# Euler Angles

The Euler angles are three angles that can be used to completely specify a 3-dimensional rotation matrix. Since this is a confusing matter, it is best to write out how they are defined and how this is related to the conventions used in the lectures.

In the lecture we defined the rotation of a Cartesian system of axes, such that the rotated base vectors (which bare a ‘) are related to the original base axe vectors (which don’t bare a ‘) by the relation

$$\vec{e}_i{}' \equiv \vec{e}_j R_{ji}$$

where the $R_{ji}$ are the entries of some rotation matrix, i.e. a real matrix that satisfies $R^T R=1$.

The infinitesimal rotation matrices $J_i$ are chosen such that they correspond to an infinitesimal rotation around the $i^{th}$ axis in the counter-clockwise direction of the Cartesian axis system. Their matrix form is:

$$J_1 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; J_2 \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; J_3 \equiv \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

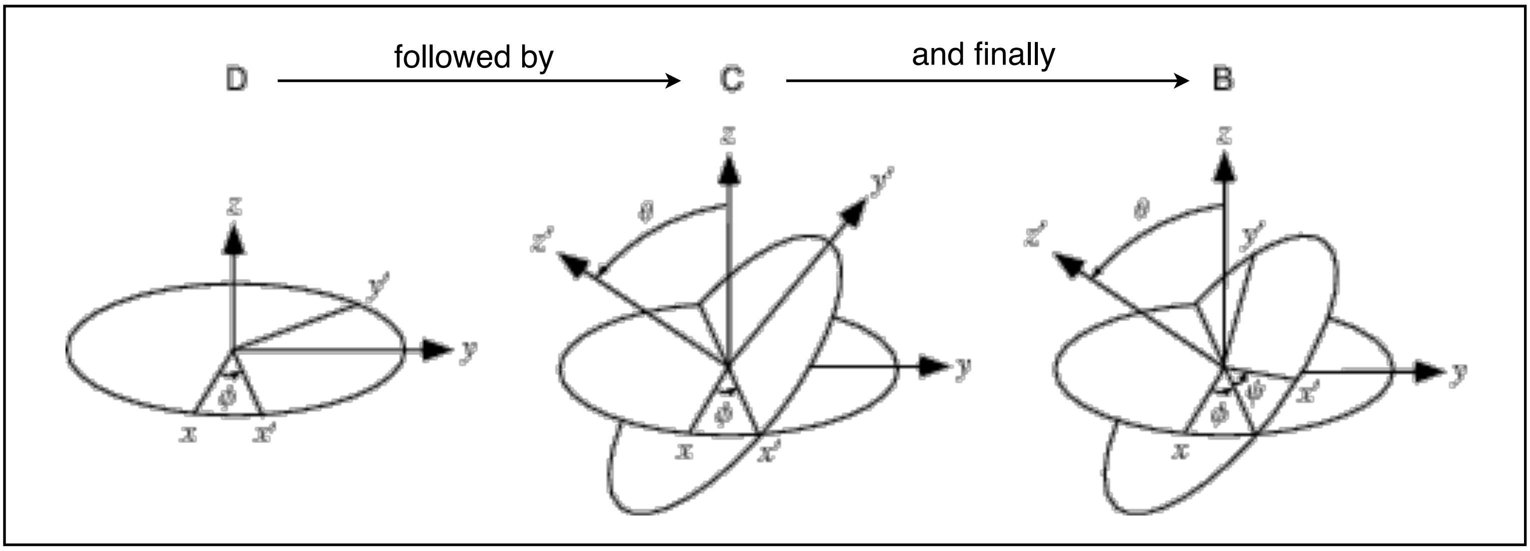
And satisfy:

$$[J_i, J_j] = \varepsilon_{ijk} J_k \qquad \text{and} \qquad \mathrm{Tr}\,(J_i J_j) = -2\delta_{ij}$$

The finite, counter-clockwise rotations around the $i^{th}$ axis are then given by

$$e^{\varphi\ J_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) \\ 0 & \sin(\varphi) & \cos(\varphi) \end{pmatrix}, \; e^{\varphi\ J_2} = \begin{pmatrix} \cos(\varphi) & 0 & \sin(\varphi) \\ 0 & 1 & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) \end{pmatrix}, \; e^{\varphi\ J_3} = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The standard definition of the Euler angles is pictorially defined by the following three successive rotations:




In the first step, indicated by D in the figure, we do a counter-clockwise rotation along an angle $\phi$, to define an intermediate Cartesian system of axes $\vec{e}_i{}'$. This relation can be written as:

$$\vec{e}_i{}' = \vec{e}_j \left(e^{\phi J_3}\right)_{ji}$$

In the second step, indicated by C in the figure, we do a rotation around the $\vec{e}_3{}'$ axis with angle $\theta$ as to define a new Cartesian system of axes $\vec{e}_i{}''$; the relation is given by:

$$\vec{e}_i{}'' = \vec{e}_j{}' \left(e^{\theta J_1}\right)_{ji}$$

In the third step, denoted B in the figure, we do a counter-clockwise rotation with angle $\psi$ around the axis $\vec{e}_3{}''$ in order to create the final Cartesian axis system $\vec{e}_i{}'''$. The relation between these two Cartesian systems reads:

$$\vec{e}_i{}''' = \vec{e}_j{}'' \left(e^{\Psi J_3}\right)_{ji}$$

Putting it all together we see that the final cartesian system, with base vectors $\vec{e}_i{}'''$, is given in terms of the original Cartesian system, with base vectors $\vec{e}_i$, is given by

$$\vec{e}_i{}''' = \vec{e}_j \left(e^{\phi J_3}\ e^{\theta J_1}\ e^{\psi J_3}\right)_{ji}$$

The angles $\psi$, $\theta$ and $\phi$ are called the Euler angles. The above derivation of the transformation in terms of the Euler angles is equivalent to the statement that any rotation in 3-dimensional space can be written as

$$R(\psi, \theta, \phi) = e^{\phi J_3}\ e^{\theta J_1}\ e^{\psi J_3}$$

Equivalently, we can take the above matrix as the definition of the Euler angles; the figure then gives the geometric interpretation of the angles.

More information: see e.g. Goldstein's section (4-4), the wikipedia pages on Euler and his angles, or Mathworld's discussion on the Euler angles.