8) <u>Hoja 5</u>

1) $\Gamma(\vec{L}^2) = \Gamma(L_i)\Gamma(L_i)$

$$= -\varepsilon_{ijk}\varepsilon_{imn}\, q_j \partial_k\, q_m \partial_n$$

$$= -(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})\, q_j \partial_k\, q_m \partial_n$$

$$= q_j \partial_k\, q_k \partial_j - q_j \partial_k\, q_j \partial_k$$

$$\partial_k q_k = (\partial_k q_k) + q_k \partial_k = 3 + \vec{q}\cdot\vec{\partial}$$

$$\partial_k q_j = (\partial_k q_j) + q_j \partial_k = \delta_{kj} + q_j \partial_k$$

$$= 3\,\vec{q}\cdot\vec{\partial} + q_j\, \vec{q}\cdot\vec{\partial}\, \vec{\partial}_j - q_j(\delta_{kj} + q_j \partial_k)\partial_k$$

$$= 3\,\vec{q}\cdot\vec{\partial} + q_j\left(\partial_j\, \vec{q}\cdot\vec{\partial} - \partial_j(\vec{q}\cdot\vec{\partial})\right) - \vec{q}\cdot\vec{\partial} - \vec{q}^{\,2}\vec{\partial}^{\,2}$$

$$\partial_j(\vec{q}\cdot\vec{\partial}) = \partial_j$$

$$= 3\,\vec{q}\cdot\vec{\partial} + \vec{q}\cdot\vec{\partial}\,\vec{q}\cdot\vec{\partial} - \vec{q}\cdot\vec{\partial} - \vec{q}\cdot\vec{\partial} - \vec{q}^{\,2}\vec{\partial}^{\,2}$$

$$= \vec{q}^{\,2}\vec{\partial}^{\,2} + \vec{q}\cdot\vec{\partial}\,\vec{q}\cdot\vec{\partial} + \vec{q}\cdot\vec{\partial}$$

$\Rightarrow$ $\vec{x}\cdot\vec{\partial}$ es un escalar bajo rotaciones con lo cual no puede tener términos como ∂_θ, ∂_φ

esto se ve de $\vec{q}\cdot\vec{\partial}\, q_i = q_i$ $\rightarrow$ no induce ninguna rotación es decir que con lo cual $\vec{q}\cdot\vec{\partial} = f(r)\partial_r$

$$\Rightarrow \vec{q}\cdot\vec{\partial}\, r^2 = \vec{q}\cdot\vec{\partial}\, \vec{q}^{\,2} = 2q_i\, \vec{q}\cdot\vec{\partial}\, q_i = 2r^2$$

$$\hookrightarrow \vec{q}\cdot\vec{\partial}\, r = r \Rightarrow f(r)\partial_r r = r \Rightarrow \underline{f(r) = r}$$

$$\boxed{\vec{q}\cdot\vec{\partial} = r\partial_r}$$

② Ya que $[L_i, S_j] = 0$

tenemos $$(\vec{L}+\vec{S})^2 = \vec{L}^2 + \{L_i, S_i\} + \vec{S}^2$$

$$= \vec{L}^2 + 2\vec{L}\cdot\vec{S} + \vec{S}^2$$

$$\Rightarrow \quad \vec{L}\cdot\vec{S} = \frac{1}{2}\left((\vec{L}+\vec{S})^2 - \vec{L}^2 - \vec{S}^2\right)$$

~~Si los autoestados~~

los autoestados de $\vec{L}$ son los de V_ℓ

los autoestados de $\vec{S}$ son los de $V_{\frac{1}{2}} = \mathrm{span}_{\mathbb{C}}(|\uparrow\rangle\, |\downarrow\rangle)$

Entonces de clase sabemos de la serie de C-G. $V_\ell \otimes V_{\frac{1}{2}} = V_{\ell+\frac{1}{2}} \oplus V_{\ell-\frac{1}{2}}$

que los autovalores de $(\vec{L}+\vec{S})^2$ son $(\ell+\frac{1}{2})(\ell+\frac{3}{2})$

o $(\ell-\frac{1}{2})(\ell+\frac{1}{2}) = \ell^2 - \frac{1}{4}$.

Llama los estados $|j_1, j_2; j, m\rangle$

Entonces $$\vec{L}\cdot\vec{S}\,|\ell, \tfrac{1}{2}; \ell+\tfrac{1}{2}, m\rangle = \frac{1}{2}\left((\ell+\tfrac{1}{2})(\ell+\tfrac{3}{2}) - \ell(\ell+1) - \tfrac{1}{2}\cdot\tfrac{3}{2}\right)|\ell, \tfrac{1}{2}; \ell+\tfrac{1}{2}, m\rangle$$

$$= \frac{1}{2}\left(\cancel{\ell^2} + 2\ell + \cancel{\tfrac{3}{4}} - \cancel{\ell^2} - \ell - \cancel{\tfrac{3}{4}}\right)|\ell, \tfrac{1}{2}; \ell+\tfrac{1}{2}, m\rangle$$

$$= \frac{\ell}{2}\,|\ell, \tfrac{1}{2}; \ell+\tfrac{1}{2}, m\rangle$$

y $$\vec{L}\cdot\vec{S}\,|\ell, \tfrac{1}{2}; \ell-\tfrac{1}{2}, m\rangle = \frac{1}{2}\left((\ell-\tfrac{1}{2})(\ell+\tfrac{1}{2}) - \ell(\ell+1) - \tfrac{3}{4}\right)|\ell, \tfrac{1}{2}; \ell-\tfrac{1}{2}, m\rangle$$

$$= \frac{1}{2}\left(\cancel{\ell^2} - \tfrac{1}{4} - \cancel{\ell^2} - \ell - \tfrac{3}{4}\right)|\ell, \tfrac{1}{2}; \ell-\tfrac{1}{2}, m\rangle$$

$$= \frac{1}{2}(-\ell-1)\,|\ell, \tfrac{1}{2}; \ell-\tfrac{1}{2}, m\rangle = -\frac{\ell+1}{2}\,|\ell, \tfrac{1}{2}; \ell-\tfrac{1}{2}, m\rangle.$$

3) En el segundo ej. acabamos de hallar el espectro de $\vec{L}\cdot\vec{S}$ y con tal de decir que $l=\frac{1}{2}$

Estamos listo.

mas podemos usar:

$$|\tfrac{1}{2},\tfrac{1}{2}:1,1\rangle = |\uparrow\uparrow\rangle$$

$$|\tfrac{1}{2},\tfrac{1}{2}:1,0\rangle = \tfrac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

$$|\tfrac{1}{2},\tfrac{1}{2}:1,-1\rangle = |\downarrow\downarrow\rangle$$

$$|\tfrac{1}{2},\tfrac{1}{2}:0,0\rangle = \tfrac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$\left.\begin{aligned} |\uparrow\downarrow\rangle &= \tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2},\tfrac{1}{2}:1,0\rangle + |\tfrac{1}{2},\tfrac{1}{2}:0,0\rangle\right) \\ |\downarrow\uparrow\rangle &= \tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2},\tfrac{1}{2}:1,0\rangle - |\tfrac{1}{2},\tfrac{1}{2}:0,0\rangle\right)\end{aligned}\right.$$

$$\Rightarrow \quad \vec{S}_1\cdot\vec{S}_2\,|\tfrac{1}{2},\tfrac{1}{2}:1,m\rangle = \tfrac{1}{4}\,|\tfrac{1}{2},\tfrac{1}{2}:1,m\rangle$$

$$\vec{S}_1\cdot\vec{S}_2\,|\tfrac{1}{2},\tfrac{1}{2}:0,0\rangle = -\tfrac{3}{4}\,|\tfrac{1}{2},\tfrac{1}{2}:0,0\rangle\,.$$

a) $P_{|\uparrow\uparrow\rangle\to|\downarrow\downarrow\rangle}(t) = \left|\langle\downarrow\downarrow|\,e^{\frac{t}{i\hbar}H}\,|\uparrow\uparrow\rangle\right|^2 = 0$ as $|\uparrow\uparrow\rangle$ is an eigenstate of H.

b) $P_{|\uparrow\downarrow\rangle\to|\downarrow\uparrow\rangle}(t) = \left|\langle\downarrow\uparrow|\,e^{\frac{t}{i\hbar}H}\,|\uparrow\downarrow\rangle\right|^2$

$$= \frac{1}{4}\left|\left(\langle\tfrac{1}{2},\tfrac{1}{2}:1,0| - \langle\tfrac{1}{2},\tfrac{1}{2}:0,0|\right)e^{\frac{t}{i\hbar}H}\left(|\tfrac{1}{2},\tfrac{1}{2}:1,0\rangle + |\tfrac{1}{2},\tfrac{1}{2}:0,0\rangle\right)\right|^2$$

$$= \frac{1}{4}\left|\langle\tfrac{1}{2},\tfrac{1}{2}:1,0|e^{\frac{t}{i\hbar}H}|\tfrac{1}{2},\tfrac{1}{2}:1,0\rangle - \langle\tfrac{1}{2},\tfrac{1}{2}:0,0|e^{\frac{t}{i\hbar}H}|\tfrac{1}{2},\tfrac{1}{2}:0,0\rangle\right|^2$$

$$= \frac{1}{4}\left|e^{\frac{t}{i}\frac{d^2}{4\hbar}} - e^{\frac{t}{i\hbar}\cdot -\frac{3}{4}d^2}\right|^2 = \frac{1}{4}\left|\underbrace{e^{-\frac{td^2}{i\hbar 4}}}_{\text{drops out.}}\left(e^{-\frac{itd^2}{2\hbar}} - e^{\frac{itd^2}{2\hbar}}\right)\right|^2$$

$$= \frac{1}{4}\left|-2i\sin\left(\frac{d^2}{2\hbar}t\right)\right|^2 = \sin^2\left(\frac{d^2}{2\hbar}t\right)$$

4) Los estado de V_1 son $|1,1\rangle$, $|1,0\rangle$ y $|1,-1\rangle$

y tenemos

$$\boxed{\begin{array}{ll} J_3|1,m\rangle = m|1,m\rangle & \\ J_+|1,1\rangle = 0 & J_-|1,1\rangle = |1,0\rangle \\ J_+|1,0\rangle = |1,1\rangle & J_-|1,0\rangle = |1,-1\rangle \\ J_+|1,-1\rangle = |1,0\rangle & J_-|1,-1\rangle = 0 \end{array}}$$

$C_+(1,0) = \frac{1}{\sqrt{2}}\sqrt{1\cdot 2} = 1$ $\qquad C_-(1,1) = 1$

$C_+(1,-1) = \frac{1}{\sqrt{2}}\sqrt{2\cdot 1} = 1$ $\qquad C_-(1,0) = 1$

El estado $|1,1:1,1\rangle = \alpha|1,1\rangle\otimes|1,0\rangle + \beta|1,0\rangle\otimes|1,1\rangle$

por comportamiento de $\Delta(J_3)$

y satisface $\Delta(J_+)|1,1:1,1\rangle = 0$

$$\Rightarrow 0 = \alpha\underbrace{J_+|1,1\rangle}_{0}\otimes|1,0\rangle + \alpha|1,1\rangle\otimes J_+|1,0\rangle + \beta J_+|1,0\rangle\otimes|1,1\rangle + \beta|1,0\rangle\otimes\underbrace{J_+|1,1\rangle}_{0}$$

$$= \alpha|1,1\rangle\otimes|1,1\rangle + \beta|1,1\rangle\otimes|1,1\rangle \Rightarrow \boxed{\beta = -\alpha}$$

normalizando $\boxed{|1,1:1,1\rangle = \frac{1}{\sqrt{2}}\left(|1,1\rangle\otimes|1,0\rangle - |1,0\rangle\otimes|1,1\rangle\right)}$

And the

$$|1,1:1,0\rangle = \Delta(J_-)\,|1,1:1,1\rangle$$

$$= \frac{1}{\sqrt{2}}\Big(J_-|1,1\rangle\otimes|1,0\rangle + |1,1\rangle\otimes J_-|1,0\rangle - J_-|1,0\rangle\otimes|1,1\rangle - |1,0\rangle\otimes J_-|1,1\rangle \Big)$$

$$= \frac{1}{\sqrt{2}}\Big(|1,0\rangle\otimes|1,0\rangle + |1,1\rangle\otimes|1,-1\rangle - |1,-1\rangle\otimes|1,1\rangle - |1,0\rangle\otimes|1,0\rangle \Big)$$

$$= \frac{1}{\sqrt{2}}\Big(|1,1\rangle\otimes|1,-1\rangle - |1,-1\rangle\otimes|1,1\rangle \Big)$$

y $|1,1:1,-1\rangle = \Delta(J_-)\,|1,1:1,0\rangle$

$$= \frac{1}{\sqrt{2}}\Big(|1,0\rangle\otimes|1,-1\rangle - |1,-1\rangle\otimes|1,0\rangle \Big)$$

Good

$$|1,1:0,0\rangle = \alpha|1,1\rangle\otimes|1,-1\rangle + \beta|1,0\rangle\otimes|1,0\rangle + \gamma|1,-1\rangle\otimes|1,1\rangle$$

$$0 = \Delta(J_+)\,|1,1:0,0\rangle = \alpha|1,1\rangle\otimes|1,0\rangle + \beta|1,1\rangle\otimes|1,0\rangle + \beta|1,0\rangle\otimes|1,1\rangle + \gamma|1,0\rangle\otimes|1,1\rangle$$

$$\Rightarrow \quad \beta = -\alpha \qquad \gamma = -\beta$$

$$|1,1:0,0\rangle = \alpha\Big(|1,1\rangle\otimes|1,-1\rangle - |1,0\rangle\otimes|1,0\rangle + |1,-1\rangle\otimes|1,1\rangle \Big)$$

norm. $\boxed{\alpha = \frac{1}{\sqrt{3}}}$

5) Dado $V_m^{(1)}$ y $K_m^{(1)}$ queremos construir un $\Phi_m^{(2)}$.

Obviamente. $\quad \Phi_2^{(2)} = V_1^{(1)} K_1^{(1)}$

$$[J_+, \Phi_2^{(2)}] = \underbrace{[J_+, V_1^{(1)}]}_{0} K_1^{(1)} + V_1^{(1)} \underbrace{[J_+, K_1^{(1)}]}_{0} = 0 .$$

then we know that

$$[J_-, \Phi_2^{(2)}] = C_-(2,2)\, \Phi_1^{(2)} \qquad C_-(2,2) = \frac{1}{\sqrt2}\left(4 \cdot 1\right)^{\frac12} = \sqrt2$$

$$= \sqrt2\, \Phi_1^{(2)}$$

$$\Rightarrow \Phi_1^{(2)} = \frac{1}{\sqrt2}[J_-, \Phi_2^{(2)}] = \frac{1}{\sqrt2}\underbrace{[J_-, V_1^{(1)}]}_{V_0^{(1)}} K_1^{(1)} + \frac{1}{\sqrt2} V_1^{(1)} \underbrace{[J_-, K_1^{(1)}]}_{K_0^{(1)}}$$

$$\boxed{\Phi_1^{(2)} = \frac{1}{\sqrt2}\left(V_0^{(1)} K_1^{(1)} + V_1^{(1)} K_0^{(1)}\right)}$$

$$C_-(2,1) = \frac{1}{\sqrt2}\left(3 \cdot 2\right)^{\frac12} = \sqrt3$$

$$[J_-, \Phi_1^{(2)}] = C_-(2,1)\, \Phi_0^{(2)} = \sqrt3\, \Phi_0^{(2)}$$

$$\Rightarrow \Phi_0^{(2)} = \frac{1}{\sqrt3}[J_-, \Phi_1^{(2)}]$$

$$= \frac{1}{\sqrt6}\left(V_{-1}^{(1)} K_1^{(1)} + V_0^{(1)} K_0^{(1)} + V_0^{(1)} K_0^{(1)} + V_1^{(1)} K_{-1}^{(1)}\right)$$

$$\boxed{\Phi_0^{(2)} = \frac{1}{\sqrt6}\left(V_1^{(1)} K_{-1}^{(1)} + 2 V_0^{(1)} K_0^{(1)} + V_{-1}^{(1)} K_1^{(1)}\right)}$$

$$\left[J_-, \Phi_0^{(2)}\right] = C_-(2,0)\,\Phi_{-1}^{(2)} = \sqrt{3}\,\Phi_{-1}^{(2)} \qquad \frac{1}{\sqrt{2}}(2\cdot 3)^{\frac{1}{2}} = \sqrt{3}$$

$$\Phi_{-1}^{(2)} = \frac{1}{\sqrt{3}}\left[J_-, \Phi_0^{(2)}\right]$$

$$= \frac{1}{3\sqrt{2}}\left(V_0^{(1)} K_{-1}^{(1)} + 2V_{-1}^{(1)} K_0^{(1)} + 2V_0^{(1)} K_{-1}^{(1)} + V_{-1}^{(1)} K_0^{(1)} \right)$$

$$\Phi_{-1}^{(2)} = \frac{1}{\sqrt{2}}\left(V_0^{(1)} K_{-1}^{(1)} + V_{-1}^{(1)} K_0^{(1)} \right)$$

$$\left[J_-, \Phi_{-1}^{(2)}\right] = C_-(2,-1)\,\Phi_{-2}^{(2)} = \sqrt{2}\,\Phi_{-2}^{(2)} \qquad \frac{1}{\sqrt{2}}(1\cdot 4)^{\frac{1}{2}} = \sqrt{2}$$

$$\Phi_{-2}^{(2)} = \frac{1}{\sqrt{2}}\left[J_-, \Phi_{-1}^{(2)}\right] = \frac{1}{2}\left(V_{-1}^{(1)} K_{-1}^{(1)} + V_{-1}^{(1)} K_{-1}^{(1)} \right) = V_{-1}^{(1)} K_{-1}^{(1)}$$

$\Rightarrow \quad \Phi_{-2}^{(2)} = V_{-1}^{(1)} K_{-1}^{(1)}$ and obviously $\left[J_-, \Phi_{-2}^{(2)}\right] = 0.$

6

a)

$$[J_3, K_\pm K_\mp] = [J_3, K_\pm]K_\mp + K_\pm [J_3, K_\mp]$$

$$= \pm K_\pm K_\mp + K_\pm(\mp K_\mp) = 0 \Longrightarrow \boxed{[J_3, C] = 0}$$

$$[K_+, C] = [K_+, J_3]J_3 + J_3[K_+, J_3] - K_+[K_+, K_-] - [K_+, K_-]K_+$$

$$= -K_+J_3 - J_3K_+ - K_+(-J_3) - (-J_3)K_+$$

$$= 0$$

$$(K_\pm)^\dagger = K_\mp \Rightarrow 0 = [K_+, C]^\dagger = [C^\dagger, K_+^\dagger] = [C, K_-].$$

b)

$$C|c,\xi\rangle = (J_3^2 - K_+K_- - K_-K_+)|c,\xi\rangle$$

$$= (J_3^2 - K_+K_- - (K_+K_- + [K_-, K_+]))|c,\xi\rangle$$

$$= \xi^2 (J_3^2 - 2K_+K_- + J_3)|c,\xi\rangle$$

$$= \xi(\xi - 1)|c,\xi\rangle \Longrightarrow \boxed{C = \xi(\xi - 1)}$$

c) El estado $K_+|c,\xi\rangle \sim |c,\xi+1\rangle$ ssi

$$0 \neq \left| K_+|c,\xi\rangle \right|^2 = \langle c,\xi| K_-K_+ |c,\xi\rangle = \langle c,\xi| [K_-, K_+] |c,\xi\rangle$$

$$= \langle c,\xi| J_3 |c,\xi\rangle = \xi \Longrightarrow \boxed{\xi > 0}$$

Now suppose that $(K_+)^p|c,\xi\rangle$ is a well-def'd state

$$\boxed{\begin{array}{l} K_+^p|c,\xi\rangle = \alpha_{(p)}|c,\xi+p\rangle \\ \langle c,\xi+p|c,\xi+p\rangle = 1 \end{array}}$$

Then $\exists_{p\in\mathbb{N}^+}\ K_+|c,\xi+p\rangle = 0$.

$$\Rightarrow 2\left|K_+|c,\xi+p\rangle\right|^2 = \langle c,\xi+p|2K_-K_+|\xi+p\rangle$$

$$\Rightarrow C = J_3^2 - K_+K_- - K_-K_+ = J_3^2 - (K_-K_+ + [K_+,K_-]) - K_-K_+$$

$$= J_3^2 - 2K_-K_+ + J_3 = J_3^2 + J_3 - 2K_-K_+$$

$$\Rightarrow 2K_-K_+ = J_3^2 + J_3 - C$$

$$\Rightarrow 2\left|K_+|c,\xi+p\rangle\right|^2 = \langle c,\xi+p|J_3^2 + J_3 - C|c,\xi+p\rangle$$

$$= \left[(\xi+p)^2 + \xi + p - \xi(\xi-1)\right]\langle c,\xi+p|c,\xi+p\rangle$$

$$= \xi^2 + 2p\xi + p^2 + \xi + p - \xi^2 + \xi$$

$$= 2(p+1)\xi + (p+1)p = (p+1)(2\xi+p)$$

But $p\in\mathbb{N}^+$ and $\xi > 0$

So that
$$(p+1)(2\xi+p) \neq 0$$

which implies that

$$\nexists_{p\in\mathbb{N}^+}\ K_+^{p+1}|c,\xi\rangle = 0$$

unless $\xi = 0$

but then you have a state
$$\boxed{\begin{array}{l} J_3|0,0\rangle = 0 \\ K_\pm|0,0\rangle = 0 \end{array}}$$