given, is a closed algebra in itself. It is, however, not very enlightening to be working with polynomials of generators when one is working with algebras that look like normal Lie algebras. Therefore one introduces auxiliar generators, which are not needed but make the structure of the algebra clearer [14, 66].

However, a multitude of possible choices for these auxiliar operators exists. One would like that these generators behave like co-roots in the Lie algebra case and, thus, that they satisfy some kind of relation like eq.(3.9). One such choice [14, 57, 66] can be found by introducing a generalized commutator on the roots. This commutator is, for roots α and β, defined by

$$[e_\alpha, e_\beta]_\star \;=\; e_\alpha e_\beta - q^{\pm(\alpha,\beta)} e_\beta e_\alpha \,, \tag{3.19}$$

[4]The original form [19, 27] deviates from this one, but fortunately the forms are related by kosher transformations.

where the $-$ sign occurs if one uses clockwise and the $+$ sign if one uses anti-clockwise ordening[5]. Analogous to the definitions (3.9) one can now introduce generators, that in the limit $q \to 1$ behave as generators of the Lie algebra, where one has to pay attention to the ordening.

$$\begin{aligned} e_3 \equiv [e_1, e_2]_\star &= e_1 e_2 - q^{-\alpha_{12}} e_2 e_1 \equiv [e_1, e_2]_q , \\ e_{-3} \equiv [e_{-2}, e_{-1}]_\star &= e_{-2} e_{-1} - q^{-1} e_{-1} e_{-2} = [e_{-2}, e_{-1}]_{q^{-1}} , \\ e_4 \equiv [e_1, e_3]_\star &= e_1 e_3 - q^{-\alpha_{13}} e_3 e_1 \equiv [e_1, e_3] , \\ e_{-4} \equiv [e_{-3}, e_{-1}]_\star &= e_{-3} e_{-1} - e_{-1} e_{-3} = [e_{-3}, e_{-1}] . \end{aligned} \tag{3.20}$$

By using these operators the q-Serre relations can be written as

$$\begin{aligned} 0 = [e_1, e_4]_\star &= [e_1, [e_1, [e_1, e_2]_q]]_{q^{-1}} , \\ 0 = [e_3, e_2]_\star &= [[e_1, e_2]_q, e_2]_{q^{-1}} , \end{aligned} \tag{3.21}$$

and, of course, analogous relations for the e_{-i}'s. By now it should be clear why one went through all this trouble by defining these auxiliar operators.

Upon using equations (3.15,3.18,3.20) one can calculate the remaining commutation relations, and thus complete the form of this q-deformed C_2, generally denoted $U_q(C_2)$. A, tedious, calculation then results in the complete algebra, which reads $(i, j = 1, 2)$

$$\begin{array}{lll} [e_i, e_{-j}] = \delta_{ij} [h_i]_q & , & [h_i, h_j] = 0 , \\ [h_i, e_{\pm j}] = \pm \alpha_{ij} e_{\pm j} & , & \\ [e_3, e_{-3}] = [h_3]_q & , & h_3 = h_1 + h_2 , \\ [e_4, e_{-4}] = [h_4]_q & , & h_4 = h_2 + 2h_1 , \\ [e_1, e_4]_{q^{-1}} = 0 & , & [e_{-4}, e_{-1}]_q = 0 , \\ [e_4, e_3]_{q^{-1}} = 0 & , & [e_{-3}, e_{-4}]_q = 0 , \\ [e_3, e_2]_{q^{-1}} = 0 & , & [e_{-2}, e_{-3}]_q = 0 , \\ [e_1, e_{-3}] = -q^{-h_1} e_{-2} & , & [e_3, e_{-1}] = -e_2 q^{h_1} , \\ [e_1, e_{-4}] = -q^{h_1} e_{-3} & , & [e_4, e_{-1}] = -q^{h_1} e_3 , \\ [e_2, e_{-3}] = e_{-1} q^{h_2} & , & [e_3, e_{-2}] = q^{-h_2} e_1 , \\ [e_2, e_4] = (1 - q^{-1}) e_3^2 & , & [e_{-2}, e_{-4}] = (q - 1) e_{-3}^2 \\ [e_2, e_{-4}] = -(1 - q) e_{-1}^2 q^{h_2} & , & [e_4, e_{-2}] = -(1 - q^{-1}) q^{-h_2} e_1^2 , \\ [e_3, e_{-4}] = e_{-1} q^{h_3} & , & [e_4, e_{-3}] = q^{-h_3} e_1 . \end{array} \tag{3.22}$$

Seeing this, it ought to be clear that this really is an extention of a normal Lie algebra.

The original algebra (3.15,3.16) can be equipped with a Hopfian structure by subdueing it with a few maps, as can be found in chapter 2. The maps that deserve attention (the remaining maps can always be defined) are the coproduct and the antipode, which take care of the action on tensor spaces and the inverse in the algebra. These mappings can then be enlarged to the algebra (3.22) by using the (anti-)homomorphic character of the (antipode) coproduct.

First of all, the coproduct for the system (3.15,3.16) is given by [19, 27]

$$\begin{aligned} \Delta(h_i) &= h_i \otimes 1 + 1 \otimes h_i , \\ \Delta(e_{\pm i}) &= q^{\frac{h_i}{2}} \otimes e_{\pm i} + e_{\pm i} \otimes q^{-\frac{h_i}{2}} , \end{aligned} \tag{3.23}$$

[5] It is evident that also this is a convention. This convention was chosen, on behalf of consistency with the last section

and it acts like a homomorphism. Then we can use eqs.(3.20,3.23) to calculate

$$\begin{aligned} \Delta(e_3) &= \Delta(e_1e_2 - qe_2e_1) = \Delta(e_1)\Delta(e_2) - q\Delta(e_2)\Delta(e_1) \\ &= e_3 \otimes q^{\frac{h_3}{2}} + q^{-\frac{h_3}{2}} \otimes e_3 - \lambda q^{-\frac{h_2}{2}} e_1 \otimes e_2 q^{\frac{h_1}{2}} \,, \end{aligned} \tag{3.24}$$

where we've abbreviated $q - q^{-1}$ to λ. In the same way the remaining coproducts are found to be

$$\begin{aligned} \Delta(e_{-3}) &= e_{-3} \otimes q^{\frac{h_3}{2}} + q^{-\frac{h_3}{2}} \otimes e_{-3} + \lambda q^{-\frac{h_1}{2}} e_{-2} \otimes e_{-1} q^{\frac{h_2}{2}} \,, \\ \Delta(e_4) &= e_4 \otimes q^{\frac{h_4}{2}} + q^{-\frac{h_4}{2}} \otimes e_4 + \lambda(1-q^{-1}) q^{\frac{h_2}{2}} e_1^2 \otimes e_2 q^{h_1} \\ &\quad -\lambda q^{-\frac{h_3}{2}} e_1 \otimes e_3 q^{\frac{h_1}{2}} \,, \\ \Delta(e_{-4}) &= e_{-4} \otimes q^{\frac{h_4}{2}} + q^{-\frac{h_4}{2}} \otimes e_{-4} + \lambda(q-1) q^{-h_1} e_{-2} \otimes e_{-1}^2 q^{\frac{h_2}{2}} \\ &\quad +\lambda q^{-\frac{h_1}{2}} e_{-3} \otimes e_{-1} q^{\frac{h_3}{2}} \,. \end{aligned} \tag{3.25}$$

The counit, ϵ, has to be trivial on the generators of the Drinfel'd-Jimbo algebra. One can see this immediately since the only one-dimensional representation is trivial (see chapter 2 for more details). Since they are trivial, they will not be dealt with whilst discussing the Hopf structure of the resulting algebra after contraction.

The antipode is an anti-homomorphic mapping and can be found with relative ease. Since the 'Cartan subalgebra', $\{h\}$, is abelian and the coproduct is the same as in the Lie algebra, it is paramount that

$$S(h_i) = -h_i \,. \tag{3.26}$$

The antipode for the e's can be found by using (2.22) on $e_{\pm i}$, i.e.

$$\begin{aligned} 0 &= m \cdot (id \otimes S) \cdot \Delta(e_{\pm i}) \implies \\ & S(e_{\pm i}) = -q^{\frac{h_i}{2}} e_{\pm i} q^{-\frac{h_i}{2}} = -q^{\pm \frac{\alpha_{ii}}{2}} e_{\pm i} \,. \end{aligned} \tag{3.27}$$

In the same, but now anti-homomorphic, way we can extend the antipode from the alg.(3.15) to the alg.(3.22) by putting

$$\begin{aligned} S(e_3) &= S(e_1e_2 - qe_2e_1) = S(e_2)S(e_1) - qS(e_1)S(e_2) \\ &= q^{\frac{3}{2}}(e_2e_1 - qe_1e_2) \,, \\ S(e_{-3}) &= q^{-\frac{3}{2}}(e_{-1}e_{-2} - q^{-1}e_{-2}e_{-1}) \,, \\ S(e_{\pm 4}) &= -q^{\pm 2} e_{\pm 4} \,. \end{aligned} \tag{3.28}$$

The fact that the antipodes for $e_{\pm 3}$ cannot be given in terms of $e_{\pm 3}$ needn't bother us. It is possible to extend the algebra with the antipodes, as was done in [34], but this leads to nothing new. Moreover, after the contraction, this problem vanishes.

Alas, the Hopf structure is known and this section has nothing more to offer, thus we can go on with our resolved application, which is in fact a contraction.

3.3 The construction of the LNR algebra

The general idea was to use a contraction of a q-deformed algebra, in order to get a kind of q-deformed Poincaré algebra. At this point the most straightforward thing to do would be to rescale and contract the $U_q(C_2)$ algebra as was done in chapter (1.4). However, by doing so, we'd end up with an algebra with a parameter, q, which bears no mass-scale. Of course there is nothing wrong with this idea. But one might as well use the q to introduce a mass-scale, which could act as a kind of cut-off, and thus give rise to a better renormalization behaviour of physical theories, then a mass-less q. The introduction of a mass-scale may even reduce the number of fundamental constants as it can act as a cut-off for the momenta. Studies of field theories with a cut-off confirm this [53].

Up to now q could be any complex number except 0. Looking to the coproduct one notices that, although the observables describing one-particle states may be real, the observables describing poly-particle states needn't be. Since this property is bound to remain, even, after contraction, one has, from a physicists point of view, to impose the condition that q is real. The algebra (3.22) with the reality condition on q, is then said to be the q-deformation of $so(2,3,\Re)$, denoted by $U_q(so(2,3,\Re))$.

Led by the above considerations Celeghini *et. al.*, [12], proposed the parameterisation[6]

$$q = \exp\left(\frac{1}{\kappa R}\right) , \tag{3.29}$$

where κ is a mass-scale and R is the contraction parameter that was used in chapter (1.4). Upon doing, then, the Wigner-Inönü contraction one obtains a mass-scale in the fundamental theory, which is bound to have some effects on physics. It is obvious that κ has to be a very large mass-scale. Led by these and the forthcoming ideas, [18] came to a lowest bound on κ

$$\kappa \succeq 10^{12}\text{GeV} . \tag{3.30}$$

3.3.1 The result of the contraction

The contraction itself, based on the rescaling (1.50), the identification (3.14) and eq.(3.22), is straightforward and tedious, so that it will be skipped and only the result will be given. In this case the (J, K, P) notation will be used, which gives more insight into the structure of the algebra. Furthermore, we introduce $X_\pm = X_1 \pm iX_2$.

Let us elucidate a bit on the technical details of the contraction by looking at the commutator $[K_3, P_3]$ for general q and R. Due to the identification (3.14), the definitions (1.22,1.50) and the alg.(3.22) we can write

$$\begin{aligned} [K_3, P_3] &= -[M_{03}, R^{-1}M_{43}] \\ &= \tfrac{i}{2R}[e_3 + e_{-3}.e_3 - e_{-3}] \\ &= \tfrac{i}{R}[e_3, e_{-3}] \\ &= \tfrac{i}{R}[h_3]_q = \tfrac{i}{R}[RP_0]_q . \end{aligned} \tag{3.31}$$

Upon plugging eq.(3.29) into the above expression and taking the limit $R \to \infty$, which automatically takes care of the simultaneous limit $q \to 1$, we can obtain

$$\lim_{R\to\infty}[K_3, P_3] = \lim_{R\to\infty} iR^{-1}\frac{e^{P_0/\kappa} - e^{-P_0/\kappa}}{e^{\frac{1}{R\kappa}} - e^{-\frac{1}{R\kappa}}} = i\kappa \sinh\left(\frac{P_0}{\kappa}\right) . \tag{3.32}$$

[6] Note that a multitude of possible parameterisations exists. For a review one is referred ot [33, 34].

The form of the LNR algebra, after contraction, can then be seen to be

$$
\begin{array}{lll}
[J_+,J_-]=2J_3 & , & [J_3,J_\pm]=\pm J_\pm\ , \\
[K_+,K_-]=2J_3\cosh(\frac{P_0}{\kappa})+\frac{i}{\kappa}(P_3K_3+K_3P_3)-\frac{1}{2\kappa^2}P_3^2 & , & \\
[K_3,K_\pm]=\mp e^{\mp\frac{P_0}{\kappa}}J_\pm\pm\frac{1}{2i\kappa}K_\pm P_3+\frac{1}{2\kappa}K_3P_\mp & , & \\
[J_3,K_3]=0 & , & [J_3,K_\pm]=\pm K_\pm\ , \\
[J_+,K_3]=-K_+-\frac{1}{2\kappa}J_3P_- & , & [J_-,K_3]=K_--\frac{1}{2\kappa}P_+J_3\ , \\
[J_+,K_-]=2K_3-\frac{1}{2\kappa}P_+J_+-\frac{i}{\kappa}P_3J_3 & , & [J_-,K_+]=-2K_3-\frac{1}{2\kappa}J_-P_-+\frac{i}{\kappa}P_3J_3\ , \\
[J_\pm,K_\pm]=-\frac{1}{2\kappa}J_\pm P_\mp & , & [P_\mu,P_\nu]=0\ , \\
[J_i,P_0]=0 & , & [J_i,P_j]=i\epsilon_{ijk}P_k\ , \\
[K_3,P_0]=iP_3 & , & [K_3,P_3]=i\kappa\sinh(\frac{P_0}{\kappa}) \\
[K_3,P_2]=-\frac{1}{2\kappa}P_1P_3 & , & [K_3,P_1]=\frac{1}{2\kappa}P_2P_3\ , \\
[K_\pm,P_0]=iP_1\mp P_2 & , & [K_\pm,P_3]=\frac{1}{2i\kappa}P_\mp P_3 \\
[K_\pm,P_2]=\mp\kappa\sinh(\frac{P_0}{\kappa})+\frac{1}{2\kappa}P_3^2 & , & [K_\pm,P_1]=i\kappa\sinh(\frac{P_0}{\kappa})\mp\frac{i}{2\kappa}P_3^2\ .
\end{array}
\tag{3.33}
$$

This is the original form in which the LNR algebra was put [33]. A few remarks are in order. First of all one can see that, if $\kappa\to\infty$, one recovers the Poincaré algebra, which in all cases is a '*conditio sine qua non*'. Secondly, there exists an exact $su(2)$ subalgebra, under which the momenta transform as vectors.

There exists a bijective map [22] which puts the algebra (3.33) in a more accessible form, and it is this form that is going to be used throughout the rest of this chapter. This mapping is found by defining

$$
\begin{array}{rl}
\tilde{P}_\mu=P_\mu, & \tilde{J}_\mu=J_\mu\ , \\
\tilde{K}_3=K_3-\frac{i}{2\kappa}(P_1J_1+P_2J_2)+\frac{i}{4\kappa}P_3\,, & \tilde{K}_\pm=K_\pm+\dfrac{i}{2\kappa}J_\pm P_3-\dfrac{1}{4\kappa}P_\mp\,, \\
\tilde{K}_1=K_1+\frac{i}{4\kappa}(J_1P_3+P_3J_1)\,, & \tilde{K}_2=K_2+\dfrac{i}{4\kappa}(J_2P_3+P_3J_2)\,.
\end{array}
\tag{3.34}
$$

By using this mapping on the algebra (3.33) one finds that the 'tilded' generators satisfy[7]

$$
\begin{aligned}
[J_i,P_j]= &\ \ i\epsilon_{ijk}P_k\ ,\ [J_i,P_0]=0\ , \\
[J_i,J_j]= &\ \ i\epsilon_{ijk}J_k\ ,\ [J_i,K_j]=i\epsilon_{ijk}K_k\ , \\
[K_i,P_0]= &\ \ iP_i\ ,\ [K_i,P_j]=i\delta_{ij}\kappa\sinh(\frac{P_0}{\kappa})\,, \\
[K_i,K_j]= &\ -i\epsilon_{ijk}\left[J_k\cosh(\frac{P_0}{\kappa})-\frac{1}{4\kappa^2}P_k\left(\vec{P}\cdot\vec{J}\right)\right]\ .
\end{aligned}
\tag{3.35}
$$

At this point one should notice that the 'new' boosts are also vectors under $su(2)$, and that the limit is paramount. Bacry [6] arrived at the same algebra by using methods analogous to the ones that will be used in chapter 4.

3.3.2 The addition of observables: coproducts and antipodes

The coproducts and the antipodes can be found in a similar way. One uses the coproducts for $U_q(C_2)$ and contracts them into suitable coproducts for the algebra (3.33). Then upon using

[7]Since the forthcoming algebra is going to be used instead of alg.(3.33), we'll drop the tildes from now on.

the transformation (3.34) one finds the coproducts for the algebra (3.35). The derivation of the antipodes follows the same path, but one has to keep in mind the anti-homomorphic nature of the antipode. After a small calculation one can behold the resulting coproducts and antipodes; for those who don't want to do the calculation, behold

$$\begin{aligned} \Delta(J_i) &= J_i \otimes 1 + 1 \otimes J_i \ , & S(J_i) &= -J_i \ , \\ \Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0 \ , & S(P_\mu) &= -P_\mu \ , \\ \Delta(P_i) &= P_i \otimes \exp\left(\tfrac{P_0}{2\kappa}\right) + \exp\left(-\tfrac{P_0}{2\kappa}\right) \otimes P_i \quad , \end{aligned}$$

$$\begin{aligned} \Delta(K_i) = \quad & K_i \otimes \exp\left(\frac{P_0}{2\kappa}\right) + \exp\left(-\frac{P_0}{2\kappa}\right) \otimes K_i + \frac{1}{2\kappa}\epsilon_{ijk} \exp\left(\frac{P_0}{2\kappa}\right) J_j \otimes P_k \\ & + \frac{1}{2\kappa}\epsilon_{ijk} P_j \otimes J_k \exp\left(\frac{P_0}{2\kappa}\right) \ , \\ S(K_i) = \quad & -K_i + \frac{3i}{2\kappa} P_i \ . \end{aligned} \tag{3.36}$$

From these coproducts it is clear that when the observables on one-particle states have real eigenvalues, also the observables on polyparticle states have real eigenvalues. However glad we may be with this result a minor deficiency gives rise to some concern. The problem is that the addition of observables isn't symmetric with respect to the interchanging of two particles. Actually this problem was to be expected since the algbra $U_q(C_2)$ isn't co-commutative, i.e. the coproduct defined on it isn't symmetric. It has been proposed that, when working with poly-particle states, one should use a sum over all the permutations between the particles. But in that case each term of the sum is well-defined in the Hopf sense, i.e. each term is a valid addition compatible with the LNR-algebra, whereas the complete sum is not. On the other hand, if one uses a non-symmetric coproduct it would defy the idea of identical particles, since the observables change their values upon interchanging two 'normally' identical particles.

3.3.3 The Casimirs of LNR

In general, the task of finding the Casimirs of an algebra can be a hard nut to crack. In this case one is greatly helped by the fact that one knows the limit $\kappa \to \infty$, so that one can make a few educated guesses.

The mass^2-Casimir can be found by noting that the LNR-algebra contains an exact $su(2)$ subalgebra. This means that every function $f(P_0, \vec{P}^2)$ is invariant under $su(2)$, i.e.

$$[J_i, f(P_0, \vec{P}^2)] \;=\; 0 \ . \tag{3.37}$$

Furthermore one can calculate the action of a boost on such a function to be

$$[K_i, f(P_0, \vec{P}^2)] \;=\; iP_i \left[\frac{\partial f}{\partial P_0} + 2\frac{\partial f}{\partial \vec{P}^2}\, \kappa \sinh\left(\frac{P_0}{\kappa}\right) \right] \ . \tag{3.38}$$

Putting this equation to zero, we can find the most general solution to be $f(P_0, \vec{P}^2) = f(4\kappa^2 \sinh^2\left(\frac{P_0}{2\kappa}\right) - \vec{P}^2)$. Having in mind the Poincaré limit, we are forced to put

$$4\kappa^2 \sinh^2\left(\frac{P_0}{2\kappa}\right) - \vec{P}^2 \;=\; \mu^2 \,, \tag{3.39}$$

which we naturally interpret as the mass-shell condition.

The deformed Pauli-Lubanski fourvector can be found by using an educated guess. The guess is that we look at

$$\begin{aligned} W_0 = &\ \vec{J}\cdot\vec{P}\,, \\ W_i = &\ \kappa\sinh\left(\frac{P_0}{\kappa}\right) J_i + \epsilon_{ijk}P_jK_k\,, \end{aligned} \tag{3.40}$$

and the education can be found in [22]. It is then easy to show that the above defined W's satisfy

$$\begin{aligned} [P_\mu, W_\nu] = &\ 0\,,\ [K_i, W_0] \ = iW_i\,, \\ [J_i, W_0] = &\ 0\,,\ [J_i, W_j] \ = i\epsilon_{ijk}W_k\,, \\ [K_i, W_j] = &\ i\delta_{ij}\cosh\left(\frac{P_0}{\kappa}\right) W_0 - \frac{i}{4\kappa^2}W_0\left(\vec{P}^2\delta_{ij} - P_iP_j\right)\,, \\ [W_0, W_i] = &\ -i\epsilon_{ijk}P_jW_k\,, \\ [W_i, W_j] = &\ i\epsilon_{ijk}\left[\kappa W_k\sinh\left(\frac{P_0}{\kappa}\right) - P_kW_0\cosh\left(\frac{P_0}{\kappa}\right) + \frac{1}{4\kappa^2}\vec{P}^2P_kW_0\right]\,. \end{aligned} \tag{3.41}$$

And finally, it is almost trivial to see, by use of eq.(3.41), that the deformed Pauli-Lubanski Casimir is given by

$$\mathcal{C} \ = \left[\cosh\left(\frac{P_0}{\kappa}\right) - \frac{\vec{P}^2}{4\kappa^2}\right] W_0^2 - \vec{W}^2 \ = \ -s(s+1)\mu^2\left(1 + \frac{\mu^2}{4\kappa^2}\right)\,, \tag{3.42}$$

where s denotes the spin of the representation. This deformed Pauli-Lubanski vector satisfies an orthogonality relation, like the Pauli-Lubanski vector, which reads

$$\underline{P}_\mu = \left(\kappa\sinh\left(\frac{P_0}{\kappa}\right), \vec{P}\right)\,,\ \underline{P}_\mu W^\mu \ = \ 0\,. \tag{3.43}$$

3.3.4 Mapping the Poincaré algebra onto the LNR algebra

In the theory of quantum groups it is common that there exists a mapping between the undeformed Lie algebra and the deformed algebra [17, 67]. This mapping, however, is in most cases not bijective so that a quantum group is not completely trivial, i.e. it is not just a redefinition of the generators. In the case of the LNR-algebra, such a mapping exists also [45]. It is a mapping from the Poincaré algebra, its generators will be denoted by a tilde, onto the LNR-algebra, and it reads

$$\begin{aligned} J_i = &\ \tilde{J}_i\,, \quad P_i = \tilde{P}_i\,, \quad P_0 = 2\kappa\sinh\left(\frac{\tilde{P}_0}{2\kappa}\right)\,, \\ K_i = &\ \frac{1}{2}\left[\sqrt{1 + \frac{\tilde{P}_0^2}{4\kappa^2}}, \tilde{K}_i\right] + \frac{\sqrt{1 + \frac{\tilde{P}_0^2}{4\kappa^2}} - \sqrt{1 + \frac{m^2}{4\kappa^2}}}{\tilde{P}_0^2 - m^2}\epsilon_{ijk}\tilde{P}_j\left(\tilde{P}_0\vec{\tilde{J}} - \vec{\tilde{K}}\times\vec{\tilde{P}}\right)_k\,, \end{aligned}$$

where m is the Poincaré-'mass' of the represenation used in the mapping. This mapping is however not invertible, and the LNR-algebra is therefore kosher.

3.3.5 A D-dimensional LNR-algebra

As well as the Poincaré algebra can be defined in higher dimensions, see chapter (1.1), also the LNR-algebra can be extended to higher dimensions [37, 44]. We already know that the rotation subalgebra of alg.(3.35) is exactly $su(2) \simeq so(3)$. This subalgebra will, in the D-dimensional case, be enlarged to $so(D-1)$, whose generators will be denoted by M_{ij} $(i, j = 1 \ldots D-1)$. Their mutual commutation relations have been given in chapter (1.1). Next one introduces D commuting generators of translation P_0, P_i, of which P_0 is a scalar, and P_i is a vector under $so(D-1)$. The boosts will be introduced as a vector under $so(D-1)$. Up to now, everything goes analogously to the LNR-algebra, and the commutation relations can be found in chapter (1.1). The D-dimensional algebra can then be completed by finding a suitable extention of the commutator between two boosts [6, 31]. The D-dimensional algebra obtains the form

$$
\begin{aligned}
[M_{ij}, M_{kl}] =& \quad i\left(\delta_{ik} M_{jl} + \delta_{jl} M_{ik} - \delta_{il} M_{jk} - \delta_{jk} M_{il}\right) , \\
[M_{ij}, P_k] =& \quad i\left(\delta_{ik} P_j - \delta_{jk} P_i\right) , \\
[M_{ij}, K_k] =& \quad i\left(\delta_{ik} K_j - \delta_{jk} K_i\right) , \\
[M_{ij}, P_0] =& \quad 0 \, , \; [K_i, P_0] = i P_i \, , \; [K_i, P_j] = i\delta_{ij} \kappa \sinh\left(\frac{P_0}{\kappa}\right) , \\
[K_i, K_j] =& \quad -i\left[M_{ij} \cosh\left(\frac{P_0}{\kappa}\right) - \frac{1}{4\kappa^2}\left(M_{ij} \vec{P}^2 + \sum_{k=1}^{D-1} P_i M_{jk} P_k - \sum_{k=1}^{D-1} P_j M_{ik} P_k\right)\right] .
\end{aligned}
\tag{3.44}
$$

A quick glance at the alg.(3.35) and the above algebra, tells us that the above algebra indeed is a D-dimensional version of alg.(3.35) and that the above algebra reduces to the alg.(3.35) when $D = 4$.

The algebra (3.44) can be equipped with a Hopfian structure, which, seeing the great similarity between alg.(3.35) and alg.(3.44), shouldn't deviate much from the form (3.36). In fact it isn't hard to prove that

$$
\begin{aligned}
\Delta(P_0) =& \quad P_0 \otimes 1 + 1 \otimes P_0 \, , \; S(P_0) = -P_0 \, , \\
\Delta(P_i) =& \quad P_i \otimes \exp\left(\frac{P_0}{2\kappa}\right) + \exp\left(-\frac{P_0}{2\kappa}\right) \otimes P_i \, , \; S(P_i) = -P_i , \\
\Delta(M_{ij}) =& \quad M_{ij} \otimes 1 + 1 \otimes M_{ij} \, , \; S(M_{ij}) = -M_{ij} \, , \\
\Delta(K_i) =& \quad K_i \otimes \exp\left(\frac{P_0}{2\kappa}\right) + \exp\left(-\frac{P_0}{2\kappa}\right) \otimes K_i + \frac{1}{2\kappa} \sum_{j=1}^{D-1} P_j \otimes M_{ij} \exp\left(\frac{P_0}{2\kappa}\right) \\
& -\frac{1}{2\kappa} \sum_{j=1}^{D-1} \exp\left(-\frac{P_0}{2\kappa}\right) M_{ij} \otimes P_j \, , \\
S(K_i) =& \quad -K_i + \frac{i(D-1)}{2\kappa} P_i \, ,
\end{aligned}
\tag{3.45}
$$

indeed induces a Hopfian structure on the algebra (3.44). Enough has been said about the LNR-algebra, therefore we end this section and go on to see some physical consequences of this algebra.

3.4 Some physical consequences of the LNR-algebra

In order to look into some of the changes induced by the LNR-algebra as the symmetry of spacetime, we'll look at some fundamental things that change. These things will include the Dirac equation , deformed Heisenberg uncertainty principle and we'll have a look at the resulting Minkowski space which necessarily has to be non-commutative.

3.4.1 The Dirac equation

In Poincaré invariant Field Theory, the construction of a Dirac equation (De) is well-known. There exist however a lot of possible ways to derive it. One way is to follow Diracs account (see e.g. [9]) and split the Klein-Gordon equation into a matrix equation. Although this method works also in the LNR case [22, 33], there exists a, probably more legitimate, method, based on representation theory, that gives a deformed Dirac equation (dDe) which coincides in the massless case with the one given by [22, 33]. The method which is going to be used is an extension of the procedure described in [11, 48].

It ought to be known that the De describes a physical particle which carries spin $\frac{1}{2}$. This means that the Hilbert space, used to describe the particle quantummechanically, can be written as $\mathcal{H}_{phys} = \mathcal{H}_{coord} \otimes \mathcal{H}_{spin}$, where $\mathcal{H}_{coord}$ is the Hilbert space representing the (Minkowski) coordinates and $\mathcal{H}_{spin}$ is the Hilbert space for the spin degrees of freedom. In this case the $\mathcal{H}_{spin}$ will be four dimensional since we include parity transformations in our theory[8]. The representation on $\mathcal{H}_{phys}$ is then given by the Poincaré coproduct working on the tensor space. One then has to find an invariant on the physical space, which turns out to be equal to the De.

The same method can, and will, be followed in the LNR case. A main ingredient of this method lies in the action of the operators on $\mathcal{H}_{spin}$. One can try to find such a representation by hand, but there are nicer ways. One first looks for a representation of the algebra (3.15), such a four dimensional representation can be found in [48], contracts this according to eq.(3.29), uses the transformation (3.32) and one finds a representation of the algebra (3.35). Another way is by noting that in the Poincaré case a four dimensional representation exists where the momenta are trivial, i.e. $P_\mu = 0$. By looking at the alg.(3.35) one sees that in that case one ends up with the $so(1,3)$ algebra, for which a four dimensional representation already exists [9]. So by cunningness we ended up with a faithful representation [36]

$$\begin{aligned} j_i &= \frac{i}{4}\epsilon_{ilm}\gamma_l\gamma_m \,, \\ k_i &= \frac{i}{2}\gamma_0\gamma_i \,. \end{aligned} \tag{3.46}$$

The fact that one can use a 'classical' representation in this case stems from a general statement [17], which puts forward that the lowest dimensional representations of Hopf extensions of Lie algebras are the same as the Lie ones. The only difference in representations occur in the higher dimensional ones, due to the different coproducts[9].

---

[8]Remember that the action of parity on a general Lorentz representation acts like $\mathcal{P} \cdot D^{(j,k)} = D^{(k,j)}$ so that a parity invariant representation of a spin $\frac{1}{2}$ particle must be given by $D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$.

[9]The higher dimensional representations can be obtained by the same method as in group theory, but now by using (3.36) as the tensor-product [17].

An off-mass-shell representation of the LNR-algebra on momentum space can be obtained with relative ease, i.e. the representation of the $su(2)$ subalgebra and the commutative translation algebra is known [25]. The representation of the boosts is then not hard to find. A representation of the LNR-algebra on the momentum space is

$$\begin{aligned} P_0 = &\ p_0 \ , \ P_i \ = p_i \ , \\ J_i = &\ -i\epsilon_{ijk} p_j \frac{\partial}{\partial p_k} \ , \\ K_i = &\ i\left[\kappa \sinh\left(\frac{p_0}{\kappa}\right)\frac{\partial}{\partial p_i} + p_i \frac{\partial}{\partial p_0}\right] \ , \end{aligned} \tag{3.47}$$

Now that we know the representation of the LNR algebra on the two sub-systems, we can use eq.(3.36) to define the action of the LNR algebra on $\mathcal{H}_{phys}$. This representation reads

$$\begin{aligned} \mathcal{P}_0 = &\ p_0 \ , \ \mathcal{P}_i \ = p_i \, , \\ \mathcal{J}_i = &\ J_i + j_i \ , \\ \mathcal{K}_i = &\ K_i + \exp\left(-\frac{p_0}{2\kappa}\right) k_i - \frac{1}{2\kappa}\epsilon_{ilm} j_l p_m \, . \end{aligned} \tag{3.48}$$

It is known that a spin $\frac{1}{2}$ representation of the Poincaré algebra leads to $\mathcal{W}^2 = -\frac{3}{4}m^2$ as the eigenvalue of the Pauli-Lubanski Casimir on the spin $\frac{1}{2}$ representation. In this case a straightforward calculation, which consists of using eq.(3.48) in eqs.(3.40,3.42) and some γ calculus [23], leads to

$$\mathcal{C} \ = \ -\frac{3}{4}\mu^2\left(1 + \frac{\mu^2}{4\kappa^2}\right) \ , \tag{3.49}$$

where μ is the mass of the employed representation. Clearly this result states that the representation describes a spin $\frac{1}{2}$ particle, as can be seen by comparing eq.(3.49) with eq.(3.42). Moreover, in the limit one recovers the above mentioned 'classical' relation.

As was said before, the dDe will be put forward as a central element to the representation on $\mathcal{H}_{phys}$. Once again this task can be very hard hadn't it been for our knowledge of the limiting result. One can guess and then show that

$$\mathcal{D} \ = \ \gamma_0 \kappa \sinh\left(\frac{P_0}{\kappa}\right) - \exp\left(-\frac{P_0}{2\kappa}\right)\gamma_i P_i - \frac{1}{2\kappa}\gamma_0 P_i P_i \ , \tag{3.50}$$

is indeed central to the rep.(3.48), and thus constitutes the dirac operator we wanted. A peculiar property of this dDe occurs when taking the square of this operator. One then finds that

$$\mathcal{C} \ = \ -\frac{3}{4}\mathcal{D}^2 \ , \tag{3.51}$$

so that the dDe is the square-root of the Pauli-Lubanski Casimir rather than the square-root of the mass Casimir. The dDe can then be found by combining eqs.(3.49,3.50,3.51) into

$$\mathcal{D}\Psi \ = \ \mu\left(1 + \frac{\mu^2}{4\kappa^2}\right)^{\frac{1}{2}}\Psi \ . \tag{3.52}$$

One should note however, that this dDe works only in momentum space. This is due to the fact that in this case the connection between momentum and normal space isn't clear.

As was advocated by Majid and Ruegg [42], the corresponding Minkowski space has to be non-commutative since it is the dual of the translation algebra, whose coproduct is non-co-commutative. In fact, they found that the generators of the Minkowski space have to satisfy

$$[X_i, X_j] = 0 \,,\; [X_i, X_0] = \frac{X_i}{\kappa} \,, \tag{3.53}$$

with as an invariant 'length'

$$X_0^2 - \vec{X} \cdot \vec{X} + \frac{3}{\kappa} X_0 \,. \tag{3.54}$$

The dDe on this space has then to be found by introducing differential operators on this space and doing the whole construction again.

3.4.2 A possible position operator

In this section we want to find an appropriate generalization of the Newton-Wigner position operator as was found in section (1.4). Here we'll follow Maggiore [40] who tries to find an appropriate extention of eq.(1.60) in the spin 0 representation. This spin 0 representation is the easiest example, since it excludes the $\vec{J} \cdot \vec{P}$ part in the alg.(3.35), which would make things a bit more difficult. Also in this case one can define more different physical position operators as one can see in Bacry's account [6], where kosher position operators for spinning particles are introduced.

Since it is intended to generalize the position operator, we need to define an inproduct on the space of functions corresponding to the LNR mass-shell condition (3.39). Therefore we define

$$\begin{aligned}(\Phi, \Psi) &= \int \frac{d^4p}{(2\pi)^3} \theta(p_0) \delta\left(\vec{p}^2 + \mu^2 - 4\kappa^2 \sinh^2\left(\frac{p_0}{2\kappa}\right)\right) \Phi^*(p)\Psi(p) \\ &= \int \frac{d^3p}{(2\pi)^3 2\kappa \sinh(p_0/\kappa)} \Phi^*(p)\Psi(p)\end{aligned} \tag{3.55}$$

as the inproduct on the physical Hilbert space.

On this Hilbert space we can find a representation of the LNR-algebra, which one can find by using an educated guess, to be

$$\begin{aligned}P_\mu &= p_\mu \,, \\ J_i &= -i\epsilon_{ijk} p_j \frac{\partial}{\partial p_k} \,, \\ K_i &= i\kappa \sinh\left(\frac{p_0}{\kappa}\right) \frac{\partial}{\partial p_i} \,.\end{aligned} \tag{3.56}$$

It should be noted, however, that the occurrence of the $\vec{J} \cdot \vec{P}$ part in the algebra will give rise to a more complicated form of the representation on mass-shell.

A short reflection on the nature of an extention of the Newton-Wigner operator gives some obvious restrictions. First of all the operator should be Hermitean according to the inproduct (3.55) and should reduce to eq.(1.60) in the limit $\kappa \to \infty$. Secondly, seeing that the deformation respects the $so(3)$ nature of all generators, it should be clear that the position operator should be a vector under $so(3)$. Finally, since the representation and the inproduct contain only functions of $\frac{p_0}{\kappa}$, it is only natural to demand that the extention should be done by functions of this variable. These requirements can be used to write down the most general form of the deformed Newton-Wigner operator, i.e.

$$Q_i = i\left[A\left(\frac{p_0}{\kappa}\right)\frac{\partial}{\partial p_i} - B\left(\frac{p_0}{\kappa}\right)\frac{p_i}{2p_0^2}\right] , \tag{3.57}$$

with the conditions $A(0) = B(0) = 1$. Now threefourth of the requirements has been taken care of, and we only need to impose Hermiticity with respect to the inproduct. This can be calculated straightforwardly, and implies

$$B = \frac{p_0^2}{\kappa \sinh(p_0/\kappa)}\left[\frac{1}{\kappa}\coth\left(\frac{p_0}{\kappa}\right)A - \frac{dA}{dp_0}\right] . \tag{3.58}$$

From this equation it's clear that we end up with one degree of freedom in our construction, which we don't want.

One can, however, calculate some useful commutation relations. The most important ones can be found to be

$$\begin{aligned} [Q_i, Q_j] =& \ -\frac{A}{\kappa \sinh(p_0/\kappa)}\frac{dA}{dp_0} i\epsilon_{ijk}J_k , \\ [Q_i, P_j] =& \ iA\delta_{ij} , \\ \tfrac{dQ_i}{dt} =& \ i[P_0, Q_i] = A\frac{p_i}{\kappa \sinh(p_0/\kappa)} . \end{aligned} \tag{3.59}$$

Having these results in our hands, it would be very tempting to put $A = 1$, and count our blessings that almost nothing has changed. The only change occurs in the velocity[10]. By using eq.(3.59) with $A = 1$ and eq.(3.39) one can find the velocity as a function of p to be

$$v^2(p) = \frac{p^2}{\mu^2}\left(1 + \frac{p^2}{\mu^2}\right)^{-1}\left(1 + \frac{p^2 + \mu^2}{4\kappa^2}\right)^{-1} . \tag{3.61}$$

From this result something weird emerges when looking at the ultra-violet behaviour, namely

$$v|_{ultra-violet} \sim \frac{1}{p} , \tag{3.62}$$

which seems to make no sense at all. This implies that one cannot use $A = 1$ as a good choice for the construction of a deformed position operator.

Maggiore [40] put forward a position operator for which the velocity is allowed to take any value it wants between 0 and 1, as it is the case in special relativity. The price one has to pay for this feature is a non-commutative position operator, which seeing eq.(3.53) ought to be expected, and a κ-deformed Dirac algebra. However, this κ-deformed Dirac algebra will enable us to make an estimation of κ by looking at string theory..

Maggiore proposed to deform the classical relation $p_i = p_0 v_i$ to

$$p_i = 2\kappa \sinh\left(\frac{p_0}{2\kappa}\right) v_i , \tag{3.63}$$

[10]One should note that this result is equivalent to the 'classical' definition of speed in special relativity, i.e.

$$v_i = \left.\frac{\partial p_0}{\partial p_i}\right|_{mass-shell} , \tag{3.60}$$

but now with eq.(3.39) as the appropriate mass-shell condition.

which seems to be natural since the replacement $p_0 \to 2\kappa \sinh(\frac{p_0}{2\kappa})$ also occurs in the definition of the mass-shell eq.(3.39). By identifying eq.(3.63) with $\dot{Q}_i$ in eq.(3.59), it turns out that

$$A = \cosh\left(\frac{p_0}{2\kappa}\right) . \tag{3.64}$$

With the help of eq.(3.58) one can then calculate

$$Q_i = i \cosh\left(\frac{p_0}{2\kappa}\right)\left[\frac{\partial}{\partial p_i} - \frac{p_i}{8\kappa^2 \sinh^2(p_0/2\kappa)}\right] , \tag{3.65}$$

which in its turn will lead to

$$\begin{aligned} [Q_i, Q_j] &= -\frac{1}{4\kappa^2}\epsilon_{ijk}J_k , \\ [Q_i, P_j] &= i\delta_{ij}\cosh\left(\frac{p_0}{2\kappa}\right) \\ &= i\delta_{ij}\left(1 + \frac{E^2}{4\kappa^2}\right)^{\frac{1}{2}} , \end{aligned} \tag{3.66}$$

where E^2 is short for $p^2 + \mu^2$. One should note that eq.(3.66) is the maximal κ-deformation of the Dirac algebra with commuting P_i, which is closed under the Jacobi identities [41].

The link with string theory is now readily found by looking at the Heisenberg uncertainty principle following from eq.(3.66). The general expression of the Heisenberg uncertainty principle

$$\Delta(\mathcal{A})\Delta(\mathcal{B}) \geq \frac{1}{2}|\langle[\mathcal{A}, \mathcal{B}]\rangle| , \tag{3.67}$$

can then be used to derive, up to order κ^{-2},

$$\Delta(Q_i)\Delta(P_j) \geq \frac{1}{2}\delta_{ij}\left(1 + \frac{\langle E^2\rangle}{8\kappa^2}\right) . \tag{3.68}$$

With the aid of the usual definitions $(\Delta(\mathcal{A}))^2 = \langle(\mathcal{A} - \langle\mathcal{A}\rangle)^2\rangle$, so that $\langle\mathcal{A}^2\rangle = \mathcal{A}^2 + (\Delta\mathcal{A})^2$, the above result can be rewritten as

$$\Delta(Q_i)\Delta(P_j) \geq \frac{1}{2}\delta_{ij}\left(1 + \frac{E^2 + \Delta(P)^2}{8\kappa^2}\right) . \tag{3.69}$$

In the context of string theories such a deformed Heisenberg principle occurs due to metric fluctuations [3, 29][11]. In these theories the Heisenberg uncertainty relation reads

$$\Delta(X)\Delta(P) \geq \frac{1}{2}\left(1 + \alpha^{'}(\Delta(P))^2\right) , \tag{3.70}$$

where $\alpha^{'}$ is the, so-called, inverse string tension. The expression of this inverse string tension in more fundamental values is somewhat model dependent, but a comparison between eq.(3.69), in the region $E \ll \kappa$, and a closed, quantized bosonic string [16, 28], leads to

$$\kappa \sim 2 \cdot 10^{15} GeV . \tag{3.71}$$

One should note that the two estimations given in this chapter are not in contradiction since the one given by [18] is a lowest bound on κ. Furthermore, the estimations of κ were only given to stress what was obvious all along: κ has to be a very big mass-scale.

[11] Since this thesis doesn't deal with string-related matters, we'll just state the results and refer to the literature.

3.5 Outlook and conclusions

From the foregoing sections one can see that the application of Hopf algebras to spacetime symmetries surely leads to new effects. These new effects, however, are hardly measurable since the defining mass-scale κ is at least bigger than $10^{12} GeV$, which is a bit too high for present experiments. From a theoretical point of view a lot can be done about the physical implications of the LNR algebra.

In these quests for these implications one is however obstructed by the fact that one doesn't always know the correct way how to generalize some ideas [7, 32, 40]. A prime example of this problem can be found in section (3.4.2) where there is a whole family of possible position operators. Some of these operators can be discarded as being non-physical but the rest may all be well-defined position operators and one has to call upon experiment to sift the correct one from the flock. Another problem occurs when trying to find Lagrangians for different fields. The problem is that when such a Lagrangian is found, it probably will contain higher orders of the derivatives, which makes the usual program of gauging groups impossible. This means that gauging as a way of obtaining interactions is out of the question and one has to resort to Yukawa couplings.

Finally let us get the idea that the same way of making a Hopf analogue of a spacetime symmetry can also be used on the other symmetry algebras: The Galilei and the ($D = 4$) Conformal algebra. The Galilei algebra ($\pi_{0,3}$), which describes the symmetry of non-relativistic quantum mechanics, has been deformed [22] by using the limit $c \to \infty$ on the LNR algebra. The conformal algebra was deformed [35] by using the fact that the complexification of the conformal algebra is isomorphic to $sl(4, \mathbb{C})$, which is simple. Since the conformal group is simple, there is no introduction of a mass-scale like κ, but this was to be expected since we're talking about the conformal algebra.

One should note, however, that there are more ways to contract the Drinfel'd-Jimbo family, and thus there are more possible deformations of the Poincaré algebra. For a review one is referred to [34, 35].

Chapter 4

Minimal deformed Poincaré containing an exact Lorentz

[1]In the recent years quantum deformations of space-time symmetries have attracted an appreciable amount of interest [33, 59]. Among other reasons explaining this intensive research activity is the fact that q-deformations of space-time seem to lead to some lattice pattern. Defining quantum field theories on a lattice is a well-known regularization procedure and has been used very extensively. One of the negative aspects of the most popular way to do that, which uses equidistance lattices, is that it spoils the Lorentz invariance of the theory [2]. By deforming the symmetries of space-time one can hope for improvement of the ultraviolet properties of a quantum field theory defined on it. More specifically, it is fair to hope that the space-time symmetries, although deformed, are still remembered by the theory, while in the worst case some kind of regularization is built in. Next it is natural to search for deformations of space-time symmetries that maintain as much as possible of those required by the known renormalizable field theories describing elementary particle physics. In this spirit we examine here deformations of the Poincaré algebra (PA), which do not affect the Lorentz algebra. Among other things, this will permit us to define field theories for particles with usual spin, which in turn is expected to facilitate their quantization.

There exists a well-established formalism to obtain the q-deformed counterpart of a simple Lie algebra developed by Drinfeld and Jimbo [19, 27]. However, the PA is not simple since it is the semi-direct sum of the Lorentz and the translation subalgebras: $\mathcal{P} \simeq O(3,1) \ni T_4$. As a result another scheme must be found in order to construct a deformed Poincaré algebra (dPA).

There exist, mainly, two possible constructions. The direct one involves the employment of the q-deformed Lorentz algebra, which is isomorphic to $SL_q(2, C)$ and associates to it a q-deformed four-vector, which is interpreted as the four-momentum [59, 49, 54]. This construction obviously is not aligned with the strategy we would like to develop here.

The other construction is based on the fact that the PA can be obtained by a Wigner-Inönü contraction of the simple anti-de Sitter algebra $O(3,2)$ [12, 33, 34]. Thus, one first constructs the quantum group $O_q(3,2)$ using the Drinfeld-Jimbo method and after the introduction of

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[1]This chapter has been published in collaboration with A.A. Kehagias and G. Zoupanos [31]

[2]There exist Lorentz-invariant formulations of field theories on random lattices [15].

the anti-de Sitter radius R, the contraction

$$R \to \infty \qquad , \qquad iR \log q \to \kappa^{-1} \tag{4.1}$$

is performed. A very interesting feature of this construction is the introduction of a dimensionful parameter κ and the ordinary PA is recovered in the limit $\kappa \to \infty$. The same result has been obtained by Bacry [6] who directly considered general deformations for the sets of generators (K_i, K_j), (K_i, P_0) and (K_i, P_j). Unfortunately, this dPA does not contain the Lorentz algebra either.

In addition, the light-cone depends on the energy of the photons. As a result the light-cone is not well defined since its angle is not uniquely specified and depends on the energy [6]. This creates problems with causality and the natural way to avoid such problems is again to demand the existence of the exact Lorentz subalgebra in the dPA. In that case causality is guaranteed as in the ordinary case.

4.1 Deformed Poincaré containing the exact Lorentz algebra

Let us recall some well-known properties of the PA. This is a ten-dimensional Lie algebra whose generators can be labelled J_i, K_i, P_i and P_0 $(i = 1, 2, 3)$. It is defined by the following commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k \, , \tag{4.2}$$
$$[J_i, K_j] = i\epsilon_{ijk} K_k \, , \tag{4.3}$$
$$[J_i, P_j] = i\epsilon_{ijk} P_k \, , \tag{4.4}$$
$$[J_i, P_0] = 0 \, , \tag{4.5}$$
$$[P_i, P_j] = 0 \, , \tag{4.6}$$
$$[P_i, P_0] = 0 \, , \tag{4.7}$$
$$[K_i, P_0] = iP_i \, , \tag{4.8}$$
$$[K_i, P_j] = iP_0 \delta_{ij} \, , \tag{4.9}$$
$$[K_i, K_j] = -i\epsilon_{ijk} J_k \, . \tag{4.10}$$

As may been seen from eqs.(4.2)–(4.10), J_i, K_i and P_i are three-vectors generating rotations, boosts and translations, respectively; P_0 corresponds to the energy and, according to eq.(4.5), is a scalar. The PA contains the Lorentz algebra as a subalgebra, which is generated by J_i's and K_i's and the commutation relations (4.2),(4.3),(4.10).

The PA has two quadratic Casimir invariants. They are the lengths of the momentum four-vector $P_\mu = (P_0, P_i)$:

$$P_\mu P^\mu = P_0^2 - \vec{P} \cdot \vec{P} \, , \tag{4.11}$$

and the Pauli-Lubanski four-vector $W_\mu = (J_i P_i, P_0 J_i + \epsilon_{ijk} P_j K_k)$:

$$W_\mu W^\mu = W_0^2 - \vec{W} \cdot \vec{W} \, . \tag{4.12}$$

These invariants label the representations of the PA.

The enlargement of the Lorentz group to the Poincaré group was proposed [63] as a way of describing the quantum states of relativistic particles without using the wave equations.

The states of a free particle are then given by the unitary irreducible representations of the Poincaré group. One of our main aims in the following is to show that the above enlargement of the Lorentz algebra is not unique. The solution that we propose can be seen as a deformation of the Poincaré algebra. Clearly this deformation will be defined in the enveloping algebra of the PA. The latter is generated by all possible polynomials of the generators, and the deformed algebra we are looking for will form a closed subset of this enveloping algebra. There are many ways to choose this subset, of which we have already mentioned two in the Introduction. Moreover we demand that the dPA thus constructed have a Casimir invariant of the form

$$f(P_0) - \vec{P} \cdot \vec{P}\,, \tag{4.13}$$

which will correspond to the (mass)^2 in the ordinary case.

A very important feature of the Lorentz-invariant Casimir (4.13), which is a deformation of the Casimir (4.11), is that it only changes the P_0-part of the latter. At first sight one might think that it is impossible to define a Lorentz-invariant Casimir without deforming also its $\vec{P}$-part. In fact the authors in ref.[33, 34, 6], who also demanded the existence of a Casimir invariant of the form (4.13), were led considering a deformation of the Lorentz algebra as well. Therefore one might find it quite striking that such a deformation can be done in a Lorentz-invariant way.

Let us discuss explicitly how the above features can be realized. Keeping the Lorentz algebra unchanged we enlarge the set of generators (J_i, K_j) with a 3-vector P_i and a scalar P_0. In this way the commutation relations (4.4),(4.5) remain unchanged as compared to the ordinary PA. Next we choose, for the purposes of the present paper, to keep also the commutation relations among the generators (P_0, P_i) (4.6),(4.7) as in the ordinary PA. The reason for the latter choice is that we are interested here in introducing ultraviolet regularizations. Introducing non-commutativity in the P_i's would provide an infrared regularization. Proceeding in this way we write down generalized commutation relations (as compared to the ordinary PA) for the generators (P, K_i):

$$[K_i, P_0] \;=\; i\alpha_i(P_0, \vec{P})\,, \tag{4.14}$$

$$[K_i, P_j] \;=\; i\beta_{ij}(P_0, \vec{P})\,. \tag{4.15}$$

Here α_i, β_{ij} are functions of P_0 and P_j.

The generators of the dPA have to satisfy the Jacobi identities. Applying the Jacobi identity to the set $(J_i,\, K_i,\, P_0)$, we conclude that α_i must be a vector, i.e.

$$[J_i, \alpha_j] \;=\; i\epsilon_{ijk}\alpha_k\,. \tag{4.16}$$

Correspondingly, the Jacobi identity for the set $(J_i,K_j,\, P_k)$ determines that β_{ij} transforms as a symmetric tensor under SU(2):

$$[J_i, \beta_{jl}] \;=\; i\epsilon_{ijk}\beta_{kl} + i\epsilon_{ilk}\beta_{jk}\,. \tag{4.17}$$

From eqs.(4.16),(4.17) it follows that α_i and β_{jk} can be chosen as

$$\begin{aligned} \alpha_i(P_0, \vec{P}) &= \alpha(P_0, P^2)P_i\,, \\ \beta_{jk}(P_0, \vec{P}) &= \beta(P_0, P^2)\delta_{jk} + \gamma(P_0, P^2)P_jP_k\,. \end{aligned} \tag{4.18}$$

Furthermore, we have assumed the expression (4.13) is invariant under the dPA and thus commutes with all generators. One can easily check that (4.13) indeed commutes with J_i, P_i, P_0. The commutator of K_i with expression (4.13) gives

$$\begin{aligned} f^{'}(P_0) &= 2\frac{\beta(P_0,P^2)}{\alpha(P_0,P^2)}\,, \\ \gamma(P_0,P^2) &= 0\,, \end{aligned} \tag{4.19}$$

where $f^{'}$ denotes the derivative of f with respect to P_0, and, consequently:

$$\begin{aligned} \alpha_i(P_0,P^2) &= \alpha(P_0)P_i\,, \\ \beta_{jk}(P_0,P^2) &= \beta(P_0)\delta_{jk}\,. \end{aligned} \tag{4.20}$$

The functions α and β in eq.(4.20) are not independent. One may easily see that from the Jacobi identity for the set (K_i, K_j, P_j), i.e.

$$[K_i,[K_j,P_k]] + [K_j,[P_k,K_i]] + [P_k,[K_i,K_j]] \;=\; 0\,, \tag{4.21}$$

we obtain, for $i \neq j$:

$$3(\beta^{'}(P_0)\alpha(P_0) \;-\; 1)P_k \;=\; 0\,. \tag{4.22}$$

As a result, in order to close the algebra, we must require that

$$\alpha(P_0)\beta^{'}(P_0) \;=\; 1\,. \tag{4.23}$$

In this way a minimally deformed Poincaré algebra is defined by the commutation relations

$$[J_i,J_j] = i\epsilon_{ijk}J_k\,, \tag{4.24}$$
$$[J_i,K_j] = i\epsilon_{ijk}K_k\,, \tag{4.25}$$
$$[J_i,P_j] = i\epsilon_{ijk}P_k\,, \tag{4.26}$$
$$[K_i,K_j] = -i\epsilon_{ijk}J_k\,, \tag{4.27}$$
$$[J_i,P_0] = 0\,, \tag{4.28}$$
$$[P_i,P_j] = [P_i,P_0] \;=\; 0\,, \tag{4.29}$$
$$[K_i,P_0] = i\alpha(P_0)P_i\,, \tag{4.30}$$
$$[K_i,P_j] = i\beta(P_0)\delta_{ij}\,, \tag{4.31}$$

where $\alpha(P_0)$, $\beta(P_0)$ satisfy eq.(4.23).

It should be noted that more general deformations can be performed as well [6]. However, the algebra (4.23)–(4.31) has the characteristic feature that it contains the exact Lorentz algebra.

Let us turn now to the Casimir invariants of the algebra (4.24)–(4.31). There exist two 'quadratic' Casimir invariants. One corresponds to the $(\text{mass})^2$ of the ordinary Poincaré algebra, and in view of eqs.(4.13),(4.19),(4.23) is given by

$$\beta^2(P_0) - \vec{P}\cdot\vec{P} \;=\; \mu^2\,. \tag{4.32}$$

The other corresponds to the length of the Pauli-Lubanski four-vector

$$W_0 = \vec{J}\cdot\vec{P}\,, \tag{4.33}$$
$$W_i = \beta(P_0)J_i \,+\, \epsilon_{ijk}P_jK_k\,. \tag{4.34}$$

One can easily verify that the vector (4.33),(4.34) satisfies the following commutation relations:

$$\begin{aligned}
[W_0, W_i] &= i\epsilon_{ijk}\beta(P_0)J_jP_k + iK_i(\vec{P}\cdot\vec{P}) - iP_i(\vec{K}\cdot\vec{P}) + \beta(P_0)P_i \\
[W_i, W_j] &= i\epsilon_{ijk}(\beta^2(P_0)J_k - W_0P_k) + i\beta(P_0)(P_iK_j - P_jK_i)\,, \\
[J_i, W_j] &= i\epsilon_{ijk}W_k\,, \\
[J_i, W_0] &= 0\,, \\
[K_i, W_0] &= iW_i\,, \\
[K_i, W_j] &= iW_0\delta_{ij}\,, \\
[P_i, W_0] &= [P_i, W_j] \;=\; 0\,, \\
[P_0, W_0] &= [P_0, W_j] \;=\; 0\,.
\end{aligned} \tag{4.35}$$

Consequently, all generators commute with

$$\mathcal{W}^2 \;=\; W_0^2 \,-\, \vec{W}\cdot\vec{W}\,, \tag{4.36}$$

i.e. the second Casimir invariant of the deformed algebra. The eigenvalues of $\mathcal{W}^2$ are

$$\mathcal{W}^2 \;=\; -\mu^2\, s(s+1)\,, \tag{4.37}$$

where $s = 0, \frac{1}{2}, 1, \ldots$ is the spin, and will label the representations of the dPA.

Notice that the form of the β function is not specified by the dPA. Therefore one can impose some physical requirements on the form of the β. An obvious one is to recover the ordinary PA in the low-energy region. This means that the low-energy behaviour of β has to be

$$\beta(P_0) \;\sim\; P_0. \tag{4.38}$$

Another constraint will be introduced by demanding that there exist an upper cut-off in the mass spectrum and, consequently, due to eq.(4.32), to the momentum for μ^2 positive. This constraint means that the equation

$$\beta^{'}(P_0) = 0 \tag{4.39}$$

has a solution for a finite P_0. An obvious realization of such a function is provided by

$$\beta(P_0) \;=\; M\sin\left(\frac{P_0}{M}\right)\,, \tag{4.40}$$

which, modulo periodicity, may restrict the energy P_0 in the interval

$$-\frac{\pi M}{2} \le P_0 \le \frac{\pi M}{2}.$$

Note that β is an odd function and, thus, the representations of the dPA can be classified according to

$$sign(\beta) \;=\; \frac{\beta(P_0)}{\mid\beta(P_0)\mid}\,. \tag{4.41}$$

Let us make further remarks concerning eq.(4.23). This equation holds independently of the representations. In the case that eq.(4.39) holds, we may define states denoted by $|M\rangle$ which are annihilated by β'. Observe that $|M\rangle$ can be defined as a limiting case, i.e.,

$$\lim_{\beta\to M} \mid\mu\rangle = |M\rangle\,. \tag{4.42}$$

In that case we see that eq.(4.21) is satisfied only if the commutator $[K_i, K_j]$ also vanishes and thus leaves the $\alpha(P_0)$ unconstrained. The commutator $[K_i, \mu^2]$ acting on $|M\rangle$ gives zero only if P_i also annihilates $|M\rangle$. As a result, the dPA acting on these states is realized by the algebra generated by the set (J_i, K_i, P_i), which satisfies the commutation relations (4.24)–(4.26),(4.29) and

$$\begin{aligned}[][K_i, K_j] &= 0\,, \\ [K_i, P_j] &= i\beta_0\delta_{ij}\,, \\ [K_i, P_0] &= 0\,, \end{aligned} \tag{4.43}$$

where β_0 is the eigenvalue of $\beta(P_0)$ on $|M\rangle$. It is clear now that this algebra is no longer the dPA.

The states $|M\rangle$ which are annihilated by β', carry zero momentum and have maximum energy, which cannot be changed by the action of the boosts. A representation for the generators of the dPA in momentum space is given by

$$P_0 = p_0\,, \tag{4.44}$$

$$P_i = p_i\,, \tag{4.45}$$

$$J_i = -i\epsilon_{ijk}p_j\frac{\partial}{\partial p_k}\,, \tag{4.46}$$

$$K_i = i\left(p_i\alpha(p_0)\frac{\partial}{\partial p_0} + \beta(p_0)\frac{\partial}{\partial p_i}\right)\,. \tag{4.47}$$

It is not difficult to verify that the above generators indeed satisfy the commutation relations (4.24)–(4.31), if we take eq.(4.23) into account.

Let us finally make some remarks concerning the additivity properties of momentum $\vec{P}^{(12)}$ and energy $P_0^{(12)}$ of a system $S^{(12)}$ composed out of two non-interacting systems $S^{(1)}, S^{(2)}$ with momenta $\vec{P}^{(1)}, \vec{P}^{(2)}$) and energies $P_0^{(1)}, P_0^{(2)}$, respectively. Among the new ones, only the $\vec{P}^{(12)}$, $\vec{J}^{(12)}$ and $\vec{K}^{(12)}$ have the usual additivity property, i.e.:

$$\begin{aligned}\vec{P}^{(12)} &= \vec{P}^{(1)} + \vec{P}^{(2)}, \\ \vec{J}^{(12)} &= \vec{J}^{(1)} + \vec{J}^{(2)}, \\ \vec{K}^{(12)} &= \vec{K}^{(1)} + \vec{K}^{(2)}. \end{aligned}$$

As far as the energy is concerned, although it is still conserved, the energy $P_0^{(12)}$ of $S^{(12)}$ is no longer the sum of the energies of the two subsystems $S^{(1)}$ and $S^{(2)}$. Instead we have[3]

$$\sin\frac{P_0^{(1)}}{M} + \sin\frac{P_0^{(2)}}{M} = 2\sin\frac{P_0^{(12)}}{2M}. \tag{4.48}$$

4.2 Space-time of the deformed Poincaré algebra

The dPA defined above has been constructed solely in momentum space. However, one can recover space-time, in the spirit of Gel'fand, as the spectrum of appropriate self-adjoint

[3]Actually this is not a legitimate coproduct since it isn't a homomorphism on the algebra. One can easily show that when the range of β is $[0, \infty)$, including a possible multivaluedness, that the Hopfian structure is given by $\Delta(P_0) = \beta^{-1}(\beta(P_0) \oplus \beta(P_0))$ and the usual Hopf structure for the rest of the generators and maps. In appendix B, we'll investigate some algebras with a kosher Hopfian structure, and look into the consequences for a ϕ^4 model.

operators acting on momentum space. In order to do that we will assume that there exists a set of commuting operators T, X_i $(i = 1, 2, 3)$ that satisfy

$$\begin{aligned} [X_i, X_j] &= [T, X_i] = 0, \\ [X_i, P_j] &= i\delta_{ij}, \\ [X_i, P_0] &= 0, \\ [T, P_0] &= i\alpha(P_0). \end{aligned} \tag{4.49}$$

These operators act on functions on the momentum space and their eigenvalues (t, x_i) label the space-time points. From eqs.(4.49) we see that a representation of T, X_i is given by

$$\begin{aligned} T &= i\alpha(p_0)\frac{\partial}{\partial p_0}, \\ X_i &= i\frac{\partial}{\partial p_i}. \end{aligned} \tag{4.50}$$

This particular representation satisfies the commutation relations

$$\begin{aligned} [J_i, X_j] &= i\epsilon_{ijk}X_k, \\ [J_i, T] &= 0, \\ [K_i, X_j] &= -iT\delta_{ij}, \\ [K_i, T] &= -iX_i. \end{aligned} \tag{4.51}$$

It is now clear that the set (T, X_i) transforms as a Lorentz four-vector. As a result Lorentz transformations leave invariant the quadratic expression

$$T^2 - \vec{X}^2. \tag{4.52}$$

Let us next find the domain $D(T, X_i)$ where these operators are self-adjoint. The function $\beta(p_0)$ has local extrema at the points $p_0 = \pm\frac{\pi M}{2}$. We define the inner product for functions f and g as

$$\langle f \mid g \rangle = \int d^4p\beta'(p_0)\, f^*(p_0, \vec{p})\, g(p_0, \vec{p}). \tag{4.53}$$

The operator T is self-adjoint with respect to this inner product for the functions $f(p_0, \vec{p})$ that satisfy

$$f\left(\frac{\pi M}{2}, \vec{p}\right) = f\left(-\frac{\pi M}{2}, \vec{p}\right) = 0. \tag{4.54}$$

The eigenvalues of the operator T will be real and will correspond to the possible values of time measurements. These eigenvalues are specified by solving

$$-i\frac{\partial}{\partial p_0}f(p_0, \vec{p}) = t\beta'(p_0)\, f(p_0, \vec{p}) \tag{4.55}$$

with the condition (4.54). The general solution of eq.(4.55) can be written as

$$f(p_0, \vec{p}) = \sum_n \left[C_n(\vec{p})\cos\left(\beta(p_0)\frac{(2n+1)\pi}{2M}\right) + D_n(\vec{p})\sin\left(\beta(p_0)\frac{n\pi}{M}\right)\right]. \tag{4.56}$$

where $M \equiv \beta(\frac{\pi M}{2})$. The Fourier transform of the above solution in t-space is given by

$$\begin{aligned} f(t,\vec{x}) &= \sum_n \Big\{ C_n^{'}(\vec{x}) \left[\delta\left(t + \frac{(2n+1)\pi}{2M}\right) + \delta\left(t - \frac{(2n+1)\pi}{2M}\right) \right] \\ &\quad + D_n^{'}(\vec{x}) \left[\delta\left(t + \frac{n\pi}{M}\right) - \delta\left(t - \frac{n\pi}{M}\right) \right] \Big\} . \end{aligned} \tag{4.57}$$

This leads to a lattice pattern for the t-axis with lattice-spacing $\frac{1}{M}$.

Let us now examine the X_i spectrum. We recall that, in momentum space, three types of four-momenta exist: time-like ($\mu^2 > 0$), space-like ($\mu^2 < 0$) and null ones ($\mu^2 = 0$). For non space-like momenta we see that P^2 is always less than or equal to β^2. If $\beta^{'} = 0$ for some p_0, then there exists, as we have already discussed, a highest state with maximum energy $\frac{\pi M}{2}$; P^2 is then always less than M^2, and thus

$$-M \ \le p \le M. \tag{4.58}$$

In this respect, there also exist three types of X_i: those that act on functions $f(p_0, \vec{p})$, with $(p_0, \vec{p})$ time-like, space-like or null four-momenta, respectively. For time-like momenta, proceeding as before, one can find that the coefficients C_n in eq.(4.57) are given by

$$\begin{aligned} C'_m(\vec{x}) &= \prod_{i=1}^{3} \sum_{n_i} \Big\{ E_{m,n_i} \left[\delta\left(x_i - \frac{(2n_i+1)\pi}{2M}\right) + \delta\left(x_i + \frac{(2n_i+1)\pi}{2M}\right) \right] \\ &\quad + F_{m,n_i} \left[\delta\left(x_i + \frac{n_i\pi}{M}\right) - \delta\left(x_i - \frac{n_i\pi}{M}\right) \right] \Big\} . \end{aligned} \tag{4.59}$$

and a similar expression for $D'_m(\vec{x})$. For space-like momenta these coefficients are continuous functions of x_i. As a result the space-time portrait looks like the following. At each space-time point we can draw the curves given by equating expression (4.52) to zero. These curves specify the lightcone at that point and are invariant under Lorentz transformations. As in the ordinary case we define time-like, space-like and null regions. The fundamental difference is that for non-space-like regions a lattice structure emerges, whereas space-like ones are continuous [39, 30].

Let us define the inner product of two vectors $|f\rangle$, $|g\rangle$ on the Hilbert space of a massive particle by

$$\begin{aligned} \langle f \mid g \rangle &= \int \frac{d^4p}{(2\pi)^4} \delta(\beta^2(P_0) - p^2 - \mu^2)\theta(p_0) f^*(p) g(p) \\ &= \int \frac{d^3p}{(2\pi)^3 2\beta'\beta} f^*(\vec{p}) g(\vec{p}) \, , \end{aligned} \tag{4.60}$$

where p_0 is the positive solution of $\beta^2(p_0) - p^2 = \mu^2$. A representation of the generators of the dPA is given by

$$\begin{aligned} P_i &= p_i \, , \\ J_i &= -i\epsilon_{ijk} p_j \frac{\partial}{\partial p_k} \, , \\ K_i &= i \left(\beta(p_0) \, \frac{\partial}{\partial p_i} + \frac{\alpha^{'}}{2} p_i \right) . \end{aligned} \tag{4.61}$$

These operators act on functions of the Hilbert space and they are self-adjoint with respect to the inner product (4.60). Furthermore we can define a position operator by

$$Q_i \equiv \frac{1}{2\beta} K_i + K_i \frac{1}{2\beta} , \tag{4.62}$$

which is Hermitean with respect to the inner product (4.60) [6, 47, 40]. It is easy to check that Q_i is a vector, i.e.

$$[J_i, Q_j] = i\epsilon_{ijk} Q_k , \tag{4.63}$$

and moreover

$$\begin{aligned} [Q_i, Q_j] &= 0, \\ [Q_i, P_j] &= i\delta_{ij}. \end{aligned} \tag{4.64}$$

Using the representation (4.60) of the generators (P_i, J_j, K_k), the Q_i are found to be

$$Q_i = i \left(\frac{\partial}{\partial p_i} + \frac{\alpha' \beta - 1}{2\beta^2} p_i \right) . \tag{4.65}$$

In the Heisenberg representation the time evolution of the position operator is governed by

$$\dot{Q}_i = i\beta'(P_0) [P_0, Q_i] = \frac{P_i}{\beta} , \tag{4.66}$$

from which we conclude that

$$p_i = \beta v_i , \tag{4.67}$$

where v_i is the velocity of the particle. Note that we have formally written $\dot{Q}_i$ as the velocity of the particle and that the normalization $\dot{T} = 1$ has been used. For massless particles we have $\beta^2 = \vec{P}^2$, so that

$$v = 1 , \tag{4.68}$$

where v is the length of the velocity vector. Equation (4.67) yields, for a massive particle with mass μ

$$v_{mas.\,part.} < \left(1 - \frac{\mu^2}{M^2} \right)^{\frac{1}{2}} . \tag{4.69}$$

4.3 Representations of the deformed Poincaré group

The irreducible representations of the PA are associated with relativistic one-particle states [52, 63]. Here we shall construct such representations for the dPA presented in section 2 and examine their relation with the corresponding representations of the ordinary PA.

Consider the quadratic Casimir invariant

$$\beta^2(P_0) - P^2 = \mu^2.$$

As in the ordinary PA there exist three classes of unitary representations: those with $\mu^2 > 0, \mu^2 = 0$ and $\mu^2 < 0$. Let us denote the corresponding representations in each class as [T],[0] and [S], respectively.

For the [T]-class we observe that we can choose a four-momentum P_μ to be

$$P_\mu \propto (\pm 1, 0, 0, 0). \tag{4.70}$$

and there thus exist two subclasses in [T] denoted by [$T_\pm$] which correspond to the $\pm$ choice of P_0. It is not difficult to see that the vector (4.70) is invariant under the little group SO(3) of this representation. Therefore these representations are classified according to Casimirs of SO(3), namely $S^2 = s(s+1)$, where s is the spin with values $s = 0, 1/2, 1, ...$ As a result the one-particle states in the [T]-class are uniquely determined by the mass μ, the spin s and the $sign(\beta)$.

In the [0]-class one can choose P_μ to be

$$P_\mu = (\pm 1, 0, 0, 1). \tag{4.71}$$

There thus exist in this case also two subclasses [$0_\pm$]. The little group in this case is the two-dimensional Euclidean group E(2). Therefore the representations are here classified according to the Casimirs of E(2). These are the length of the two-dimensional vectors and the helicities of the helicity group SO(2), namely $0, 1/2, 1, ...$ Since the length of a two-vector can be zero or positive, the possible representations in the [0]-class can be [$0^0{}_\pm$] and [$0^+{}_\pm$].

Finally, for the [S]-class we can choose

$$P_\mu \propto (0, 0, 0, 1). \tag{4.72}$$

For this class the little group is the Lorentz group in three dimensions SO(1,2). The representations will be classified according to the Casimirs of SO(1,2).

From the above discussion it is clear that there exists a correspondence between the representations of our dPA and the representations of the undeformed PA. This result would be immediately obtained if one noted that, by making the transformation $P_0 \to \beta(P_0)$, the dPA is reduced to PA for the set of generators $(K_i, J_i, P_i, \beta(P_0))$. However this transformation is not invertible, given the fact that β is a multivalued function of P_0.

This last observation does not hold only in the present case. For instance in the case of $SU_q(2)$ it is known that, in general, a redefinition of the SU(2) generators yields the $SU_q(2)$ algebra and vice versa, when q is not a root of unity [2, 14]. Similarly in our case the redefinition of the generators (K_i, J_i, P_i, P_0) of the dPA results in the ordinary PA. However this redefinition leaves the Lorentz subalgebra unaltered and thus the representation of the dPA are the same as in the usual PA. The only Casimir that is different in the two cases is the one defined in (4.32). Therefore the only difference in the one-particle states will appear in the masses labelling the representations.

4.4 Discussion

The hope that quantum theories with improved ultraviolet properties can be constructed has motivated a considerable research interest in deformations of space-time symmetries. However, in these studies one has to take into account that so far the laws of physics are required to be invariant under Lorentz transformations, while the notion of an elementary particle can be understood in the framework of the representation theory of the Poincaré group. In this framework, consistent quantum field theories have been constructed; these describe successfully the non-gravitational interactions of elementary particles. In order to keep as much as possible of these features we were led to consider here minimal deformations of the space-time symmetries. Specifically in the present paper we were looking for deformations of the Poincaré algebra that leave the Lorentz one invariant. As a result the representations of the constructed dPA are the same as in the ordinary case, modulo the mass. This fact is expected to play a catalytic role in the quantization of the field theories that one might attempt to construct. Moreover, the present construction introduces an upper limit in the energy. The deformed quadratic Casimir (4.32) and the upper bound in the energy provide also an upper bound in the momentum for free particles. On the other hand, in quantum field theories, loop diagrams involve exchanges of virtual particles that do not obey the on mass-shell condition (4.32). Then in these diagrams the fact that the energy is bounded relaxes by one unit the degree of their divergences. For instance, four-dimensional quantum field theories are expected to behave as three-dimensional ones from the renormalization point of view. This momentum space regularization can also be seen in the space-time as a regularization due to higher derivative terms obtained from the expansion of β in powers of P_0. The latter leads to a non-local theory, which however is local in the momentum space. We shall discuss in detail the question of quantum field theories defined in the newly constructed framework in another publication.

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Chapter 5

The Wess-Zumino algebra

It is known that 4-vectors transform, under the Lorentz group, as the $(\frac{1}{2},0)\otimes(0,\frac{1}{2})$ representation. This means that any 4-vector can be written as the product of two spinors ζ^a, $\zeta^{\bar{a}}$. The first transforms under the $(\frac{1}{2},0)$ representation of $SL(2,\mathbb{C})$, and the other as the complex conjugated representation $(0,\frac{1}{2})\simeq(\frac{1}{2},0)^*$. The Wess-Zumino approach to a q-deformed Poincaré algebra relies on this fact.

The plan is to impose commutation relations on the components of the spinor. Due to these relations we can define invariance transformations that form a complete set, which in its turn will lead to commutation relations between different spinors. Then we'll look at infinitesimal transformations that are consistent with the commutation relations of the spinors. Out of the spinors we'll then make fourvectors by the above mentioned method, which will enable us to define a deformed Minkowski space on which we then define a deformed Lorentz algebra. As a way of extending the deformed Lorentz algebra to a deformed Poincaré algebra we'll look at the differentials on the deformed Minkowski space, which will be interpreted as the translations. The knowledge of the algebra of translations enables us to define the action of the Lorentz algebra on the translations, which completes the construction of the Wess-Zumino algebra (WZ-algebra). All that remains to be done after the construction of the WZ-algebra, is to define a Hopf structure on this algebra. Needless to say that this is indeed possible.

5.1 On to a q-Minkowski space

Let's introduce a q-spinor $\zeta^a=\begin{pmatrix}x\\y\end{pmatrix}$, whose components satisfy the, so-called, quantum-plane relation (QP) [43], i.e.

$$x\cdot y \;=\; qy\cdot x \qquad\qquad : q\in\mathbb{C}_{/\{0\}}. \tag{5.1}$$

By introducing a 'metric' ϵ on these spinors of the form [10, 49]

$$\epsilon_{ab} \;=\; \begin{pmatrix}0 & q^{-\frac{1}{2}}\\ -q^{\frac{1}{2}} & 0\end{pmatrix},\; \epsilon^{ab} \;=\; \begin{pmatrix}0 & -q^{-\frac{1}{2}}\\ q^{\frac{1}{2}} & 0\end{pmatrix}, \tag{5.2}$$

eq.(5.1) can be written as

$$\epsilon_{ab}\zeta^a\zeta^b \;=\; 0\,. \tag{5.3}$$

The 'covariant' q-spinor can then be defined as

$$\zeta_a = \epsilon_{ab}\zeta^b = \begin{pmatrix} q^{-\frac{1}{2}}y \\ -q^{\frac{1}{2}}x \end{pmatrix}, \tag{5.4}$$

and its components satisfy $\zeta_1\zeta_2 = q^{-1}\zeta_2\zeta_1$.

Normally, the group $SL(2,\mathbb{C})$ manifests itself as the group that leaves the length, something like eq.(5.3), invariant. It is then straightforward to introduce a deformed $SL(2,\mathbb{C})$ by imposing invariance of $\epsilon_{ab}\zeta_1^a\zeta_2^b$. In order to find this invariance group we introduce a transformation on the q-spinors by

$$\zeta^{a\prime} = M^{a\cdot}_{\cdot b}\zeta^b = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{5.5}$$

where the elements of M commute with ζ. Invariance then clearly implies

$$\epsilon_{ab}M^{a\cdot}_{\cdot c}M^{b\cdot}_{\cdot d} = D_q\,\epsilon_{cd}\,, \tag{5.6}$$

where D_q is a multiplicative factor, which also occurs in conformal transformations. Upon doing the calculations one arrives at the following equations

$$\begin{aligned} ac &= qca\,, \\ bd &= qdb\,, \\ D_q &= ad - qcb\,, \\ &= da - q^{-1}bc\,. \end{aligned} \tag{5.7}$$

Seeing the two last equations one sees that more relations have to exist for the two last equations to be compatible. A consistent choice is to say that eq.(5.3) not only has to be invariant under left multiplication but also under right multiplication, which amounts up to saying that, as in the $SL(2,\mathbb{C})$ case, the transposed representation is equivalent to the defining representation. This method then implies that one has to interchange b and c in the previous calculation, which gives

$$\begin{aligned} ab &= qba\,, \\ cd &= qdc\,, \\ D_q &= ad - qbc\,, \\ &= da - q^{-1}cb\,. \end{aligned} \tag{5.8}$$

Upon combining eqs.(5.7,5.8) one finds a consistent set of relations which is enough to order any polynomial in a, b, c and d. The relations are called the Manin-plane [43] and are explicitly given by

$$\begin{array}{llll} ac = qca & , & ab = qba\,, & \\ bd = qdb & , & cd = qdc\,, & \\ bc = cb & , & ad = da + \lambda bc\,, & \\ D_q = ad - qcb = da - q^{-1}bc & , & & \end{array} \tag{5.9}$$

and the abreviation $\lambda = q - q^{-1}$ has been introduced. The above system has some nice and valuable properties. The most important one is that matrix multiplication preserves the above relations [10, 43, 51, 62]. I.e. take a second matrix M', whose matrix elements satisfy eq.(5.9) and that commute with the elements of M, then the elements of MM' obey the relations in eq.(5.9). Furthermore, it's obvious that this system allows a consistent alfabetical, actually any, ordering and that in the limit $q \to 1$ the elements commute.

The factor D_q coincides with the so-called q-determinant and is central to the Manin-plane, i.e. $D_q M^{a\cdot}_{\cdot b} = M^{a\cdot}_{\cdot b} D_q$. Since D_q lies in the center of the enveloping algebra of the alg.(5.9), we boldly take $D_q = 1$ and call the set (5.9) enhanced by the condition $D_q = 1$, the defining relations for the invariance set $SL_q(2,\mathbb{C})$[1].

One might object to the fact that when $q \to 1$ we are dealing with commuting spinors, whereas for a second-quantized theory one needs anti-commuting spinors. This objection, however, can be rejected as being insignificant due to the fact that if one puts $xy = -qyx$ in stead of eq.(5.1), one ends up with the same algebra, i.e. alg.(5.9). On this point one may, should I be found untrustable, consult [43].

The question arises how two different copies of q-spinors commute. Seeing the fact that the elements of one q-spinor don't commute, it is to be expected that two different copies don't commute either. In order to investigate this, write the commutation relations between two copies as

$$\zeta_1^a \zeta_2^b \;=\; Q^{ab}{}_{cd} \zeta_2^c \zeta_1^d \,, \tag{5.11}$$

which, seeing eq.(5.1), isn't as bad an 'Ansatz' as one might feel. This equation is, however, far too general to handle and thus one imposes, analogous to eq.(5.6), that the 'Ansatz' has to be invariant under an $SL_q(2,\mathbb{C})$ transformation. From this one then finds

$$Q^{ab}{}_{cd} M^{c\cdot}_{\cdot e} M^{d\cdot}_{\cdot f} \;=\; M^{a\cdot}_{\cdot c} M^{b\cdot}_{\cdot d} Q^{cd}{}_{ef} \,. \tag{5.12}$$

The most general solution to eq.(5.12) can then be written as [43]

$$Q^{ab}{}_{cd} \;=\; kR^{ab}{}_{cd} \;=\; k \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}^{ab}_{\;\;cd} \,, \tag{5.13}$$

where the indexation $(ab) = (11), (12), (21), (22)$ has been used. From this solution one can write down all the commutation relations between the components of the q-spinors, which are explicitly given by

$$\begin{aligned} x_1 x_2 &= kq x_2 x_1 \,, \\ x_1 y_2 &= k y_2 x_1 + k\lambda x_2 y_1 \,, \\ y_1 x_2 &= k x_2 y_1 \,, \\ y_1 y_2 &= kq y_2 y_1 \,. \end{aligned} \tag{5.14}$$

[1] Although this set isn't a vector space we can define a consistent coproduct, counit and antipode on it, i.e. they satisfy the axioms. It can be shown that

$$\begin{aligned} \Delta(M^{a\cdot}_{\cdot b}) &= M^{a\cdot}_{\cdot c} \otimes M^{c\cdot}_{\cdot b} \,, \\ \epsilon(M^{a\cdot}_{\cdot b}) &= \delta^{a\cdot}_{\cdot b} \,, \\ S(M^{a\cdot}_{\cdot b}) &= \epsilon^{ac} M^{d\cdot}_{\cdot c} \epsilon_{db} \,, \end{aligned} \tag{5.10}$$

together with the normal definitions for mappings that are not defined by the equations above, define a structure similar to a Hopf algebra. Note that, as it is in the case of normal Lie algebras, $S(M) = M^{-1}$.

The unknown k can be eliminated by imposing the condition that eq.(5.11) also holds when $\zeta_1 = \zeta_2 = \zeta$. One then obviously obtains $k = q^{-1}$.

If one imposes associativity on the multiplication of q-spinors, i.e.

$$\left(\zeta_1^a \zeta_2^b\right) \zeta_3^c \;=\; \zeta_1^a \left(\zeta_2^b \zeta_3^c\right) , \tag{5.15}$$

and one orders the whole expression to $\zeta_3\zeta_2\zeta_1$ one arrives at [10]

$$R^{ab}{}_{ij} R^{jc}{}_{kf} R^{ik}{}_{de} \;=\; R^{bc}{}_{ij} R^{ai}{}_{dk} R^{kj}{}_{ef} \,. \tag{5.16}$$

This equation is known as the Yang-Baxter equation and plays a major role in the construction of QP's. The whole construction up to now can be reversed. First one finds a solution to the Yang-Baxter equation and then one defines the commutation relations between the generators of the QP, by virtue of eqs.(5.11,5.12) [51, 60].

As was already mentioned, one needs a kind of conjugated q-spinor in order to arrive at q-fourvectors. Therefore one needs a prescription how to take the conjugated. In this approach a conjugation including order-reversal is introduced [10, 50]. From the quantum plane condition eq.(5.1) we then find

$$\overline{x}{\cdot}\overline{y} \;=\; \overline{y{\cdot}x} \;=\; \overline{\frac{1}{q} x{\cdot}y} \;=\; \frac{1}{q^*}\overline{y}{\cdot}\overline{x} \,. \tag{5.17}$$

At this point it is both convenient and wise to make a remark about the q. Seeing the idea behind this scheme, one expects that the Lorentz boosts, which still have to be defined, will include the q. One wants, however, the observables to have real eigenvalues in every inertial system, therefore we see that we have to choose q to be real. Then the relation (5.17) results in

$$\overline{x}{\cdot}\overline{y} \;=\; \frac{1}{q}\overline{y}{\cdot}\overline{x} \,. \tag{5.18}$$

It shouldn't be difficult to find the invariance transformations for eq.(5.18) [2]. Seeing the quantum-plane relation, eq.(5.4), and the condition for the covariant q-spinors, it is tempting to introduce the spinor $\overline{\zeta}_a = \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}$. On this spinor we define the appropriate metric, which are given by

$$\overline{\epsilon}^{ab} \;=\; \begin{pmatrix} 0 & q^{\frac{1}{2}} \\ -q^{-\frac{1}{2}} & 0 \end{pmatrix} , \overline{\epsilon}_{ab} \;=\; \begin{pmatrix} 0 & -q^{\frac{1}{2}} \\ q^{-\frac{1}{2}} & 0 \end{pmatrix} . \tag{5.19}$$

Then upon making the transformation $\overline{\zeta}'_a = \overline{\zeta}_b \overline{M}^{b\cdot}{}_{\cdot a}$, where $\overline{M} = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}$, we find that the elements of the conjugated transformation matrix have to satisfy:

$$\begin{aligned} \overline{a}\,\overline{b} &= q^{-1}\overline{b}\,\overline{a} \,, \\ \overline{a}\,\overline{c} &= q^{-1}\overline{c}\,\overline{a} \,, \\ \overline{b}\,\overline{d} &= q^{-1}\overline{d}\,\overline{b} \,, \\ \overline{c}\,\overline{d} &= q^{-1}\overline{d}\,\overline{c} \,, \\ \overline{b}\,\overline{c} &= \overline{c}\,\overline{b} \,, \\ \overline{a}\,\overline{d} &= \overline{d}\,\overline{a} - \lambda\overline{b}\,\overline{c} \,. \end{aligned} \tag{5.20}$$

[2] In fact everything goes completely analogously to eqs.(5.1-5.10), one only has to make the substitution $q \to q^{-1}$.

The relations between different sets of conjugated q-spinors can be found by using eq.(5.11) and the definition of conjugation. Then upon using the symmetry of the R-matrix, e.g. $R^{ij}{}_{kl} = R^{kl}{}_{ij}$, one finds

$$\overline{\zeta}_{2a}\overline{\zeta}_{1b} = q^{-1}R^{cd}{}_{ba}\overline{\zeta}_{1d}\overline{\zeta}_{2c}\ , \tag{5.21}$$

whose explicit result reads

$$\begin{array}{rclcrcl} \overline{x}_1\overline{x}_2 &=& \overline{x}_2\overline{x}_1 &,& \overline{x}_1\overline{y}_2 &=& q\overline{y}_2\overline{x}_1 - q\lambda\overline{x}_2\overline{y}_1\ , \\ \overline{y}_1\overline{y}_2 &=& \overline{y}_2\overline{y}_1 &,& \overline{y}_1\overline{x}_2 &=& q\overline{x}_2\overline{y}_1\ . \end{array} \tag{5.22}$$

There exists a nice mapping from the unbarred to the barred $SL_q(2,\mathbb{C})$ elements which stems from the 'classical' $SL(2,\mathbb{C})$. In that case one has the equivalence

$$(D^{(0,\frac{1}{2})})^{-1} = (D^{(\frac{1}{2},0)})^{\dagger}\ , \tag{5.23}$$

If one looks at the inverse of the matrix $\overline{M}$, defined by

$$(\overline{M}^{-1})^{a\cdot}_{\cdot b} = \overline{\epsilon}^{ac}\overline{M}^{d\cdot}_{\cdot c}\overline{\epsilon}_{db} = \begin{pmatrix} \overline{d} & -q\overline{b} \\ -q^{-1}\overline{c} & \overline{a} \end{pmatrix}^{a\cdot}_{\cdot b}\ , \tag{5.24}$$

where one should look at eq.(1.4) to see the equivalence in notation. Upon making the identification $M^{a\cdot}_{\cdot b} \to \overline{M}^{-1}{}^{a\cdot}_{\cdot b}$ and substituting this into eq.(5.9) we arrive at eq.(5.20), which in turn tells us that $\overline{M}^{-1}$ satisfies eq.(5.12). Also it tells us that the equivalence (5.23) also holds in the $SL_q(2,\mathbb{C})$ case and that the $\overline{\zeta}^a$'s satisfy eq.(5.11).

Having found this result, one sees a natural way how to define the commutation relations between the barred and unbarred elements of $SL_q(2,\mathbb{C})$. Namely, both M and $\overline{M}^{-1}$ satisfy eq.(5.12) with the same R-matrix. Therefore we'll propose [10, 50]

$$R^{ab}{}_{cd}(\overline{M}^{-1})^{c\cdot}_{\cdot e}M^{d\cdot}_{\cdot f} = M^{a\cdot}_{\cdot c}(\overline{M}^{-1})^{b\cdot}_{\cdot d}R^{cd}{}_{ef}\ , \tag{5.25}$$

as the defining relation for the remaining commutation relations. Explicitly, eq.(5.25) can be given by

$$\begin{array}{rclcrcl} a\overline{a} &=& \overline{a}\,a - q\lambda\overline{c}c &,& a\overline{b} &=& q^{-1}\overline{b}a - \lambda\overline{d}c\ , \\ a\overline{c} &=& q\overline{c}a &,& a\overline{d} &=& \overline{d}a\ , \\ b\overline{a} &=& q^{-1}\overline{a}b - \lambda\overline{c}d &,& b\overline{b} &=& \overline{b}b + q\lambda(a\overline{a} - \overline{d}d)\ , \\ b\overline{c} &=& \overline{c}b &,& b\overline{d} &=& q\overline{d}b + q\lambda a\overline{c}\ , \\ c\overline{a} &=& q\overline{a}c &,& c\overline{b} &=& \overline{b}c\ , \\ c\overline{c} &=& \overline{c}c &,& c\overline{d} &=& q^{-1}\overline{d}c\ , \\ d\overline{a} &=& \overline{a}d &,& d\overline{b} &=& q\overline{b}d + q\lambda c\overline{a}\ , \\ d\overline{c} &=& q^{-1}\overline{c}d &,& d\overline{d} &=& \overline{d}d + q\lambda\overline{c}c\ . \end{array} \tag{5.26}$$

Of course eq.(5.26) implies commutation relations between the q-spinors and their conjugated q-spinors. Since eq.(5.25) looks like eq.(5.12), it defines the commutation relations between a normal q-spinor and a conjugated object that transforms according to the inverse conjugated representation. This object can be found by making a transformation on the $\overline{\zeta}^a$-spinor, as can be seen as follows

$$\overline{\zeta}^{a\prime} = \overline{\epsilon}^{ab}\overline{\zeta}'_b = \overline{\epsilon}^{ab}\overline{M}^{c\cdot}_{\cdot b}\overline{\epsilon}_{cd}\overline{\zeta}^d = (\overline{M}^{-1})^{a\cdot}_{\cdot b}\overline{\zeta}^b\ , \tag{5.27}$$

which is just what we were looking for. Now clearly eq.(5.26) is a consequence of the invariance of[3]

$$\zeta_1^a \overline{\zeta}_2^b \;=\; R^{ab}{}_{cd} \overline{\zeta}_2^c \zeta_1^d \,, \tag{5.28}$$

whose explicit result is, up to conjugation,

$$\begin{array}{lll} x_1\overline{x}_2 \;=\; \overline{x}_2 x_1 - q\lambda y_2 \overline{y}_1 & , & y_1\overline{y}_2 \;=\; \overline{y}_2 y_1 \\ x_1\overline{y}_2 \;=\; q\overline{y}_2 x_1 & , & y_1\overline{x}_2 \;=\; q\overline{x}_2 y_1 \,. \end{array} \tag{5.29}$$

From now on, one should be able to order any polynomial in the spinors.

As was mentioned in the introduction, we need bilinear combinations of the spinors, in order to make something like four-vectors which we obviously will need. Therefore define the following bilinears

$$\begin{array}{lll} A \;=\; \overline{x}y & , & B \;=\; \overline{y}x \\ C \;=\; \overline{x}x & , & D \;=\; \overline{y}y \,, \end{array} \tag{5.30}$$

which behave under conjugation like

$$\overline{A} \;=\; B \,,\, \overline{B} = A \,,\, \overline{C} = C \,,\, \overline{D} = D \,. \tag{5.31}$$

From the commutation relations between the spinors, it should be clear that also the bilinears enjoy such a property. A calculation of such a kind is necessarily based on the commutation relations between spinors and bilinears. These can be found by using eqs.(5.1,5.18) and eq.(5.29). The result of this calculation turns out to be

$$\begin{array}{lll} xA \;=\; qAx - q\lambda Dy & , & xB \;=\; qBx \,, \\ yA \;=\; qAy & , & yB \;=\; q^{-1}By \,, \\ \overline{x}A \;=\; q^{-1}A\overline{x} & , & \overline{x}B \;=\; q^{-1}B\overline{x} + \lambda D\overline{y} \,, \\ \overline{y}A \;=\; qA\overline{y} & , & \overline{y}B \;=\; q^{-1}B\overline{y} \,, \\ xC \;=\; Cx - q\lambda Dx & , & xD \;=\; q^2 Dx \,, \\ yC \;=\; Cy & , & yD \;=\; Dy \,, \\ \overline{x}C \;=\; C\overline{x} + q^{-1}\lambda D\overline{x} & , & \overline{x}D \;=\; q^{-2}D\overline{x} \,, \\ \overline{y}C \;=\; C\overline{y} & , & \overline{y}D \;=\; D\overline{y} \,. \end{array} \tag{5.32}$$

Using these relations it is very easy to find the commutation relations between the bilinears. Let's, as an example, look at the relation for AB, i.e.

$$\begin{array}{lll} AB & = & \overline{x}(yB) \;=\; q^{-1}\overline{x}By \;=\; q^{-1}(\overline{x}B)y \\ & = & q^{-2}B\overline{x}y + q^{-1}\lambda D\overline{y}y \;=\; q^{-2}BA + q^{-1}\lambda D^2 \,. \end{array}$$

For future convenience however, we'd like to have this relation in the form of a commutator, therefore write

$$\begin{array}{ll} = & BA - q^{-1}\lambda(BA - D^2) \;=\; BA - q^{-1}\lambda(\overline{y}x\overline{x}y - \overline{y}y\overline{y}y) \\ = & BA - q^{-1}\lambda\overline{y}(x\overline{x} - y\overline{y})y \;=\; BA - q^{-1}\lambda\overline{y}(\overline{x}x - q^2\overline{y}y)y \\ = & BA - q^{-1}\lambda CD + q\lambda D^2 \,, \end{array}$$

[3] A possible constant is wisely chosen to be one.

where in the last step eq.(5.29) was used. In the same way the complete set of commutation relations can be deduced, which results in

$$\begin{array}{rclcrcl} AB &=& BA - q^{-1}\lambda CD + q\lambda D^2 &,& BC &=& CB - q^{-1}\lambda BD\,, \\ AC &=& CA + q\lambda AD &,& BD &=& q^2 DB\,, \\ AD &=& q^{-2}DA &,& CD &=& DC\,. \end{array} \tag{5.33}$$

On this little coordinate algebra a central element can be found which corresponds to the invariant length. In fact this length is trivial because it is the distance between the same spinor, and thus is zero. It can be found by looking at the derivation of the commutator AB where the identity $BA - D^2 = CD - q^2D^2$ was used. By rewriting this identity and using eq.(5.33) we then find that

$$\mathcal{L} \equiv AB - q^{-2}CD = 0\,. \tag{5.34}$$

In this case the centrality of $\mathcal{L}$ is trivial, but it can be shown [59] that this also holds if one constructs the bilinears out of different spinors.

The question remains why to call $\mathcal{L}$ an invariant length? This can be seen by introducing coordinates as usual by defining

$$\begin{array}{rclcrcl} X_0 &=& (C+D) &,& C &=& \frac{1}{2}(X_0+X_3)\,, \\ X_3 &=& (C-D) &,& D &=& \frac{1}{2}(X_0-X_3)\,, \\ X_1 &=& (A+B) &,& A &=& \frac{1}{2}(X_1-iX_2)\,, \\ X_2 &=& i(A-B) &,& B &=& \frac{1}{2}(X_1+iX_2)\,. \end{array} \tag{5.35}$$

Upon using this in eq.(5.34) we find something which looks like the invariant length in Minkowski-space, i.e.

$$\mathcal{L} \sim X_1^2 + X_2^2 + q^{-2}X_3^2 - q^{-2}X_0^2 + i[X_1, X_2]\,. \tag{5.36}$$

In the limit $q = 1$ this is exactly the Minkowski length as can be seen by $i[X_1, X_2] = \lambda(qD^2 - q^{-1}CD)$. The above properties invite us to interpret the space of bilinears as a quantized, i.e. a q-deformed, Minkowski space.

5.2 The q-Lorentz algebra

Now that we have defined, in a consistent manner, a quantized Minkowski space, we can introduce on it infinitesimal transformations, which will be interpreted as rotations and boosts. First the rotations are constructed, by generalizing the action of normal rotations on spinors to q-spinors. Then the same method will be used to construct boost-like operators, which will lead to a generalized Lorentz algebra, denoted qL.

5.2.1 The contruction of a $su_q(2)$ algebra

In the normal case, the action of a generator of the group $SU(2)$ on a basis of the carrier-space of a representation is defined by

$$J(\vec{n})\Phi^a J(\vec{n})^{-1} = \Gamma(J(\vec{n}))^{a}_{\cdot b}\Phi^b\,, \tag{5.37}$$

where $\vec{n}$ is the axis of rotation, Φ^a are the basis-functions of a representation and Γ is the representation of the group. In the case of a infinitesimal rotation this can be written as

$$J\cdot\Phi^a \;=\; \Phi^a\cdot J \;+\; \Gamma(J)^{a}_{\cdot b}\Phi^b \,, \tag{5.38}$$

where Γ still is a representation but now of the Lie algebra. In the case of the construction of a rotation algebra on the q-spinors, we'll use something like eq.(5.38) as the 'Ansatz' to the construction of the algebra.

Therefore let us look at the general transformation J defined on the q-spinors by

$$\begin{aligned} Jx &= axJ + \alpha x + \beta y \quad , \quad J\overline{x} = \overline{a}\,\overline{x}J + \overline{\alpha}\,\overline{x} + \overline{\beta}\,\overline{y} \,, \\ Jy &= dyJ + \gamma x + \delta y \quad , \quad J\overline{y} = \overline{d}\,\overline{y}J + \overline{\gamma}\,\overline{x} + \overline{\delta}\,\overline{y} \,, \end{aligned} \tag{5.39}$$

where $a, d, \alpha, \beta, \gamma, \delta$ and their barred counterparts are arbitrary $\mathbb{C}$-numbers. Furthermore, a bar needn't, in this case, mean complex conjugation. Constraints on these actions can be found by imposing consistency with the commutation relations for the q-spinors, e.g. $J(xy - qyx) = 0$ and the same for eqs.(5.18,5.29). Another constraint can be found by imposing invariance of eq.(5.34) which, in the light of the limit $q = 1$, is both justifiable and wishable. The resulting equations leave room for three independent generators[4], whose action on the q-spinors are given by

$$\begin{aligned} J^+x &= qxJ^+ + y \quad &, \quad J^+\overline{x} &= q^{-1}\overline{x}J^+ \,, \\ J^+y &= q^{-1}yJ^+ \quad &, \quad J^+\overline{y} &= q\overline{y}J^+ - q^{-1}\overline{x} \,, \\ J^-x &= qxJ^- \quad &, \quad J^-\overline{x} &= q^{-1}\overline{x}J^- - q\overline{y} \,, \\ J^-y &= q^{-1}yJ^- + x \quad &, \quad J^-\overline{y} &= q\overline{y}J^- \,, \\ J^3x &= axJ^3 + \alpha x \quad &, \quad J^3\overline{x} &= a^{-1}\overline{x}J^3 - a^{-1}\alpha\overline{x} \,, \\ J^3y &= dyJ^3 + \tfrac{d-1}{a-1}\alpha y \quad &, \quad J^3\overline{y} &= d^{-1}\overline{y}J^3 + \tfrac{d^{-1}-1}{a-1}\alpha\overline{y} \,. \end{aligned} \tag{5.40}$$

We see that the action of J^3 is plagued by a few unwelcome degrees of freedom. α can be eliminated by normalizing the generator, i.e. we just put $\alpha = 1$. a and d can be fixed by imposing some kind of closure to the generators. In order to clarify this 'kind of closure' let us look at the action of $J^+$and J^-on the component x

$$\begin{aligned} J^+J^-x &= q^2xJ^+J^- \;+\; qyJ^- \;, \\ J^-J^+x &= q^2xJ^-J^+ \;+\; q^{-1}yJ^- \;+\; x \,. \end{aligned}$$

In order to make this look like a commutator we need to eliminate the terms linear in J^-. This can be done very easily by writing

$$(q^{-1}J^+J^- \;-\; qJ^-J^+)x \;=\; q^2x(q^{-1}J^+J^- \;-\; qJ^-J^+) \;-\; qx \,. \tag{5.41}$$

Seeing this equation and eq.(5.40) it is very tempting to use the above equation to define J^3, so that we would obtain

$$\begin{aligned} J^3 &= q^{-1}J^+J^- \;-\; qJ^-J^+ \;, \\ J^3x &= q^2xJ^3 - qx \;. \end{aligned}$$

[4]Their names might seem a bit strange, the reason for calling them this way will become clear in a few inches.

Working out the rest of the equations one can see that the above identification is legitimate. The same calculation then totally defines the action of J^3 to be

$$\begin{array}{rclcrcl} J^3 x &=& q^2 x J^3 - qx & , & J^3\overline{x} &=& q^{-2}\overline{x}J^3 + q^{-1}\overline{x}\,, \\ J^3 y &=& q^{-2} y J^3 + q^{-1} y & , & J^3\overline{y} &=& q^2\overline{y}J^3 - q\overline{y}\,. \end{array} \tag{5.42}$$

At the same time one finds the relations between the generators, which are given by

$$\begin{array}{rcl} q^{-1}J^+J^- \;-\; qJ^-J^+ &=& J^3\,, \\ q^2 J^3 J^+ \;-\; q^{-2}J^+J^3 &=& (q+q^{-1})J^+\,, \\ q^{-2}J^3J^- \;-\; q^2 J^- J^3 &=& -(q+q^{-1})J^-\,. \end{array} \tag{5.43}$$

From the above relations we see that in the limit $q = 1$ the generators behave as if they were generators of $su(2)$, therefore this algebra is called the algebra $su_q(2)$ which appeared first in [64]. Also the name for the generators is clear, because in the limit they are the ladder operators.

The point of our interest should, however, be the action of the algebra $su_q(2)$ on the bilinears. With the help of eqs.(5.40,5.42) the action can be determined to be

$$\begin{array}{rclcrcl} J^+A &=& q^{-2}AJ^+ & , & J^+C &=& CJ^+ + q^{-1}A\,, \\ J^+B &=& q^2BJ^+ + qD - q^{-1}C & , & J^+D &=& DJ^+ - q^{-1}A\,, \\ J^-A &=& q^{-2}AJ^- + q^{-1}C - qD & , & J^-C &=& CJ^- - qB\,, \\ J^-B &=& q^2BJ^- & , & J^-D &=& DJ^- + qB\,, \\ J^3A &=& q^{-4}AJ^3 + q^{-1}(q^{-2}+1)A & , & J^3C &=& CJ^3\,, \\ J^3B &=& q^4BJ^3 - q(q^2+1)B & , & J^3D &=& DJ^3\,, \end{array} \tag{5.44}$$

A nice result of this calculation is that one can see that, as in the normal case, rotations do not change the time component X_0. This means that a future energy will be invariant under rotations.

5.2.2 On with the boosts

It is clear that, in order to construct the boosts, we need a different, more general, 'Ansatz' than eq.(5.39). This 'Ansatz' can be found by defining

$$\begin{array}{rcl} T^i x &=& a^i_j x T^j + b^i_j y T^j + \alpha^i x + \beta^i y\,, \\ T^i y &=& c^i_j x T^j + d^i_j y T^j + \gamma^i x + \delta^i y\,, \\ T^i \overline{x} &=& \overline{a}^i_j \overline{x} T^j + \overline{b}^i_j \overline{y} T^j + \overline{\alpha}^i \overline{x} + \overline{\beta}^i \overline{y}\,, \\ T^i \overline{y} &=& \overline{c}^i_j \overline{x} T^j + \overline{d}^i_j \overline{y} T^j + \overline{\gamma}^i \overline{x} + \overline{\delta}^i \overline{y}\,, \end{array} \tag{5.45}$$

where a, b, c, d, etc., are,once again, $\mathbb{C}$-numbers, a bar needn't denote complex-conjugation and the indices i, j take values from 1 to some N. Upon imposing the same constraints as in the preceeding section, normalising and rescaling some generators, one can find two pairs of generators, denoted (T^1, T^2) and (S^1, S^2), which still include four degrees of freedom. The action on the spinors is, as can be shown by 'a student who enjoys gory details', given by

$$\begin{array}{lll} T^1x = aq^{-1}xT^1 + (aq^{-1}-1)\lambda^{-1}x & , & T^1\overline{x} = a^{-1}\overline{x}T^1 + q\lambda\overline{y}T^2 + (a^{-1}-1)\lambda^{-1}\overline{x}\,, \\ T^1y = dyT^1 + (d-1)\lambda^{-1}y & , & T^1\overline{y} = d^{-1}q\overline{y}T^1 + (d^{-1}q-1)\lambda^{-1}\overline{y}\,, \\ T^2x = axT^2 + \lambda yT^1 + y & , & T^2\overline{x} = a^{-1}q\overline{x}T^2\,, \\ T^2y = dq^{-1}yT^2 & , & T^2\overline{y} = d^{-1}\overline{y}T^2\,, \\ S^1x = \tilde{a}q^{-1}xS^1 & , & S^1\overline{x} = \tilde{a}^{-1}\overline{x}S^1 + \lambda\overline{y}S^2 + \overline{y}\,, \\ S^1y = \tilde{d}yS^1 & , & S^1\overline{y} = \tilde{d}^{-1}q\overline{y}S^1\,, \\ S^2x = \tilde{a}xS^2 + q\lambda yS^1 + (\tilde{a}-1)\lambda^{-1}x & , & S^2\overline{x} = \tilde{a}^{-1}q\overline{x}S^2 + (\tilde{a}^{-1}q-1)\lambda^{-1}\overline{x}\,, \\ S^2y = \tilde{d}q^{-1}yS^2 + (\tilde{d}q^{-1}-1)\lambda^{-1}y & , & S^2\overline{y} = \tilde{d}^{-1}\overline{y}S^2 + (\tilde{d}^{-1}-1)\lambda^{-1}\overline{y}\,, \end{array} \tag{5.46}$$

where $a, d, \tilde{a}$ and $\tilde{d}$ are the above mentioned degrees of freedom.

As in the $su_q(2)$-case we now impose the same 'kind of closure', which then eliminates two further degrees of freedom, namely: $\tilde{a} = qa$ and $\tilde{d} = q^{-1}d$. The resulting algebra is thus plagued by only two degrees of freedom, i.e.

$$\begin{array}{l} T^1T^2 - r^{-1}qT^2T^1 = (r^{-1}q-1)\lambda^{-1}T^2\,, \\ S^2S^1 - rqS^1S^2 = (rq-1)\lambda^{-1}S^1\,, \\ S^1T^1 - r^{-1}q^{-1}T^1S^1 = (r^{-1}q^{-1}-1)\lambda^{-1}S^1\,, \\ S^2T^2 - r^{-1}q^{-3}T^2S^2 = (r^{-1}q^{-3}-1)\lambda^{-1}T^2\,, \\ S^1T^2 - r^{-2}q^{-2}T^2S^1 = 0\,, \\ T^1S^2 - S^2T^1 = rq^2\lambda S^1T^2\,, \end{array} \tag{5.47}$$

where the abbreviation $r = \frac{a}{d}$ has been used extensively. An obvious fact of the above algebra is that there are four instead of three boosts. Furthermore, if the generators constructed above are truly boosts, why don't the rotation operators appear in the right-hand-side of eq.(5.47)? At this point it could be remarked that, thus, the boosts are combinations of normal boosts and rotations. Also it looks like there are sets of generators corresponding to different values of r, which would result in an infinite dimensional algebra, which doesn't seem to be too physical. These questions, however, will be dealt with after the completion of the q-deformed Lorentz algebra.

In order to finish the construction of the qLA, it is necessary to deduce the commutation relations of the rotations with the boosts, which isn't great fun to do... As a way to simplify the following expressions, it is best to rescale the rotations by $J^+ \to aJ^+$ and $J^- \to a^{-1}J^-$, which will not change expression (5.43) but will change the expressions (5.40,5.44). The resulting algebra is then found to be

$$\begin{array}{l} J^+T^1 - rqT^1J^+ = -T^2 + (rq-1)\lambda^{-1}J^+\,, \\ J^+T^2 - rq^{-1}T^2J^+ = 0\,, \\ J^-T^1 - r^{-1}qT^1J^- = q^2S^1 + (r^{-1}q-1)\lambda^{-1}J^-\,, \\ J^-T^2 - r^{-1}q^{-1}T^2J^- = S^2 - T^1\,, \\ J^3T^1 - T^1J^3 = 0\,, \\ q^2J^3T^2 - q^{-2}T^2J^3 = (q+q^{-1})T^2\,, \\ J^+S^1 - rq^3S^1J^+ = (1-\lambda J^3)T^1 - S^2 - J^3\,, \\ J^+S^2 - rqS^2J^+ = q^2(1-\lambda J^3)T^2 + (rq-1)\lambda^{-1}J^+\,, \\ J^-S^1 - r^{-1}q^{-1}S^1J^- = 0\,, \\ J^-S^2 - r^{-1}q^{-3}S^2J^- = -S^1 + (r^{-1}q^{-3}-1)\lambda^{-1}J^-\,, \\ q^{-2}J^3S^1 - q^2S^1J^3 = -(q+q^{-1})S^1\,, \\ J^3S^2 - S^2J^3 = 0\,. \end{array} \tag{5.48}$$

This concludes the construction, up to some minor difficulties, of the qLA.

As the 'gran finale' of this section, the action of the boosts on the bilinears will be given, where, of course, we won't bother you with the tedious calculations leading to .....

$$\begin{aligned}
T^1 A &= r^{-1} A T^1 + d\lambda D T^2 + (r^{-1}-1)\lambda^{-1} A\,,\\
T^1 B &= r B T^1 + (r-1)\lambda^{-1} B\,,\\
T^1 C &= q^{-1} C T^1 + q\lambda^2 D T^1 + aq\lambda B T^2 - (q+1)^{-1} C + q\lambda D\,,\\
T^1 D &= q D T^1 + (q^{-1}+1)^{-1} D\,,\\
T^2 A &= r^{-1} A T^2\,,\\
T^2 B &= r B T^2 + d^{-1}\lambda D T^1 + d^{-1} D\,,\\
T^2 C &= q C T^2 + a^{-1} q\lambda A T^1 + a^{-1} q A\,,\\
T^2 D &= q^{-1} D T^2\,,\\
S^1 A &= q^{-2} r^{-1} A S^1 + q^{-2} d\lambda D S^2 + q^{-2} d D\,,\\
S^1 B &= q^2 r B S^1\,,\\
S^1 C &= q^{-1} C S^1 + q\lambda^2 D S^1 + q\lambda a B S^2 + qaB\,,\\
S^1 D &= q D S^1\,,\\
S^2 A &= q^{-2} r^{-1} A S^2 + (q^{-2} r^{-1} - 1)\lambda^{-1} A\,,\\
S^2 B &= q^2 r B S^2 + q^2 d^{-1}\lambda D S^1 + (q^2 r - 1)\lambda^{-1} B\,,\\
S^2 C &= q C S^2 + q a^{-1}\lambda A S^1 + (q^{-1}+1)^{-1} C\,,\\
S^2 D &= q^{-1} D S^2 - (q+1)^{-1} D\,.
\end{aligned} \tag{5.49}$$

As a last remark in this section, let us stress the fact that the degrees of freedom in this algebra, a and d, have to be functions of q. This can be seen very clearly in the eqs.(5.47,5.48), which have to become commutation relations when $q \to 1$.

5.2.3 Some unsolved mysteries, unravelled

The first mystery that will be unravelled, is the question whether the boosts are viable deformations of the boosts and/or rotations. The way to do this, will be to look at the representations of the operators in the limit $q \to 1$. As could have been noticed before, the representations of the boosts on the spinors is, by construction, given by eq.(5.46). If one ignores the terms involving operators in the right-hand-side of eq.(5.46)[5], one finds the representation on the base $(x, y, \overline{x}, \overline{y})$ to be

$$T^1 = \lambda^{-1} \begin{pmatrix} (aq^{-1}-1) & & & \\ & (d-1) & & \\ & & (a^{-1}-1) & \\ & & & (d^{-1}q-1) \end{pmatrix},$$

[5]This ignoring is the same as saying that the boosts act on the identity by the trivial representation, i.e. they annihilate 1. Since here we are looking for the matrix-representation on a linear vectorspace and not their action on the polynomial-space, there is no need for them to work farther than one monomial. Therefore one must ignore the generators on the r.h.s. of eq.(5.46).

$$T^2 = \begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix} , \quad S^1 = \begin{pmatrix} 0 & 0 & & \\ 0 & 0 & & \\ & & 0 & 0 \\ & & 1 & 0 \end{pmatrix} ,$$

$$S^2 = \lambda^{-1} \begin{pmatrix} (aq-1) & & & \\ & (q^{-2}d-1) & & \\ & & (a^{-1}-1) & \\ & & & (d^{-1}q-1) \end{pmatrix} , \quad (5.50)$$

Due to the occurrence of the factor λ^{-1} in T^1 and S^2, one has to use l'Hôpitals rule to get the limit $q \to 1$. This means that the entities

$$a' = \frac{\partial a}{\partial q} |_{q=1} , \; d' = \frac{\partial d}{\partial q} |_{q=1} , \quad (5.51)$$

will enter the expression for the representations in the case $q = 1$. A simple calculation then yields

$$\lim_{q \to 1} T^1 = 2^{-1} \begin{pmatrix} a'-1 & & & \\ & d' & & \\ & & -a' & \\ & & & 1-d' \end{pmatrix} ,$$

$$\lim_{q \to 1} S^2 = 2^{-1} \begin{pmatrix} a'+1 & & & \\ & d'-2 & & \\ & & -a' & \\ & & & 1-d' \end{pmatrix} . \quad (5.52)$$

Comparing this with a well-known representation of the LA on the spinors, given by

$$J^+ = \begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 0 & 0 \end{pmatrix} , J^- = \begin{pmatrix} 0 & 1 & & \\ 0 & 0 & & \\ & & 0 & 0 \\ & & -1 & 0 \end{pmatrix} , J^3 = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} ,$$

$$K^+ = \begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 0 & 0 \end{pmatrix} , K^- = \begin{pmatrix} 0 & 1 & & \\ 0 & 0 & & \\ & & 0 & 0 \\ & & 1 & 0 \end{pmatrix} , K^3 = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} .$$

One can make, almost immediatly, the identification ($q = 1$)

$$T^2 = \frac{1}{2}(J^+ + K^+) \quad , \quad S^1 = \frac{1}{2}(K^- - J^-) ,$$
$$T^1 - S^2 = \frac{1}{2}(K^3 + J^3) \quad , \quad T^1 + S^2 = \frac{1}{2}((a' + d' - 1)C + (d' - a' - 1)J^3) . \quad (5.53)$$

Here C is a $u(1)$ generator, whose action on the spinors is given by $C\zeta = \zeta$, $C\overline{\zeta} = -\overline{\zeta}$. The most important conlusions to be made out of these identifications are that, first of all, the boosts consist of normal boosts and rotations and that the introduced deformation leads to a deformation of $sl(2,\mathbb{C}) \oplus u(1)$, and not to the deformation of $sl(2,\mathbb{C})$.

Two problems remain: We still have a parameter r which seems to lead to an infinite dimensional algebra and to each r there are four generators that form a closed alebra, whereas we only need three. As one can see from eq.(5.53) there is a dependence between T^1 and S^2 at the $q = 1$ level. This dependence leads us to suspect that such a dependence exists for general r, which would trivialize the latter problem.

The yellow road towards solving this problem is to take one special value for r, showing that there are only three independent generators and then to show that one can make a transformation from this particular r to general r. This obviously solves all of our problems mentioned before!

To this aim we define $a = 1$, $d = q$ which leads directly to $r = q^{-1}$. The algebra for this particular choice of r is given in appendix A. Note that with this choice of r we are dealing with a true deformation of $sl(2,\mathbb{C})$ and not with a deformation of $sl(2,\mathbb{C}) \oplus u(1)$.

In order to have a more accessible form of the alg.(5.43,5.47,5.48) with $r = q^{-1}$, we define $\mathcal{J}^3 = 1 - \lambda J^3$, $\mathcal{T}^1 = 1 + \lambda T^1$ and $\mathcal{S}^2 = 1 + \lambda S^2$. Their commutation relations and their action on the bilinears are given in eqs.(A.1,A.2,A.5,A.8).

The reduction of the four boosts to the three independent generators goes as follows. On the closed algebra formed by the generators $(\mathcal{T}^1, T^2, S^1, \mathcal{S}^2; r = q^{-1})$ one can find a central element $\mathcal{Z}$, which also commutes with the spinors. This central element can be found to be

$$\mathcal{Z} \;=\; \mathcal{T}^1\mathcal{S}^2 - q^2\lambda^2 T^2 S^1 \;. \tag{5.54}$$

Since it commutes with the spinors, e.g. $\mathcal{Z}x = x\mathcal{Z}$, it has to be a $\mathbb{C}$-number. The fact that $\mathcal{Z}$ is a $\mathbb{C}$-number allows us to write

$$\mathcal{S}^2 \;=\; (\mathcal{T}^1)^{-1}(\mathcal{Z} + q^2\lambda^2 T^2 S^1) \;, \tag{5.55}$$

which shows that there are only three independent boosts.

The redefinition of the generators with $r = q^{-1}$ to generators for general r is somewhat lengthy, so that the way of proving it will be mentioned [49]. First of all we introduce a q-generalisation of the generator C in eq.(5.53) which commutes with the generators of the $su_q(2)$ subalgebra. Note that from eq.(5.53) we have the right to introduce such an operator since in general we are dealing with a deformation of $sl(2,\mathbb{C}) \oplus u(1)$. Then one can make a redefinition on $\mathcal{T}^1, T^2, S^1, \mathcal{S}^2$ such that these generators satisfy eqs.(5.46,5.47,5.48,5.49).

5.3 q-Poincaré algebra

The Poincaré algebra $\pi_{1,3}$ is equal to the Lorentz algebra extended by its action on the translation algebra $t_{1,3}$. This translation algebra manifests itself as differentials on the Minkowski space $M_{1,3}$. It is the above idea that will be used to extend the q-Lorentz algebra, defined in the foregoing section, to the q-Poincaré algebra.

5.3.1 Differential calculus on $q - M_{1,3}$

The q-Lorentz algebra constructed in the preceeding section acts on the space of bilinears, which we'll interpret as q-Minkowski space[6]. So, in order to reach our goal we need to develop

[6]Our notion that the space of bilinears indeed forms a q-Minkowki space is consolidated by the eqs.(5.34,5.35).

a differential calculus on the space of bilinears, which the we'll then interpret as the translation algebra on the q-Minkowski space.

Let's introduce a base for the bilinears which will greatly simplify our work:

$$X^{(ab)} = \overline{\zeta}^a \zeta^b \Rightarrow \begin{cases} X^{(11)} &= \sqrt{q}B \\ X^{(12)} &= \sqrt{q}D \\ X^{(21)} &= -\frac{1}{\sqrt{q}}C \\ X^{(22)} &= -\frac{1}{\sqrt{q}}A \end{cases} \tag{5.56}$$

The commutation relations between the X's can be derived most easily by using the relations (5.11,5.21,5.28). By reshuffling the q-spinors from the left to the right we can write

$$X_1^{(ab)} X_2^{(cd)} = R^{(ab)(cd)}{}_{(ef)(gh)} X_2^{(ef)} X_1^{(gh)} , \tag{5.57}$$

and we find for the R-matrix on q-Minkowski space

$$R^{(ab)(cd)}{}_{(jl)(mi)} = q^{-2} R^{bc}{}_{ef} R^{fd}{}_{hi} R^{ae}{}_{jk} (R^{-1})^{kh}{}_{lm} , \tag{5.58}$$

where the factor q^{-2} arises due to the ambiguity, k, in the q-spinor relations between different sets of q-spinors[7], which was fixed to $k = q^{-1}$. Since the explicit form of this R-matrix isn't very enlightning, it won't be given here but in the appendix A.

Change the indexation from (ab) to i, so that we can rewrite eq.(5.57) to be

$$C^{ij}{}_{kl} X^k X^l = 0 \rightarrow C^{ij}{}_{kl} = R^{ij}{}_{kl} - \delta^i{}_k \delta^j{}_l . \tag{5.59}$$

According to [60] we can define a consistent differential calculus on a quantum space whose commutation relations are given by equations like eq.(5.59). Therefore introduce a derivative ∂_i by making the Anstaz

$$\partial_i X^j = \delta_i{}^j + Q^{jl}{}_{in} X^n \partial_l . \tag{5.60}$$

If we apply (5.60) to (5.59) we obtain

$$\begin{aligned} 0 &= \partial_m C^{ij}{}_{kl} X^k X^l \\ &= C^{ij}{}_{ml} X^l + C^{ij}{}_{kl} Q^{kn}{}_{mq} X^q \partial_n X^l \\ &= C^{ij}{}_{kn} \left(\delta_m{}^k \delta_l{}^n + Q^{kn}{}_{ml} \right) X^l , \end{aligned} \tag{5.61}$$

from which we find a consistency relation on Q, i.e.

$$C^{ij}{}_{kn} \left(\delta_m{}^k \delta_l{}^n + Q^{kn}{}_{ml} \right) = 0 . \tag{5.62}$$

Now we're in a position to generalize the exterior derivative d, which plays an important role in geometry[8]. We'll keep the basic properties of the exterior derivative, namely:

- The Leibniz rule on functions f and g: $d(fg) = (df)g + f(dg)$,
- closure or the coboundary condition: $d^2 = 0$.

[7]for a more exact statement one is refered to appendix A.

[8]The reason that it will be developed here in detail, is that with it one might study the 'geometry of the space of bilinears'. This however will not be done in this thesis, so that it's use lies foremost in its 'future convenience'.

As usual one can expand the exterior derivative as $d = \xi^i \partial_i$, where the ξ's will be the deformed analogues of the (dX^i)'s. From this expansion one finds by using closure on the coordinates that $d\xi^i = -\xi^i d$ which is an undeformed property. Upon using the Leibniz rules for d and ∂_i, eq.(5.60), and the expansion of d on a one-form one can find

$$X^i \xi^j \; = \; Q^{ij}{}_{kl} \xi^k X^l \; , \tag{5.63}$$

as the defining relations between the X's and the ξ's. Another relation can be found by using d on eq.(5.63). The resulting relation reads

$$\xi^i \xi^j \; = \; -Q^{ij}{}_{kl} \xi^k \xi^l \; . \tag{5.64}$$

We only need to specify two more sets of commutation relations, namely the $\partial\partial$- and the $\partial\xi$-relations, in order to completely define the exterior algebra on the bilinears. Write $\partial\partial = F\partial\partial$, with an obvious indexation, and use it on a coordinate X^k, which gives the relation

$$\left(\delta_m{}^k \delta_n{}^l + Q^{kl}{}_{mn}\right) \left(\delta_i{}^m \delta_j{}^n - F^{nm}{}_{ji}\right) \; = \; 0 \; . \tag{5.65}$$

Comparing this equation with eq.(5.62), one sees that F has to be something like R but with some changed indexes, i.e.

$$\partial_i \partial_j \; = \; R^{lk}{}_{ji} \partial_k \partial_l \; . \tag{5.66}$$

One should notice that the great similarity between the commutation relation of the differentials (5.66) and the ones between the coordinates (5.57) will enable us to find an isomorphism between the translations and the elements of q-Minkowski space.

The $\partial_i \xi^j$-relations can be found by using the same technique, e.g.

$$\partial_i \xi^j \; = \; \left(Q^{-1}\right)^{jn}{}_{il} \xi^l \partial_n \; . \tag{5.67}$$

This ends our construction of the exterior algbebra on the bilinears. The question ought to be whether this construction is consistent. Actually this consistency is guaranteed by the fact that the two defining matrices, R and Q, have to satisfy the Yang-Baxter equations (5.16) and all kinds of crossing relations 'à la' Yang-Baxter, e.g. $R_{12}R_{23}Q_{12} = Q_{23}R_{12}R_{23}$ [50, 60, 65].

Having found all the defining relations in the X-base we can work out all the results in the (A, B, C, D)-base, alfabet-base for short, by rescaling

$$\begin{array}{lcl} \partial_{(11)} = \frac{1}{\sqrt{q}} \partial_B & , & \partial_{(12)} = \frac{1}{\sqrt{q}} \partial_D \; , \\ \partial_{(21)} = -\sqrt{q}\, \partial_C & , & \partial_{(22)} = -\sqrt{q}\, \partial_A \; . \end{array} \tag{5.68}$$

Note that by using this rescaling the derivatives in the alfabet-base the derivatives are normalised to unity, e.g. $(\partial_A A) = 1$. By using the above rescaling we can write, from eq.(5.66), the commutation relations of the differentials in the alfabet-base to be

$$\begin{array}{lcl} \partial_A \partial_B = \partial_B \partial_A - q\lambda \partial_C \partial_C + q\lambda \partial_D \partial_C & , & \partial_B \partial_C = q^{-2} \partial_C \partial_B \; , \\ \partial_A \partial_C = q^2 \partial_C \partial_A & , & \partial_B \partial_D = \partial_D \partial_B + q\lambda \partial_C \partial_B \; , \\ \partial_A \partial_D = \partial_D \partial_A - q^3 \lambda \partial_C \partial_A & , & \partial_C \partial_D = \partial_D \partial_C \; . \end{array} \tag{5.69}$$

The rest of the needed, explicit relations can be found in appendix A.

5.3.2 The action of the q-Lorentz on the differentials

In order to define the q-Poincaré algebra, we need to specify the action of the q-Lorentz algebra on the q-differentials. This can be done by methods analogous to the ones used in sections (5.2.2) and (5.2.3). However, these methods are highly time-consuming and therefore not very cunning.

As was mentioned before, we suspect that an isomorphism exists between the q-coordinates and the q-differentials by virtue of the great similarity between eq.(5.57) and eq.(5.66). By looking at eq.(5.69) and eq.(5.33) we note that the following homomorphism exists

$$\begin{array}{ll} A \sim -\partial_B & , \quad B \sim -q^{-2}\partial_A \ , \\ C \sim q\lambda\partial_C + \partial_D & , \quad D \sim \partial_C \end{array} \tag{5.70}$$

By using this homomorphism on the action of the q-Lorentz algebra on the bilinears, eqs.(5.44,5.49), or better (A.2-A.8), we find the action on the differentials to be

$$\begin{array}{llll}
J^+\partial_A = q^2\partial_A J^+ - q\partial_C + q\partial_D & , & J^+\partial_C = \partial_C J^+ + q^{-1}\partial_B \ , \\
J^+\partial_B = q^{-2}\partial_B J^+ & , & J^+\partial_D = \partial_D J^+ - q\partial_B \ , \\
\\
J^-\partial_A = q^2\partial_A J^- & , & J^-\partial_C = \partial_C J^- - q^{-1}\partial_A \ , \\
J^-\partial_B = q^{-2}\partial_B J^- + q^{-1}\partial_C - q^{-1}\partial_D & , & J^-\partial_D = \partial_D J^- + q\partial_A \ , \\
\\
\mathcal{J}^3\partial_A = q^4\partial_A\mathcal{J}^3 & , & \mathcal{J}^3\partial_C = \partial_C\mathcal{J}^3 \ , \\
\mathcal{J}^3\partial_B = q^{-4}\partial_B\mathcal{J}^3 & , & \mathcal{J}^3\partial_D = \partial_D\mathcal{J}^3 \ , \\
\\
\mathcal{T}^1\partial_A = q^{-1}\partial_A\mathcal{T}^1 & , & \mathcal{T}^1\partial_C = q\partial_C\mathcal{T}^1 \ , \\
\mathcal{T}^1\partial_B = q\partial_B\mathcal{T}^1 - q\lambda^2\partial_C T^2 & , & \mathcal{T}^1\partial_D = q^{-1}\partial_D\mathcal{T}^1 - q^{-1}\lambda^2\partial_A T^2 \ , \\
\\
T^2\partial_A = q^{-1}\partial_A T^2 - q\partial_C\mathcal{T}^1 & , & T^2\partial_C = q^{-1}\partial_C T^2 \ , \\
T^2\partial_B = q\partial_B T^2 & , & T^2\partial_D = q\partial_D T^2 + q\lambda^2\partial_C T^2 - q\partial_B\mathcal{T}^1 \ , \\
\\
S^1\partial_A = q\partial_A S^1 & , & S^1\partial_C = q\partial_C S^1 \ , \\
S^1\partial_B = q^{-1}\partial_B S^1 - q^{-1}\partial_C\mathcal{S}^2 & , & S^1\partial_D = q^{-1}\partial_D S^1 - q^{-1}\partial_A\mathcal{S}^2 \ , \\
\\
\mathcal{S}^2\partial_A = q\partial_A\mathcal{S}^2 - q^3\lambda^2\partial_C S^1 & , & \mathcal{S}^2\partial_C = q^{-1}\partial_C\mathcal{S}^2 \ , \\
\mathcal{S}^2\partial_B = q^{-1}\partial_B\mathcal{S}^2 & , & \mathcal{S}^2\partial_D = q\partial_D\mathcal{S}^2 + q\lambda^2\partial_C\mathcal{S}^2 - q\lambda^2\partial_C S^1 \ .
\end{array} \tag{5.71}$$

One can show that the methods used in section (5.2) lead to the same result as by this method [50].

5.4 Conjugation and Hopf structure

In this section we'll try to give the derivation of the Hopf structure. By using the conjugation structure and some good 'Ansätze' we end up with a coproduct.

5.4.1 conjugation structure of the algebra

In section (5.1) we introduced a conjugation on q-spinors which included order reversal. The task of this section is to show how this conjugation afflicts the generators of the algebra, and thus opening the way to selfadjoint operators that we can use as observables.

Since the $su_q(2)$ is a subalgebra of the Wess-Zumino algebra, we expect that the conjugated elements of $su_q(2)$ are members of $su_q(2)$. We are confident about this since classically we have

$$q = 1 \;:\qquad \overline{J^\pm} = J^\mp, \qquad \overline{J^3} = J^3 \; . \tag{5.72}$$

Let's look at the representations of $su_q(2)$ on the bilinears (5.44), or (A.2). Take as an example the relation $J^+C = CJ^+ + q^{-1}A$. Under conjugation this equation goes over in $\overline{J^+}C = C\overline{J^+} - q^{-1}B$. The only equation of this type in the alg.(A.2) is $J^-C = CJ^- - qB$, so that when we match these two equations we end up with $\overline{J^+} = q^{-2}J^-$, which also holds for the other actions of J^+ on the bilinears. Upon imposing $\overline{\overline{J^+}} = J^+$ we end up with $\overline{J^-} = q^2J^+$, which is also consistent with eqs.(A.1,A.2).

Looking at the action of the $\mathcal{J}^3$ we note that this already is self-adjoint so that we find the action of $su_q(2)$ under conjugation to be

$$\overline{J^\pm} = q^{\mp 2}J^\mp \qquad , \qquad \overline{\mathcal{J}^3} = \mathcal{J}^3 \Leftrightarrow \overline{J^3} = J^3 \; . \tag{5.73}$$

This conjugation structure can be shown to be a homomorphism on the alg.(5.43)

The boosts can be dealt with in the same manner. We already know that the boosts are a combination of classical boosts and rotations. Looking at the classical level, i.e. $q = 1$, we see from eq.(5.53) that T^1 and S^2 form a conjugated pair, e.g. $\overline{T^1} \sim S^2$ etc. By the same reasoning we suspect T^2 and S^1 to be another conjugated pair. Let's see whether this is consistent.

Have a look at the relation

$$\mathcal{S}^2J^+ - J^+\mathcal{S}^2 \;=\; -q^2\lambda\mathcal{J}^3T^2 \; . \tag{5.74}$$

Under conjugation this goes over in $\overline{\mathcal{S}^2}J^- - J^-\overline{\mathcal{S}^2} = q^4\lambda\overline{T^2}\mathcal{J}^3$, which we, obviously, must put equal to

$$\mathcal{T}^1J^- - q^{-2}J^-\mathcal{T}^1 = -\lambda S^1 \; . \tag{5.75}$$

Seeing the fact that we have got to get rid of a factor $\mathcal{J}^3$ we put $\overline{\mathcal{S}^2} = f(\mathcal{J}^3)\mathcal{T}^1$ and $\overline{T^2} = g(\mathcal{J}^3)S^1$, which gives by virtue of eq.(A.2)

$$f(\mathcal{J}^3)\mathcal{T}^1J^- - f(q^{-4}\mathcal{J}^3)J^-\mathcal{T}^1 \;=\; \lambda g(\mathcal{J}^3)\mathcal{J}^3S^1 \; . \tag{5.76}$$

By comparing eq.(5.76) with eq.(5.75) we find that $f(x) = \sqrt{x}$ and $g(x) = -x^{-\frac{1}{2}}$. Applying this to the whole algebra and the representation on the bilinears (A.2–A.8), we find the consistent action under conjugation to be

$$\begin{array}{lcl} \overline{\mathcal{T}^1} = (\mathcal{J}^3)^{-\frac{1}{2}}\mathcal{S}^2 & , & \overline{T^2} = -(\mathcal{J}^3)^{-\frac{1}{2}}S^1 \; , \\ \overline{\mathcal{S}^2} = (\mathcal{J}^3)^{\frac{1}{2}}\mathcal{T}^1 & , & \overline{S^1} = -q^2(\mathcal{J}^3)^{\frac{1}{2}}T^2 \; . \end{array} \tag{5.77}$$

Note that also in this case we have $\overline{\overline{\text{boost}}} = \text{boost}$, although this is not obvious at first sight.

In the classical case the conjugation structure of the derivatives is linear, e.g. $\overline{\partial_\mu} = -\partial_\mu$. In this case, however, no such linear structure can exist. In order to show this look at, (A.10), $\partial_C A = A\partial_C - q^{-1}\lambda D\partial_B$ and $\partial_B D = q^{-2}D\partial_B$. Under conjugating these equations we find

$$\overline{\partial_C}B = B\overline{\partial_C} + q\lambda D\overline{\partial_B} \qquad , \qquad \overline{\partial_B}D = q^2D\overline{\partial_B} \; , \tag{5.78}$$

where we have used the conjugation properties of the bilinears (5.31). If we make the linear 'Ansatz' $\overline{\partial_C} = \alpha\partial_A + \beta\partial_B + \gamma\partial_C + \delta\partial_D$, use eq.(A.10) to order the differentials to the right, we obtain

$$\overline{\partial_C} = \alpha\partial_A + q\lambda\delta\partial_C + \delta\partial_D \quad , \quad \overline{\partial_B} = -q^{-2}\delta\partial_A \ . \tag{5.79}$$

Using this result in the conjugated version of the formula $\partial_B C = C\partial_B - q\lambda A\partial_C$, gives straightforwardly $\overline{\partial_C} = \overline{\partial_B} = 0$, which means that no linear conjugation structure is possible on the algebra of differentials on the quantum Minkowski space. It can be proven [4, 58] that it is impossible to have a linear conjugation properties for both the coordinates and the differentials in a quantum space. It is however possible to have either linear conjugation properties for one of both, so that one can choose either the differentials or the coordinates to be conjugating into a linear combination.

The derivation of the conjugation-structure for the differentials is not at all straightforward and involves elements of the space [50]. First we define the 'Laplacian', which is the quadratic invariant for the algebra of the differentials

$$\Box \ = \ \partial_A\partial_B - q^2\partial_C\partial_D \ . \tag{5.80}$$

We also need a kind of scaling operator which we define by

$$\Lambda \ = \ 1 - q^{-1}\lambda X^{(ab)}\partial_{(ab)} + q^{-2}\lambda^2\mathcal{L}\Box \ , \tag{5.81}$$

where the $X\partial$ term takes, owing to our normalization in the alfabet-base, the form $A\partial_A + B\partial_B + C\partial_C + D\partial_D$ and where $\mathcal{L}$ can be found in eq.(5.34). This scaling operator can be seen to satisfy

$$\Lambda X^{(ab)} = q^{-2}X^{(ab)}\Lambda \quad , \quad \Lambda\partial_{(ab)} = q^2\partial_{(ab)}\Lambda \ , \tag{5.82}$$

and is a qL scalar. With the help of these operator we can write down the conjugation structure of the differentials. Their explicit form reads

$$\begin{aligned}
\overline{\partial_A} &= -q^6\Lambda^{-1}\left(\partial_A - q^{-1}\lambda B\Box\right) \ , \\
\overline{\partial_B} &= -q^2\Lambda^{-1}\left(\partial_B - q^{-3}\lambda A\Box\right) \ , \\
\overline{\partial_C} &= -q^4\Lambda^{-1}\left(\partial_C + q^{-3}\lambda D\Box\right) \ , \\
\overline{\partial_D} &= -q^4\Lambda^{-1}\left(\partial_D + q^{-3}\lambda\left(C - q\lambda D\right)\Box\right) \ .
\end{aligned} \tag{5.83}$$

One can see immediatly that we recover the normal result when $q = 1$, which however isn't a guarantee for correctness. One can check however that the left- and right-hand side give the same result when working on the bilinears. Note that the terms proportional to the Laplacians on the righthandside of eq.(5.83) have the same transformation properties as the differentials so that the mapping is covariant.

5.4.2 the Hopf structure: A brute force method

The construction of a consistent Hopfian structure on weird algebras poses a tough nut to crack. Sometimes one is helped a great deal by a ready-made Hopf structure, see e.g. the LNR algebra presented in chapter 3, which one can convert into the structure one needs. In this case we are not so lucky, since, at first sight, there is no underlying Hopf structure, so that we have to resort to lucky guesses and cunningness. The route followed here is to a large extent the same as in the foregoing section. First the Hopf structure for the $su_q(2)$ subalgebra is deduced and then enlarged to the full qL. Then we'll have a go at the Hopf structure for the translations which will complete the Hopf structure of the q-Poincaré algebra.

Let's start with the Hopf structure for the $su_q(2)$ subalgebra. We'll make an 'Ansatz' for the coproduct for J^+, which will then, due to the conjugation structure (5.73), lead to the coproduct for the J^-. Take as a possible coproduct for $J^{\pm}$[9]

$$\Delta(J^\pm) \;=\; J^\pm \otimes g(\mathcal{J}^3) + f(\mathcal{J}^3) \otimes J^\pm \;, \tag{5.84}$$

where we have chosen $\mathcal{J}^3$ instead of J^3 because $\mathcal{J}^3$ has simpler commutation relations with the other J's than J^3, see eq(A.1). The coproduct for J^3 can be found by using the relation

$$q^{-1}J^+J^- - qJ^-J^+ = J^3 \Rightarrow q^{-1}\Delta(J^+)\Delta(J^-) - q\Delta(J^-)\Delta(J^+) = \Delta(J^3) \;, \tag{5.85}$$

which stems from the fact that the coproduct is supposed to be a homomorphism on the algebra (see eq.(2.20)). The calculation of the above entity gives

$$\begin{aligned}\Delta(J^3) \;&=\; J^3 \otimes g^2 + f^2 \otimes J^3 + q^{-1}J^+ f \otimes J^- g(4) - qJ^+ f(-4) \otimes J^- g \\ &\quad + q^{-1}J^- f(4) \otimes J^+ g - qJ^- f \otimes J^+ g(-4) \;,\end{aligned} \tag{5.86}$$

where the definitions $f = f(\mathcal{J}^3)$, $g = g(\mathcal{J}^3)$, $f(n) = f(q^n\mathcal{J}^3)$ and $g(n) = g(q^n\mathcal{J}^3)$ have been used. One should note that this experimental coproduct automatically satisfies $\overline{\Delta(J^3)} = \Delta(J^3)$ as it ought to.

By using the above coproducts in the equation $q^2J^3J^+ - q^{-2}J^+J^3 = (q+q^{-1})J^+$ we find for the left-hand side

$$\begin{aligned} l.h.s \;&=\; (q+q^{-1})\left[J^+ \otimes g^3 + f^3 \otimes J^+\right] && (5.87)\\ &\quad + \left(q^2f^2 - q^{-2}f^2(4)\right) J^+ \otimes J^3 g + J^3 f \otimes \left(q^2g^2 - q^{-2}g^2(4)\right) J^+ && (5.88)\\ &\quad + J^+J^+ \otimes J^- \left[(q+q^{-1})f(-4) \otimes g(4)g - q^3 f(-8) \otimes g^2 - q^{-3} f \otimes g^2(4)\right] && (5.89)\\ &\quad + J^- \otimes J^+J^+ \left((q+q^{-1})ff(4) \otimes g(-4) - q^3 f^2 \otimes g(-8) - q^{-3} f^2(4) \otimes g\right) && (5.90)\\ &\quad + J^-J^+ \otimes J^+ \left[qf \otimes 1 - q^3 f(-4) \otimes 1\right] \\ &\quad - J^+J^- \otimes J^+ \left[q^{-3} f(4) \otimes 1 - q^{-1} f \otimes 1\right] && (5.91)\\ &\quad + J^+ \otimes J^-J^+ \left(qf^2 \otimes g - q^3 ff(-4) \otimes g(-4)\right) \\ &\quad - J^+ \otimes J^+J^- \left(q^{-1}f^2(-4) \otimes g - q^{-3} ff(-4) \otimes g(4)\right) \;, && (5.92)\end{aligned}$$

whereas the right-hand-side is given by

$$r.h.s. = (q+q^{-1})\left[J^+ \otimes g + f \otimes J^+\right] \;. \tag{5.93}$$

Looking at these results we have to conclude that (5.89) and (5.90) have to vanish identically. The easiest way of doing this is by putting that f and g are homogeneous functions of different degrees, i.e.

$$f(q^n\mathcal{J}^3) = q^{n\alpha} f(\mathcal{J}^3), \qquad\qquad g(q^n\mathcal{J}^3) = q^{n\beta} g(\mathcal{J}^3) \;, \tag{5.94}$$

[9]How to get such an idea for such an 'Ansatz'. In this case we are helped by the Drinfel'd-Jimbo version of $su(2)$, the Cartan matrix is 1-dimensional and thus $A = 2$, eq.(3.2). in that case the coproduct for $J^\pm$, eq.(3.23), reads $\Delta(J^\pm) = J^\pm \otimes e^{J^3/2} + e^{-J^3/2} \otimes J^\pm$. So the thing we are going to look at is a generalization of this.

where α (β) is the degree of homogeneity of f (g). From (5.89) we then find a condition which the degree have to satisfy, namely

$$(q+q^{-1})q^{4\beta-4\alpha} - q^{3-8\alpha} - q^{-3-8\beta} = 0 \; . \tag{5.95}$$

This equation holds for any q when $\alpha = \frac{1}{2}, \beta = 0$ and also when $\alpha = 1, \beta = 0$. This result immediatly tells us that g has to be a constant. These solutions for α and β also take care of (5.90) which vanishes for both solutions.

If we take a look at (5.91) we see that it vanishes when $\alpha = \frac{1}{2}$, whereas $\alpha = 1$ leaves us with (5.91)$= \lambda(J^+J^- - J^-J^+)f \otimes J^+$, which we can't get rid off. From this we have to conclude that $\alpha = \frac{1}{2}$ is the only good choice we can make.

Upon using the above results in the l.h.s. we arrive at

$$l.h.s. = (q+q^{-1})\left[J^+ \otimes g^3 + f^3 \otimes J^+ + \lambda g^2 J^3 f \otimes J^+\right] \; , \tag{5.96}$$

which, when compared with eq.(5.93) and by using $\mathcal{J}^3 = 1 - \lambda J^3$, gives $g = 1$ and $f(\mathcal{J}^3) = \sqrt{\mathcal{J}^3}$. The whole calculation thus results in

$$\begin{array}{ll} \Delta(J^\pm) = J^\pm \otimes 1 + \sqrt{\mathcal{J}^3} \otimes J^\pm \; , & \\ \Delta(J^3) = J^3 \otimes 1 + \mathcal{J}^3 \otimes J^3, & \Delta(\mathcal{J}^3) = \mathcal{J}^3 \otimes \mathcal{J}^3 \; , \end{array} \tag{5.97}$$

where in the last line the usual coproduct $\Delta(1) = 1 \otimes 1$ has been used.

The rest of the Hopfian structure on $su_q(2)$ can be derived with great ease. First of all we note that the only one dimensional representation of $su_q(2)$ is trivial, so that we have to conclude that the counit has to be $\epsilon(J^\pm) = \epsilon(J^3) = 0$, from which $\epsilon(\mathcal{J}^3) = 1$ follows trivially, and which is homomorphism on $su_q(2)$. In the same way as in section (3.2) we can use eq.(2.22) to define the antipode S, e.g.

$$m \cdot (id \otimes S)\,\Delta(J^\pm) = i \cdot \epsilon(J^\pm) = 0 \;\rightarrow\; S(J^\pm) = -(\mathcal{J}^3)^{-\frac{1}{2}} J^\pm \; . \tag{5.98}$$

The complete Hopfian structure on $su_q(2)$ can then be seen to be

$$\begin{array}{lclcl} \Delta(J^\pm) = J^\pm \otimes 1 + \sqrt{\mathcal{J}^3} \otimes J^\pm & , & \epsilon(J^\pm) = 0 & , & S(J^\pm) = -(\mathcal{J}^3)^{-\frac{1}{2}} J^\pm \; , \\ \Delta(J^3) = J^3 \otimes 1 + \mathcal{J}^3 \otimes J^3 & , & \epsilon(J^3) = 0 & , & S(J^3) = -(\mathcal{J}^3)^{-1} J^3 \; , \\ \Delta(\mathcal{J}^3) = \mathcal{J}^3 \otimes \mathcal{J}^3 & , & \epsilon(\mathcal{J}^3) = 1 & , & S(\mathcal{J}^3) = (\mathcal{J}^3)^{-1} \; , \end{array} \tag{5.99}$$

and the usual form for the multiplication, inclusion and the Hopfian structure on unity.

The counit ought to be a homomorphism on the complete algebra (A.1,5.69,5.71), so that looking at the algebra we have to conclude that the counits satisfy

$$\begin{array}{l} \epsilon(T^1) = \epsilon(T^2) = \epsilon(S^1) = \epsilon(S^2) = 0 \; , \\ \epsilon(\mathcal{T}^1) = \epsilon(\mathcal{S}^2) = 1 \; , \\ \epsilon(\partial_A) = \epsilon(\partial_B) = \epsilon(\partial_C) = \epsilon(\partial_D) = 0 \; . \end{array} \tag{5.100}$$

The knowledge of this counit will come in handy when we are going to define the antipodes.

In the construction of the antipodes we'll need another ingredient. Have a look at eq.(5.54). There we introduced an element $\mathcal{Z}$ which took care of the reduction of one of the generators. We said that it had to be a number, but didn't yet have the power to appoint a value to it. Now, however, we are in a position to do so. Due to the homomorphic nature of the counit and the eq.(5.54) we know that $\epsilon(\mathcal{Z}) = 1$, from which we have to conclude that $\mathcal{Z} = 1$.

It is clear that our quest for the coproduct would take ages if we continued it as we have done above. So to speed up matters we'll omit the explicit details. Let's sketch however, briefly, the way of obtaining the results.

Due to the conjugation structure we only need to find two out of four coproduct for the boosts, $\mathcal{T}^1$ and T^2 say. Looking at their commutation relations with $\mathcal{J}^3$ we can discard some possible terms in the expansion of the coproduct[10]. Then one uses the commutation relations with $J^{\pm}$ to pin-point even more terms. And last but not least: You'll have to calculate all the rest of the commutation relations[11]....

The coproduct for the boosts can then be seen to be

$$\begin{aligned}
\Delta(\mathcal{T}^1) &= \mathcal{T}^1 \otimes \mathcal{T}^1 + \lambda^2 S^1 (\mathcal{J}^3)^{-\frac{1}{2}} \otimes T^2 \;, \\
\Delta(T^2) &= T^2 \otimes \mathcal{T}^1 + (\mathcal{J}^3)^{-\frac{1}{2}} \mathcal{S}^2 \otimes T^2 \;, \\
\Delta(S^1) &= S^1 \otimes \mathcal{S}^2 + (\mathcal{J}^3)^{\frac{1}{2}} \mathcal{T}^1 \otimes S^1 \;, \\
\Delta(\mathcal{S}^2) &= \mathcal{S}^2 \otimes \mathcal{S}^2 + \lambda^2 T^2 (\mathcal{J}^3)^{\frac{1}{2}} \otimes S^1 \;.
\end{aligned} \tag{5.101}$$

These coproducts, some algebra and the fact that $\mathcal{Z} = 1$ then result in the antipodes for the boosts, i.e.

$$\begin{aligned}
S(\mathcal{T}^1) &= \mathcal{S}^2 &, \quad S(S^1) &= -(\mathcal{J}^3)^{-\frac{1}{2}} S^1 \;, \\
S(T^2) &= -q^2 (\mathcal{J}^3)^{\frac{1}{2}} T^2 &, \quad S(\mathcal{S}^2) &= \mathcal{T}^1 \;.
\end{aligned} \tag{5.102}$$

This concludes the construction of the Hopfian structure on the quantum Lorentz algebra.

The coproduct for the differentials is found in a heuristic manner. We make an 'Ansatz' for a possible coproduct of the form

$$\Delta(\partial_i) = \partial_i \otimes 1 + \mathcal{O}_i^j \otimes \partial_j \;, \tag{5.103}$$

where we'll allow $\mathcal{O}$ to depend on generators of the qL and the scaling operator Λ, when necessary. An inspection of the derivative action shows that ∂_B and ∂_C should have a relative simple coproduct form. The coproduct for ∂_B can be found rather easily but contains an undetermined power of the scaling operator Λ, which had to be expected since it is a q-Lorentz scalar. An evaluation of ∂_B on B^2 'à la' eq.(2.6) will fix this degree of arbitraryness. The rest of the coproducts can then be generated by looking at the action of the $su_q(2)$ subalgbra on $\Delta(\partial_B)$.

$$\begin{aligned}
\Delta(\partial_A) &= \partial_A \otimes 1 + \Lambda^{\frac{1}{2}} (\mathcal{J}^3)^{\frac{1}{2}} \mathcal{T}^1 \otimes \partial_A + q^3 \lambda^2 \Lambda^{\frac{1}{2}} (\mathcal{J}^3)^{-\frac{1}{2}} J^- S^1 \otimes \partial_B \\
&\quad - \lambda \Lambda^{\frac{1}{2}} J^- \mathcal{T}^1 \otimes \partial_C - q\lambda \Lambda^{\frac{1}{2}} S^1 \otimes \partial_D \;, \\
\Delta(\partial_B) &= \partial_B \otimes 1 + \Lambda^{\frac{1}{2}} (\mathcal{J}^3)^{-\frac{1}{2}} \mathcal{S}^2 \otimes \partial_B - q\lambda \Lambda^{\frac{1}{2}} T^2 \otimes \partial_C \;, \\
\Delta(\partial_C) &= \partial_C \otimes 1 + \Lambda^{\frac{1}{2}} \mathcal{T}^1 \otimes \partial_C - q\lambda \Lambda^{\frac{1}{2}} (\mathcal{J}^3)^{-\frac{1}{2}} S^1 \otimes \partial_B \;, \\
\Delta(\partial_D) &= \partial_D \otimes 1 + \Lambda^{\frac{1}{2}} \mathcal{S}^2 \otimes \partial_D - q\lambda \Lambda^{\frac{1}{2}} (\mathcal{J}^3)^{\frac{1}{2}} T^2 \otimes \partial_A \\
&\quad - q^2 \lambda \Lambda^{\frac{1}{2}} (\mathcal{J}^3)^{-\frac{1}{2}} J^- \mathcal{S}^2 \otimes \partial_B + q\lambda^2 \Lambda^{\frac{1}{2}} J^- T^2 \otimes \partial_C \;.
\end{aligned} \tag{5.104}$$

This coproduct structure is a homomorphism on the q-Poincaré algebra and is coassociative. For a detailed discussion of the naturalness of the 'Ansatz' (5.103) and the result one is kindly referred to [50, 60].

[10] For example: the possible terms in $\Delta(\mathcal{T}^1)$ that are compatible with the commutation relation with $\mathcal{J}^3$ are $\mathcal{T}^1 \otimes \mathcal{T}^1$, $\mathcal{T}^1 \otimes \mathcal{S}^2$, $\mathcal{S}^2 \otimes \mathcal{T}^1$, $\mathcal{S}^2 \otimes \mathcal{S}^2$, $T^2 \otimes S^1$ and $S^1 \otimes T^2$, where they can be extended by functions of $\mathcal{J}^3$. Then one can discard the second, third and fourth term by looking at the expansion up to λ^2 and using ones knowledge of the $q = 1$ limit.

[11] You might as well take my word for it.

In order to prove that this coproduct indeed is coassociative one will need the Hopf structure of the scaling operator Λ. This structure is dictated by eq.(5.82) to be

$$\Delta(\Lambda) = \Lambda \otimes \Lambda, \qquad \epsilon(\Lambda) = 1, \qquad S(\Lambda) = \Lambda^{-1} \ . \tag{5.105}$$

Of course one can also calculate this straightforwardly, after we've defined the antipodes for the differentials.

Armed with the knowledge of the coproduct, counit and $\mathcal{Z} = 1$ one can calculate, by virtue of eq.(2.16,2.22), the antipode for the differentials

$$\begin{aligned} S(\partial_A) &= -\Lambda^{-\frac{1}{2}}(\mathcal{J}^3)^{-\frac{1}{2}}\left(\mathcal{S}^2\partial_A + q\lambda^2 S^1 J^- \partial_B + \lambda\mathcal{S}^2 J^- \partial_C + q\lambda S^1 \partial_D\right) \ , \\ S(\partial_B) &= -\Lambda^{-\frac{1}{2}}(\mathcal{J}^3)^{\frac{1}{2}}\left[\mathcal{T}^1\partial_B + q^3\lambda T^2 \partial_C\right] \ , \\ S(\partial_C) &= -\Lambda^{-\frac{1}{2}}\left(\mathcal{S}^2\partial_C + q^{-1}\lambda S^1 \partial_B\right) \ , \\ S(\partial_D) &= -\Lambda^{-\frac{1}{2}}\left[\mathcal{T}^1\partial_D + q\lambda T^2\partial_A + \lambda\mathcal{T}^1 J^- \partial_B + q\lambda^2 T^2 J^- \partial_C\right] \ . \end{aligned} \tag{5.106}$$

One can check that in the limit $q \to 1$ everything goes over into the things one would normally encounter (see chapter 2).

Now, we know everything about the WZ-algebra.

Appendix A

The final and explicit form of the WZ algebra

Let us present the final form of the algebra as derived in sections (5.2) and (5.3), with the choice $a = 1$ and $b = q$. Also the redefinition of some generators will be included, to wit $\mathcal{J}^3 = 1 - \lambda J^3$, $\mathcal{T}^1 = 1 + \lambda T^1$ and $\mathcal{S}^2 = 1 + \lambda S^2$. With these choices one can rewrite eqs.(5.43,5.47,5.48) to be

$$
\begin{array}{lll}
q^{-1}J^+J^- - qJ^-J^+ = J^3 = \lambda^{-1}(1 - J^3) & , & \\
q^{\pm 2}J^3J^\pm - q^{\mp 2}J^\pm J^3 = \pm(q + q^{-1})J^\pm & , & \mathcal{J}^3 J^\pm \;=\; q^{\mp 4}J^\pm \mathcal{J}^3 \, , \\
T^1T^2 - q^2T^2T^1 = (q^2 - 1)\lambda^{-1}T^2 & , & S^2S^1 - S^1S^2 = 0 \, , \\
S^2T^2 - q^{-2}T^2S^2 = (q^{-2} - 1)\lambda^{-1}T^2 & , & S^1T^1 - T^1S^1 = 0 \, , \\
T^1S^2 - S^2T^1 = q\lambda S^1T^2 & , & S^1T^2 - T^2S^1 = 0 \, , \\
J^-T^1 - q^2T^1J^- = q^2S^1 + (q^2 - 1)\lambda^{-1}J^- & , & J^+T^1 - T^1J^+ = -T^2 \, , \\
J^-T^2 - T^2J^- = S^2 - T^1 = \lambda^{-1}(\mathcal{S}^2 - \mathcal{T}^1) & , & J^+T^2 - q^{-2}T^2J^+ = 0 \, , \\
J^-S^1 - S^1J^- = 0 & , & J^+S^1 - q^2S^1J^+ = \lambda^{-1}(\mathcal{J}^3\mathcal{T}^1 - \mathcal{S}^2) \, , \\
J^-S^2 - q^{-2}S^2J^- = -S^1 + (q^{-2} - 1)\lambda^{-1}J^- & , & J^+S^2 - S^2J^+ = q^2(1 - \lambda J^3)T^2 = q^2\mathcal{J}^3T^2, \\
q^2J^3T^2 - q^{-2}T^2J^3 = (q + q^{-1})T^2 & , & J^3T^1 - T^1J^3 = 0 \, , \\
q^{-2}J^3S^1 - q^2S^1J^3 = -(q + q^{-1})S^1 & , & J^3S^2 - S^2J^3 = 0 \, , \\
\mathcal{J}^3\mathcal{T}^1 = \mathcal{T}^1\mathcal{J}^3 & , & \mathcal{J}^3T^2 = q^{-4}T^2\mathcal{J}^3 \, , \\
\mathcal{J}^3S^1 = q^4S^1\mathcal{J}^3 & , & \mathcal{J}^3\mathcal{S}^2 = \mathcal{S}^2\mathcal{J}^3 \, , \\
\mathcal{T}^1J^+ - J^+\mathcal{T}^1 = \lambda T^2 & , & \mathcal{T}^1J^- - q^{-2}J^-\mathcal{T}^1 = -\lambda S^1 \, , \\
\mathcal{T}^1T^2 = q^2T^2\mathcal{T}^1 & , & \mathcal{T}^1S^1 = S^1\mathcal{T}^1 \, , \\
\mathcal{T}^1\mathcal{S}^2 - \mathcal{S}^2\mathcal{T}^1 = q\lambda^3T^2S^1 & , & \\
\mathcal{S}^2J^+ - J^+\mathcal{S}^2 = -q^2\lambda\mathcal{J}^3T^2 & , & \mathcal{S}^2J^- - q^2J^-\mathcal{S}^2 = q^2\lambda S^1 \, , \\
\mathcal{S}^2T^2 = q^{-2}T^2\mathcal{S}^2 & , & \mathcal{S}^2S^1 = S^1\mathcal{S}^2 \, .
\end{array}
\tag{A.1}
$$

As the further developments in chapter 5 show, one only needs the action of the generators on the bilinears. Therefore only this action will be given and the action on the spinors will be flung into oblivion. In the same way as before one can rewrite eqs.(5.44,5.49) to

$$\begin{array}{lcl} J^+A = q^{-2}AJ^+ & , & J^+C = CJ^+ + q^{-1}A \ , \\ J^+B = q^2BJ^+ + qD - q^{-1}C & , & J^+D = DJ^+ - q^{-1}A \ , \\ J^-A = q^{-2}AJ^- + q^{-1}C - qD & , & J^-C = CJ^- - qB \ , \\ J^-B = q^2BJ^- & , & J^-D = DJ^- + qB \ , \\ J^3A = q^{-4}AJ^3 + q^{-1}(q^{-2}+1)A & , & J^3C = CJ^3 \ , \\ J^3B = q^4BJ^3 - q(q^2+1)B & , & J^3D = DJ^3 \ , \\ \mathcal{J}^3A = q^{-4}A\mathcal{J}^3 & , & \mathcal{J}^3C = C\mathcal{J}^3 \ , \\ \mathcal{J}^3B = q^4B\mathcal{J}^3 & , & \mathcal{J}^3D = D\mathcal{J}^3 \ , \end{array} \tag{A.2}$$

$$\begin{aligned} T^1A &= qAT^1 + q\lambda DT^2 + (q-1)\lambda^{-1}A \ , \\ T^1B &= q^{-1}BT^1 + (q^{-1}-1)\lambda^{-1}B \ , \\ T^1C &= q^{-1}CT^1 + q\lambda^2DT^1 + q\lambda BT^2 - (q+1)^{-1}C + q\lambda D \ , \\ T^1D &= qDT^1 + (q^{-1}+1)^{-1}D \ , \end{aligned} \tag{A.3}$$

$$\begin{aligned} T^2A &= qAT^2 \ , \\ T^2B &= q^{-1}BT^2 + q^{-1}\lambda DT^1 + q^{-1}D \ = \ q^{-1}BT^2 + q^{-1}D\mathcal{T}^1 \ , \\ T^2C &= qCT^2 + q\lambda AT^1 + qA \ = \ qCT^2 + qA\mathcal{T}^1 \ , \\ T^2D &= q^{-1}DT^2 \ , \end{aligned} \tag{A.4}$$

$$\begin{aligned} \mathcal{T}^1A &= qA\mathcal{T}^1 + q\lambda^2DT^2 \ , \\ \mathcal{T}^1B &= q^{-1}B\mathcal{T}^1 \ , \\ \mathcal{T}^1C &= q^{-1}C\mathcal{T}^1 + q\lambda^2D\mathcal{T}^1 + q\lambda^2BT^2 \ , \\ \mathcal{T}^1D &= qD\mathcal{T}^1 \ , \end{aligned} \tag{A.5}$$

$$\begin{aligned} S^1A &= q^{-1}AS^1 + q^{-1}\lambda DS^2 + q^{-1}D \ = \ q^{-1}AS^1 + q^{-1}D\mathcal{S}^2 \ , \\ S^1B &= qBS^1 \ , \\ S^1C &= q^{-1}CS^1 + q\lambda^2DS^1 + q\lambda BS^2 + qB \ = \ q^{-1}CS^1 + q\lambda^2DS^1 + qB\mathcal{S}^2 \ , \\ S^1D &= qDS^1 \ , \end{aligned} \tag{A.6}$$

$$\begin{aligned} S^2A &= q^{-1}AS^2 + (q^{-1}-1)\lambda^{-1}A \ , \\ S^2B &= qBS^2 + q\lambda DS^1 + (q-1)\lambda^{-1}B \ , \\ S^2C &= qCS^2 + q\lambda AS^1 + (q^{-1}+1)^{-1}C \ , \\ S^2D &= q^{-1}DS^2 - (q+1)^{-1}D \ , \end{aligned} \tag{A.7}$$

$$\begin{aligned} \mathcal{S}^2A &= q^{-1}A\mathcal{S}^2 \ , \\ \mathcal{S}^2B &= qB\mathcal{S}^2 + q\lambda^2DS^1 \ , \\ \mathcal{S}^2C &= qC\mathcal{S}^2 + q\lambda^2AS^1 \ , \\ \mathcal{S}^2D &= q^{-1}D\mathcal{S}^2 \ , \end{aligned} \tag{A.8}$$

R-matrix for the bilinears

The commutation relations between different sets of q-spinors are known to contain a degree of freedom which was fixed to $k = q^{-1}$ in section (5.1). This will lead to the occurrence of the factor q^{-2} in eq.(5.58). Define the base of bilinears as in eq.(5.56), then we can find the relations among them by using the relations of the q-spinors, i.e.

$$
\begin{aligned}
X_1^{(ab)} X_2^{(cd)} &\equiv \overline{\zeta}_1^a \zeta_1^b \overline{\zeta}_2^c \zeta_2^d \\
&= R^{bc}{}_{ij} \overline{\zeta}_1^a \overline{\zeta}_2^i \zeta_1^j \zeta_2^d = \dots \\
&= q^{-2} R^{bc}{}_{ij} R^{jd}{}_{kl} R^{ai}{}_{en} (R^{-1})^{nl}{}_{fg} \overline{\zeta}_2^e \zeta_2^f \overline{\zeta}_1^g \zeta_1^h \\
&\equiv R^{(ab)(cd)}{}_{(ef)(gh)} X_2^{(ef)} X_1^{(gh)} ,
\end{aligned} \tag{A.9}
$$

where we've used eqs.(5.11,5.28) and the fact that the $\overline{\zeta}^a$'s satisfy eq.(5.11). A direct calculation then shows that the above system reduces to eq.(5.33) when $X_1 = X_2$, from which we conclude that the above R-matrix is the correct one describing the bilinears. The exact form in the X-base is given in figure (A.1); Actually this is stuff for the 'die-hards' who want to check everything I write. Note the indexation my dear die-hards: the index runs over $(11)(11), (12)(11), (21)(11), (22)(11), (11)(12), ..., (22)(22)$

With this explicit form for the R-matrix we can calculate the other matrix, needed for the definition of the differentials on the space of bilinears, by virtue of eq.(5.62). A remark is in order at this point. Of course solving for eq.(5.62) is possible but not very wise, since clearly a great degree of freedom remains (just look at the the first row of $R-1$ and you'll see why!). We know however that Q has to satisfy the YBE and the crossing relations. It is faster to find solutions to these YB relations with a computer algebra programme, e.g. Maple, and then to see whether eq.(5.62) is satisfied. The explicit form can be found in the figure (A.2), where the same indexation was used as for the R-matrix. The action of the differentials on the alfabet-base follows directly from eq.(5.60) and the transformations (5.56) and (5.68). The explicit form reads

$$
\begin{aligned}
\partial_A A &= 1 + q^{-2} A\partial_A + \lambda^2 B\partial_B + \lambda(qD - q^{-1}C)\partial_C - q^{-1}\lambda D\partial_D , \\
\partial_A B &= B\partial_A , \\
\partial_A C &= q^{-2} C\partial_A + \lambda^2 D\partial_A + q\lambda(1 - q\lambda)B\partial_C - q\lambda B\partial_D , \\
\partial_A D &= D\partial_A - q\lambda B\partial_C , \\
\\
\partial_B A &= A\partial_B , \\
\partial_B B &= 1 + q^{-2} B\partial_B - q^{-1}\lambda D\partial_C , \\
\partial_B C &= C\partial_B - q\lambda A\partial_C , \\
\partial_B D &= q^{-2} D\partial_B , \\
\\
\partial_C A &= A\partial_C - q^{-1}\lambda D\partial_B , \\
\partial_C B &= q^{-2} B\partial_C , \\
\partial_C C &= 1 + q^{-2} C\partial_C + \lambda^2 D\partial_C - q^{-1}\lambda B\partial_B , \\
\partial_C D &= D\partial_C , \\
\\
\partial_D A &= q^{-2} A\partial_D + \lambda(qD - q^{-1}C)\partial_B , \\
\partial_D B &= B\partial_D - q^{-1}\lambda D\partial_A + \lambda^2 B\partial_C , \\
\partial_D C &= C\partial_D - q^{-1}\lambda A\partial_A + q\lambda B\partial_B - q\lambda^2(qD - q^{-1}C)\partial_C , \\
\partial_D D &= 1 + q^{-2} D\partial_D - q^{-1}\lambda B\partial_B .
\end{aligned} \tag{A.10}
$$

Here we'll put some relations that will not be needed in the rest of the thesis, but might come in handy when looking at the 'geometry' of the q-Minkowski space. The first of the superfluous relations will be the relations between the ξ's, where one should note that, seeing $\xi^{(ab)} = (dX^{(ab)})$, we use the same rescaling for the ξ's as for the X's.

$$\begin{array}{lcl} (\xi^A)^2 = (\xi^B)^2 = (\xi^D)^2 = 0 & , & (\xi^C)^2 = q\lambda\xi^B\xi^A \\ \xi^A\xi^B = -\xi^B\xi^A & , & \xi^B\xi^C = -q^{-2}\xi^C\xi^B - q\lambda\xi^D\xi^B \ , \\ \xi^A\xi^C = -q^2\xi^C\xi^A + q^3\lambda\xi^D\xi^A & , & \xi^B\xi^D = -\xi^D\xi^B \ , \\ \xi^A\xi^D = -\xi^D\xi^A & , & \xi^C\xi^D = -\xi^D\xi^C - q\lambda\xi^B\xi^A \ . \end{array} \qquad (A.11)$$

When looking at the geometry we'll also need the relations between the X's and the ξ's, well, behold

$$\begin{array}{l} A\xi^A = q^{-2}\xi^A A \ , \\ A\xi^B = \xi^B A + q\lambda\xi^D D - q^{-1}\lambda\xi^D C - q^{-1}\lambda\xi^C D + \lambda^2\xi^A B \ , \\ A\xi^C = \xi^C A + q\lambda\xi^A D - q^{-1}\lambda\xi^A C \ , \\ A\xi^D = q^{-2}\xi^D A - q^{-1}\lambda\xi^A D \ , \\ \\ B\xi^A = \xi^A B - q^{-1}\lambda\xi^D D \ , \\ B\xi^B = q^{-2}\xi^B B \ , \\ B\xi^C = q^{-2}\xi^C B + \lambda^2\xi^D B - q^{-1}\lambda\xi^B D \ , \\ B\xi^D = \xi^D B \ , \\ \\ C\xi^A = q^{-2}\xi^A C + \lambda^2\xi^A D - q^{-1}\lambda\xi^D A \ , \\ C\xi^B = \xi^B C - q^{-1}\lambda\xi^C B + q\lambda^D B \ , \\ C\xi^C = q^{-2}\xi^C C - q\lambda\xi^B A - q^2\lambda^2\xi^D D + \lambda^2\xi^D C \\ \qquad\quad +\lambda^2\xi^C D + q\lambda(1-q\lambda)\xi^A B \ , \\ C\xi^D = \xi^D C - q\lambda\xi^A B \ , \\ \\ D\xi^A = \xi^A D \ , \\ D\xi^B = q^{-2}\xi^B D - q^{-1}\lambda\xi^D B \ , \\ D\xi^C = \xi^C D - q\lambda\xi^A B \ , \\ D\xi^D = q^{-2}\xi^D D \ . \end{array} \qquad (A.12)$$

Well, this should be about enough to finish the job...

$$
R = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -q^{-2}\lambda & 0 & 0 & 0 & 0 & 0 & 0 & q^{-2} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & q^{-1}\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -q^{-1}\lambda & 0 & 0 & 0 & q^{-2} & 0 & 0 & q^{-1}\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-2} & 0 & 0 \\
0 & -q^{-1}\lambda & q^{-2} & 0 & 0 & 0 & q^{-1}\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -q^{-1}\lambda^2 & q^{-2} & 0 & 0 & 0 & 0 & 0 & q^{-1}\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & -\lambda & 0 & -q^{-2}\lambda^2 & 0 & 0 & 0 & 0 & 1 & 0 & q^{-2}\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & q^{-2} & 0 & 0 & q^{-3}\lambda & 0 & 0 & q^{-3}\lambda & 0 & 0 & \lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & q^{-1}\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -q^{-1}\lambda & 0 & 0 & 0 & q^{-2} & 0 & q^{-2}\lambda & q^{-1}\lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \tag{A.13}
$$

Figure A.1: The R-matrix for the bilinears. Mind the indexation!!

$$
Q = \begin{pmatrix}
p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -q^{-1}\lambda & 0 & 0 & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\lambda & -q^{-1}\lambda & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda^2 & 0 & -\lambda & -q^{-1}\lambda & 0 & 0 & -q^{-1}\lambda & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -q\lambda & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^{-1}\lambda & 0 & 0 & 0 & 0 & 0 & p & 0 & 0 \\
0 & -q^{-1}\lambda^2 & p & 0 & p\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -q\lambda & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \aleph & 0 & -\lambda^2 & -q^{-1}\lambda^2 & 0 & 0 & -q^{-1}\lambda^2 & p & 0 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 & 0 & -q^{-1}\lambda & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & p\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^{-1}\lambda^2 & 0 & 0 & 0 & p & 0 & p\lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p
\end{pmatrix} \tag{A.14}
$$

Figure A.2: The explicit form of the Q matrix, where $p \equiv q^{-2}$ and $\aleph = \lambda(q\lambda - 1)$

Appendix B

Deformed Poincaré Algebra and Field Theory

We examine deformed Poincaré algebras containing the exact Lorentz algebra. We impose constraints which are necessary for defining field theories on these algebras and we present simple field theoretical examples. Of particular interest is a case that exhibits improved renormalization properties.

Deformations of space-time and its symmetries have attracted a lot of attention recently [10, 31, 33, 59, 55]. The main reason from the particle physics perspective is that such deformed spaces or symmetries could be the basis to construct field theories with improved ultraviolet properties. This hope was motivated from the fact that q-deformations of space-time seem to lead to some lattice pattern which in turn could serve at least as some kind of regularization built in a theory that would be defined on such space-time[1].

Deformations of the Poincaré algebra (dPA) have been considered so far along three directions. The first consists of direct q-deformations of the Lorentz sub-algebra [10, 59, 49, 54] along the lines prescribed by Drinfeld and Jimbo [19, 27]. The second is based on the fact that the Poincaré algebra (PA) can be obtained by a Wigner-Inönü contraction of the simple anti-de Sitter algebra $O(3,2)$ [12, 24, 33]. Then one first constructs the q-deformed $O_q(3,2)$ using the Drinfeld-Jimbo method and then does the contraction. In fact it was shown in ref.[6] that the same deformation can be obtained directly by considering general deformations of the commutation relations in the PA. Unfortunately the above deformations do not preserve the Lorentz algebra. Therefore it is natural to search for those dPAs that leave the Lorentz algebra unchanged in order to facilitate the quantization of the corresponding field theories. This motivation led us in ref.[31] to consider a third direction, namely deformations of the PA that leave the Lorentz algebra invariant.

In the present paper we continue our search for the appropriate dPAs that will allow us to construct field theories with impoved ultraviolet properties. We demand that the dPAs, in addition to leaving the Lorentz algebra invariant and giving the ordinary PA in low energies, should satisfy two more constraints. First, we require that there exists a tensor product of representations (coproduct) which is necessary in order to be able to go from the irreducible representations in the Hilbert space of quantum mechanics to the reducible representations in the Fock space of free quantum fields. A second requirement is that the representations

[1]This appendix was published [32] in collaboration with A.A. Kehagias and G. Zoupanos.

of the dPA should be different from those of the ordinary PA, a usual property in q-groups [56], as a way to guarantee that the dPAs are well distinguished from ordinary PA and, in principle, with different physical implications. Finally we demonstrate using a scalar field theory defined on a specific dPA that indeed field theories with improved ultraviolet properties can be constructed.

B.1 The deformed Poincaré algebra

In ref.[31] it has been proposed to search for deformations of the PA that do not affect the Lorentz subalgebra.

The Lorentz algebra is a six-dimensional Lie algebra generated by the generators J_i, K_i of rotations and boosts correspondingly satisfying the following commutation relations

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k, \\ [J_i, K_j] &= i\epsilon_{ijk} K_k, \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k. \end{aligned} \tag{B.1}$$

Recall that by defining

$$N_i = \frac{1}{2}(J_i + iK_i)$$

one finds that N_i's and $N_i^\dagger$'s satisfy an $SU(2)\otimes SU(2)$ algebra. The enlargement of the Lorentz algebra to the Poincaré by including the energy-momentum generators (P_0, P_i) was proposed as a way to describe the quantum states of relativistic particles as unitary representations of the Poincareé group without using the wave equations [63]. One of the main points of ref.[31]was to show that this enlargement of the Lorentz algebra is not unique. Indeed, it was proposed to introduce a generalized set of commutation relations (as compared to the ordinary PA) for the generators (P_0, P_i, K_i) as follows

$$\begin{aligned} [K_i, P_0] &= i\alpha_i(P_0, \vec{P}), \\ [K_i, P_j] &= i\beta_{ij}(P_0, \vec{P}), \end{aligned} \tag{B.2}$$

where α_i, β_{ij} are functions of P_0, P_i. Then applying the Jacobi identities on the sets (J_i, K_i, P_0) and (J_i, K_j, P_k) it was found that the general form of α_i and β_{ij} is

$$\begin{aligned} \alpha_i(P_0, \vec{P}) &= \alpha(P_0, \vec{P}) P_i, \\ \beta_{ij}(P_0, \vec{P}) &= \beta(P_0, \vec{P})\delta_{ij} + \gamma(P_0, \vec{P}) P_i P_j. \end{aligned} \tag{B.3}$$

Assuming furthermore that there exists a Casimir invariant of the enlarged algebra of the form

$$f(P_0) - \vec{P}^2, \tag{B.4}$$

it was found that

$$\begin{aligned} \alpha_i(P_0, \vec{P}) &= \alpha(P_0) P_i, \\ \beta_{ij}(P_0, \vec{P}) &= \beta(P_0)\delta_{ij}. \end{aligned} \tag{B.5}$$

Moreover the closure of the enlarged algebra required that

$$\alpha(P_0)\beta'(P_0) = 1. \tag{B.6}$$

In this way a minimally deformed Poincaré algebra was constructed with commutation relations

$$\begin{aligned}
[J_i, J_j] &= i\epsilon_{ijk}J_k, \\
[J_i, K_j] &= i\epsilon_{ijk}K_k, \\
[K_i, K_j] &= -i\epsilon_{ijk}J_k, \\
[J_i, P_0] &= 0, \\
[J_i, P_j] &= i\epsilon_{ijk}P_k, \\
[P_0, P_i] &= 0, \\
[K_i, P_0] &= i\alpha(P_0)P_i, \\
[K_i, P_j] &= i\beta(P_0)\delta_{ij},
\end{aligned} \tag{B.7}$$

where $\alpha(P_0), \beta(P_0)$ satisfy eq.(B.6). Note that the ordinary PA is obtained when $\alpha(P_0) = 1$ and $\beta(P_0) = P_0$.

The above dPA has two Casimir invariants. One corresponds to the length of the Pauli-Lubanski four-vector

$$W^2 = W_0^2 - \vec{W} \cdot \vec{W}, \tag{B.8}$$

where

$$\begin{aligned}
W_0 &= \vec{J} \cdot \vec{P}, \\
W_i &= \beta(P_0)J_i + \epsilon_{ijk}P_jK_k,
\end{aligned} \tag{B.9}$$

with eigenvalues

$$W^2 = -\mu^2 s(s+1),$$

where $s = 0, 1/2, \ldots$ is the spin.

The other Casimir invariant of the dPA corresponds to the $(mass)^2$ of the ordinary PA and it is given by

$$\beta^2(P_0) - \vec{P} \cdot \vec{P} = \mu^2. \tag{B.10}$$

Let us also recall that the transformation $P_0 \to \beta(P_0)$ reduces the dPA to the ordinary PA for the set of generators $(J_i, K_i, P_i, \beta(P_0))$.

B.2 Constraints on the dPA parameter functions

From the construction of the dPA discussed above, it is clear that the functions $\beta(P_0)$ and, consequently, $\alpha(P_0)$ are not specified. An obvious physical requirement that these functions should satisfy is to let us obtain the ordinary PA as a limit of the dPA in low energies. Therefore we require that the low energy behaviour of $\beta(P_0)$ should be

$$\beta(P_0) \sim P_0$$

and therefore

$$\alpha(P_0) \sim 1.$$

One of our main aims is to define field theories on the constructed dPA hopefully with improved ultraviolet properties. A dPA with the Lorentz invariant subalgebra paves the way for an easy first quantization of such theories. Another requirement for construction of field theories is to be able to define the multiparticle states. In field theories defined on the ordinary PA the multiparticle states are constructed from the tensor product of one particle states. Here therefore we are looking for the corresponding "tensor" product which is usually called coproduct. In ref.[31] the $\beta(P_0)$ function was chosen to be

$$\beta(P_0) = M \sin(\frac{P_0}{M})$$

and the P_0 modulo periodicity was restricted to be in the interval $(-\frac{\pi M}{2}, \frac{\pi M}{2})$. This was a first attempt to improve the ultraviolet behaviour of theories defined on the dPA introducing an upper cut-off in the energy spectrum by choosing a bounded function $\beta(P_0)$. However, when considering the additivity properties of energy $P_0^{(12)}$ of a system $S^{(12)}$ composed of two non-interacting systems $S^{(1)}, S^{(2)}$ some problems were found. Specifically, although the energy is conserved the energy $P^{(12)}$ was no longer the sum of the energies $P_0^{(1)}, P_0^{(2)}$ of the two subsystems $S^{(1)}, S^{(2)}$, respectively. So it was conjectured that the law of addition of the energies should be

$$\sin(\frac{P_0^{(1)}}{M}) + \sin(\frac{P_0^{(2)}}{M}) = 2\sin(\frac{P_0^{(12)}}{2M}).$$

The above conjecture, however, does not correspond to a true coproduct of the generator P_0 [2]. On the other hand, the above choice of $\beta(P_0)$ has also a positive aspect in the sense that the transformation $P_0 \to \beta(P_0)$ is not invertible since $\beta(P_0)$ is a multivalued function of P_0.

Here we would like to discuss two more alternatives as examples of the variety of existing possibilities. Let us first assume that the function $\beta(P_0)$ is of the form

$$\beta(P_0) = M \tanh^{-1}(\frac{P_0}{M}). \tag{B.11}$$

In this case the coproduct of the generators of the dPA are found to be

$$\begin{aligned} \Delta(J_i) &= J_i \otimes 1 + 1 \otimes J_i, \\ \Delta(K_i) &= K_i \otimes 1 + 1 \otimes K_i, \\ \Delta(P_i) &= P_i \otimes 1 + 1 \otimes P_i, \\ \Delta(P_0) &= (P_0 \otimes 1 + 1 \otimes P_0)(1 \otimes 1 + \frac{P_0}{M} \otimes \frac{P_0}{M})^{-1}. \end{aligned} \tag{B.12}$$

The coproduct (B.12) is determined by using the property that the transformation $P_0 \to \beta(P_0)$ transforms the dPA to the ordinary PA, which in the present case is invertible. Then the knowledge of the coproduct $\Delta(P_0)$ within PA easily gives us the $\Delta(P_0)$ for the dPA. At this point it should be emphasized that with the present choice of $\beta(P_0)$ the dPA is not a trivial redefinition of the ordinary PA. The reason is that the function $\alpha(P_0)$ is given by

$$\alpha(P_0) = \frac{1}{\beta'(P_0)} = 1 - \frac{P_0^2}{M^2}. \tag{B.13}$$

It is then clear that states with energy $P_0^2 \geq M^2$ are not representations of the dPA since the action of the boosts K_i on them would either leave unchanged or would reduce their energy.

[2]We would like to thank L. Alvarez-Gaumé and O. Ogievetsky for pointing this to us

Therefore there is no one-to-one correspondence among the representations of dPA and PA as would be in the case of a trivial redefinition.

As a second example let us assume that the function $\beta(P_0)$ is

$$\beta(P_0) = M \tan^{-1}(\frac{P_0}{M}). \tag{B.14}$$

In this case again the coproduct is determined as before using the property that the transformation $P_0 \to \beta(P_0)$ takes the dPA to ordinary PA which is again invertible. The $\Delta(P_0)$ now becomes

$$\Delta(P_0) = (P_0 \otimes 1 + 1 \otimes P_0)(1 \otimes 1 - \frac{P_0}{M} \otimes \frac{P_0}{M})^{-1}. \tag{B.15}$$

Note however that the function $\alpha(P_0)$ now is

$$\alpha(P_0) = 1 + \frac{P_0^2}{M^2} \tag{B.16}$$

and thus it does not put any restrictions on the P_0's. Therefore there exists a one-to-one correspondence among the dPA and PA.

B.3 Scalar field theory on dPA

Here we shall examine a simple field theory such as $\lambda\phi^4$ on the dPA. The two examples for construction of multiparticle states discussed above will be examined separately. Let us consider the ordinary ϕ^4 in ordinary PA [25]. The Lagrangian is

$$L = L_0 + L_I$$

with

$$L_0 = \frac{1}{2}(\partial\phi_0)^2 - \frac{\mu_0^2}{2}\phi_0^2$$

and

$$L_I = -\frac{\lambda_0}{4!}\phi_0^4.$$

Recall that the self-energy graph at 1-loop is given by

$$-\Sigma(p^2) = -\frac{i\lambda_0}{2}\int d^4\ell \frac{i}{\ell_0^2 - \vec{\ell}^2 - \mu_0^2 + i\epsilon} \tag{B.17}$$

and it is quadratically divergent. Also the vertex corrections at 1-loop are given by

$$\Gamma(s) = (-\frac{i\lambda_0}{2})^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{i}{(\ell_0 - p_0)^2 - (\vec{\ell} - \vec{p})^2 - \mu_0^2 + i\epsilon} \frac{i}{\ell_0^2 - \vec{\ell}^2 - \mu_0^2 + i\epsilon} \tag{B.18}$$

and $\Gamma(t), \Gamma(u)$ have similar expressions where

$$s = p^2 = (p_1 + p_2)^2\,,\; t = (p_1 - p_3)^2\,,\; u = (p_1 - p_4)^2$$

are the Mandelstam variables. The vertex corrections diverge logarithmically. The divergences of the self-energy and vertex corrections are removed by the well-known procedure of introducing counterterms and performing the renormalization program. We would like to examine whether the same theory when defined on the dPA has a better ultraviolet behaviour.

Given the form of $\beta(p_0)$ the L_0 part of the Lagrangian can be expressed locally in momentum space as

$$L_0 = \frac{1}{2}\left(\tilde{\phi}_0(p)(\beta^2(p_0) - \vec{p}^2)\tilde{\phi}_0(p) - \mu_0^2\tilde{\phi}_0(p)\tilde{\phi}_0(p)\right), \tag{B.19}$$

where $\tilde{\phi}_0(p)$ is the Fourier transform of $\phi_0(x)$. Therefore the propagator in the dPA is

$$\frac{i}{\beta^2(p_0) - \vec{p}^2 - \mu_0^2 + i\epsilon},$$

which with an appropriate choice of $\beta(p_0)$ can have more convergent behaviour for large p_0 as compared to the usual one

$$\frac{i}{p_0^2 - \vec{p}^2 - \mu_0^2 + i\epsilon}.$$

The calculation of 1-loop graphs reduces in determining integrals of the form

$$I(p^2) = \int d^4\ell\beta'(\ell_0) f(\beta^2(\ell_0) - \vec{\ell}^2, \beta^2(p_0) - \vec{p}^2) \tag{B.20}$$

i.e., integrals with dPA-invariant measure. Let us start with the case that $\beta(P_0)$ has the form (B.11). In this case the integrals resulting from 1-loop corrections will be

$$\begin{aligned} I_1 &= \int_{-M}^{M} d\ell_0\beta'(\ell_0) \int_{-\infty}^{\infty} d^3\vec{\ell} f(\beta^2(\ell_0) - \vec{\ell}^2) \\ &= \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d^3\vec{\ell} f(\beta^2 - \vec{\ell}^2) \end{aligned} \tag{B.21}$$

i.e., they are exactly the same as in the ordinary PA case. Therefore the present form of $\beta(P_0)$ and the corresponding dPA invariant measure does not improve the ultraviolet properties of the 1-loop corrections to the theory.

Let us then turn to our second example which exhibits a different behaviour. In this case $\beta(P_0)$ is given by eq(B.14). Then the integrals involved in the calculations of 1-loop corrections are

$$I_2 = \int_{-\infty}^{\infty} d\ell_0\beta'(\ell_0) \int_{-\infty}^{\infty} d^3\vec{\ell} f(\beta^2(\ell_0) - \vec{\ell}^2) \tag{B.22}$$

$$= \int_{-M}^{M} d\beta \int_{-\infty}^{\infty} d^3\vec{\ell} f(\beta^2 - \vec{\ell}^2), \tag{B.23}$$

which are clearly more convergent (in view of the cut-off M) than the corresponding ones in the ordinary PA case. Therefore this choice of $\beta(P_0)$ provides us with an example of how one can improve the ultraviolet behaviour of a theory. The negative aspect of this particular choice of $\beta(P_0)$ is that the corresponding dPA has representations in one-to-one correspondence with ordinary PA and one would obtain the same results just by changing P_0 to $\beta(P_0)$ in ordinary PA. Therefore this last example cannot be considered seriously as having physical consequences but rather should be viewed as a regulator of the theory.

B.4 A satisfactory model

Here we present a choice of the function $\beta(P_0)$ which seems very promising since it satisfies the two new constraints we have demanded so far. Namely there exists a coproduct and the representations of the corresponding dPA are different from those of the ordinary PA. So we can construct a multiparticle state and the dPA under consideration is a distinct entity separate from the ordinary PA. Then in principle the theory defined on this dPA could have different physical implications as compared to the same theory defined on ordinary PA. As we shall see, the one-loop self-energy and vertex corrections of a scalar field theory defined on this dPA has improved ultraviolet properties.

The chosen function is

$$\beta(P_0) = M \sin^{-1}(\frac{P_0}{M}). \tag{B.24}$$

The coproduct of the generators of the dPA are found to be

$$\begin{aligned}
\Delta(J_i) &= J_i \otimes 1 + 1 \otimes J_i \\
\Delta(K_i) &= K_i \otimes 1 + 1 \otimes K_i \\
\Delta(P_i) &= P_i \otimes 1 + 1 \otimes P_i \\
\Delta(P_0) &= P_0 \otimes \sqrt{1 - \frac{P_0^2}{M^2}} + \sqrt{1 - \frac{P_0^2}{M^2}} \otimes P_0.
\end{aligned} \tag{B.25}$$

As far as the representations are concerned, recall that the closure of the algebra requires that eq.(B.6) should hold which in turn implies that

$$\alpha(P_0) = \sqrt{1 - \frac{P_0^2}{M^2}}. \tag{B.26}$$

It is then clear that the representations with $P_0^2 \geq M^2$ are necessarily non-unitary while the energy spectrum for the unitary representations of the dPA lies in the interval $(-M, M)$. Therefore, since there is no unitary representations describing physical states of the dPA with $P_0^2 \geq M^2$, there is no one-to-one correspondence with the PA.

Turning to the one-loop self-energy and vertex corrections of the scalar theory we find that the self-energy graph becomes now

$$\begin{aligned}
-\Sigma(p^2) &= -\frac{i\lambda_0}{2} \int_{-\infty}^{\infty} d^4\ell \beta'(\ell_0) \frac{i}{\beta^2(\ell_0) - \vec{\ell}^2 - \mu_0^2 + i\epsilon} \\
&= -\frac{i\lambda_0}{2} \int_{-M\pi/2}^{M\pi/2} d\beta \int_{-\infty}^{\infty} d^3\vec{\ell} \frac{i}{\beta^2 - \vec{\ell}^2 - \mu_0^2 + i\epsilon}
\end{aligned} \tag{B.27}$$

which is linearly divergent instead of quadratically in the usual scalar theory defined on the ordinary PA. Correspondingly, the one-loop vertex corrections take the form

$$\Gamma(s) = (-\frac{i\lambda_0}{2})^2 \int \frac{d^4\ell}{(2\pi)^4} \beta'(\ell_0) \frac{i}{(\beta(\ell_0) - \beta(p_0))^2 - (\vec{\ell} - \vec{p})^2 - \mu_0^2 + i\epsilon} \frac{i}{\beta^2(\ell_0) - \vec{\ell}^2 - \mu_0^2 + i\epsilon}$$

$$= (-\frac{i\lambda_0}{2})^2 \int_{-M\pi/2}^{M\pi/2} \frac{d\beta d^3\vec{\ell}}{(2\pi)^4} \frac{i}{(\beta(\ell_0) - \beta(p_0))^2 - (\vec{\ell} - \vec{p})^2 - \mu_0^2 + i\epsilon} \frac{i}{\beta^2(\ell_0) - \vec{\ell}^2 - \mu_0^2 + i\epsilon}. \tag{B.28}$$

and similar forms take the $\Gamma(t), \Gamma(u)$ which are all convergent.

B.5 Discussion

The aim of the present paper was to present deformations of the Poincaré algebra that preserve the Lorentz sub-algebra requiring additional constraints that would pave the way for constructing realistic theories with improved ultraviolet properties. The new constraints that we have imposed are (i) the requirement of the existence of a coproduct of the representations of the dPA and (ii) the demand that there is no one-to-one correspondence among the representations of the dPA and ordinary PA which means that the two algebras are just homomorphic.

Concerning the first requirement, one may state, as a general rule, that a coproduct for the dPA always exists if $\beta(P_0)$ is an unbounded function of P_0. In that case, one may employ the homomorphism between the dPA and PA to pull back the coproduct of the PA into the dPA. Furthermore, if $\beta(P_0)$ is an odd function of P_0, the same homomorphism can also pull back the antipode of the PA into the dPA turning the latter into a cocommutative Hopf algebra. From the physical point of view, the cocommutativity of the dPA guarantees that the addition of observables for two systems $S^{(1)}$ and $S^{(2)}$ is independent of the order of addition. Recall for comparison that in the case of non-cocommutative algebras (quantum groups), the addition depends on the order (i.e., on the "labeling"). Turning to the second requirement it guarantees that the dPA is not a simple redefinition of ordinary PA.

Although the additional constraints are necessary in order to construct a field theory on a dPA which is not a trivial redefinition of the ordinary PA they do not guarantee that the theory has better ultraviolet behaviour. Therefore from this point of view they are necessary but not sufficient. On the other hand all the constraints considered so far, including the requirement for improved ultraviolet behaviour, cannot restrict in an appreciable manner the choices of the functions $\beta(P_0)$ which differentiate the various dPAs from each other.

We should emphasize that when a dPA satisfying all the above constraints is found it has very important physical consequences. First of all, such a dPA will be characterized by a non-trivial function $\beta(P_0)$ which certainly will result in observable deviations of the special theory of relativity. There exist already some analyses [21, 18] which put limits on the characteristic mass scale M appearing in general in $\beta(P_0)$ on dimensional grounds. For instance, according to ref.[18] the lowest bound consistent with experimental observations is $M_{min} \simeq 10^{12}$GeV.

Finally, a field theory with less divergences than the usual ones requires also less counterterms to cure them. This in turn means that the theory will have less free parameters to be fixed by experiment, or equivalently the theory will have more predictive power. It is expected then that the phenomenological constraints will provide us with enough information to restrict the possible choices of the functions $\beta(P_0)$. Moreover, it is fair to hope that genuine predictions on unkown parameters would emerge as a result of the above construction [53].
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Bibliography

"Tonight I will make a tun of wine,
Set myself up with two bowls of it;
First I will divorce absolutely reason and religion,
Then take to wife the daughter of the vine."
Omar Khayyam (1048-1131)

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