Métodos Matemáticos de la Física Teórica

Representations of Finite Groups

I) Given a finite group G and a d-dimensional (unitary) representation Γ, we define the *character of the representation* by $\chi_\Gamma(\mathrm{g}) = \mathrm{Tr}\,(\Gamma(\mathrm{g}))$.

- The character of the unit element is the dimension of the representation: $\chi(\mathrm{e}) = d$.
- Equivalent representations have identical characters, *i.e.* if $\Gamma' \sim \Gamma$ then for all $\mathrm{g} \in \mathrm{G}$ we have $\chi_{\Gamma'}(\mathrm{g}) = \chi_\Gamma(\mathrm{g})$.
- Conjugate element in G have identical character, *i.e.* if $\mathrm{g}' \sim \mathrm{g}$ then $\chi_\Gamma(\mathrm{g}') = \chi_\Gamma(\mathrm{g})$.
- For 1-dimensional representations there is no distinction between the characters and the representations, meaning that the characters of 1-dimensional representations must represent the multiplications in the Cayley table.

II) The number of inequivalent irreps is equal to the number of conjugacy classes.

- Let's assume that there are r irreps, then we can index the different irreps by Γ_p where p runs from 0 to $r-1$; the associated characters are then written as $\chi_p(\mathrm{g})$ and the dimension of the irrep d_p is $\chi_p(\mathrm{e}) = d_p$.
- By convention Γ_0 is the trivial representation: for all $\mathrm{g} \in \mathrm{G}$ we have $\chi_0(g) = 1$.
- If G is Abelian, then there are ord(G) irreps.

III) We can gather the characters into an ordered table where we plot the character for a given conjugacy class in a given irrep; due to the above results, we'll find a square table such as

Γ_p	$\chi(\{\mathrm{e}\})$	$\chi(\{\mathrm{g}_1\})$	$\ldots$	$\chi(\{\mathrm{g}_{last}\})$
0	1	1	$\ldots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
r	d_r	$\ldots$	$\ldots$	$\ldots$

The above table is called the character system of the group G.
A necessary condition for two finite groups to be isomorphic is for their character systems to be equal; this is, however, not a sufficient condition.

IV) $\sum_{p=0}^{r-1} d_p^2 = \text{ord(G)}$

- If G is Abelian, there are ord(G) irreps, which due to the above result implies that all irreps of an finite Abelian group are 1-dimensional.

V) The fundamental ingredient is the orthonormality of unitary irreps which for finite groups reads

$$\sum_{g\in\text{G}} \overline{\Gamma_p(g)^i{}_j}\ \Gamma_q(g)^a{}_b \ = \ \frac{\text{ord(G)}}{d_p}\ \delta_{pq}\ \delta^{ia}\ \delta_{jb}\ . \tag{1}$$

VI) For the characters the above implies

$$\sum_{g\in\text{G}} \overline{\chi_p(g)}\ \chi_q(g) \ = \ \text{ord(G)}\ \delta_{pq}\ . \tag{2}$$

VII) The Clebsch-Gordon series reads $\Gamma_p \otimes \Gamma_q \sim \oplus_{i=1}^r\ N_{pq}{}^i\ \Gamma_i$, where

$$\text{ord(G)}\ N_{pq}{}^i \ = \ \sum_{g\in\text{G}} \chi_p(g)\ \chi_q(g)\ \overline{\chi_i(g)}\ , \tag{3}$$

and also

$$d_p d_q \ = \ \sum_{i=1}^r N_{pq}{}^i\ d_i\ . \tag{4}$$

Example: Consider $\mathbb{Z}_2$. Since this group is Abelian and of order 2, we see that there are only 2 irreps which are furthermore 1-dimensional. The character system then is

Γ	χ(e)	χ(b)
0	1	1
1	1	-1

Checking the orthonormality of the character system is straightforward.

Example: Let us consider our favorite example, namely D_3: then we know that $\text{ord}(\text{D}_3) = 6$ and that there are 3 conjugacy classes, namely $\{\text{e}\}$, $\{\text{a}, \text{a}^2\}$ and $\{\text{b}, \text{ab}, \text{a}^2\text{b}\}$. From point IV we see that the dimensions of the 3 irreps are such that $d_0^2 + d_1^2 + d_2^2 = 6$. But we already know that the trivial representation is always possible and that $d_0 = 1$, which means that $d_1^2 + d_2^2 = 5$. The only solution to this equation then is that $d_1 = 1$ and $d_2 = 2$ (let's also introduce the convention that we order the irreps by increasing dimensionality!). At this point we have determined part of the character system, namely

Γ	$\chi(\{e\})$	$\chi(\{a\})$	$\chi(\{b\})$
0	1	1	1
1	1		
2	2		

We can start filling in the blanc spots by using eq. (2): consider the 1-dimensional irrep Γ_1, then taking into account the amount of elements contained in a given conjugacy class we must have $\chi_1(\mathrm{e}) + 2\chi_1(\mathrm{a}) + 3\chi_1(\mathrm{b}) = 0$ or $2\chi_1(\mathrm{a}) + 3\chi_1(\mathrm{b}) = -1$. But since we are dealing with a 1-dimensional representation,[1] the character itself is a representation and so we must have that for instance $\chi_1(a^2) = \chi_1(a)^2$. But since "$\mathrm{a}^2$" sits in the same conjugacy class as "a" we must have $\chi_1(\mathrm{a}^2) = \chi_1(\mathrm{a})$, which when combined with the other equation implies that $\chi_1(\mathrm{a}) = 1$. Applying the same reasoning to $\chi_1(\mathrm{b})$ we see that $\chi_1(\mathrm{b}) = \pm 1$, and by orthogonality of the characters we see that we must have $\chi_1(\mathrm{b}) = -1$.

If we impose the orthonormality conditions for Γ_2 we see that we must satisfy

$$\begin{aligned} 0 &= 2 + 2\chi_2(\mathrm{a}) + 3\chi_2(\mathrm{b}) \;, \\ 0 &= 2 + 2\chi_2(\mathrm{a}) - 3\chi_2(\mathrm{b}) \;, \\ 6 &= 4 + 2\chi_2(\mathrm{a})^2 + 3\chi_2(\mathrm{b})^2 \;, \end{aligned}$$

whose solution is given by $\chi_2(\mathrm{a}) = -1$ and $\chi_2(\mathrm{b}) = 0$. In short, the complete character system is

Γ	$\chi(\{e\})$	$\chi(\{a\})$	$\chi(\{b\})$
0	1	1	1
1	1	1	-1
2	2	-1	0

In this case we can also create the full 2-dimensional representation by putting

$$\Gamma_2(\mathrm{a}) = \tfrac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} \quad , \quad \Gamma_2(\mathrm{b}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The non-trivial Clebsch-Gordon series are easily calculated to be

$$\Gamma_1 \otimes \Gamma_1 = \Gamma_0 \;,\; \Gamma_1 \otimes \Gamma_2 = \Gamma_2 \;,\; \Gamma_2 \otimes \Gamma_2 = \Gamma_2 \oplus \Gamma_1 \oplus \Gamma_0 \;.$$

[1] This in effect is the way how to deduce the 1-dimensional representations: since for 1-dimensional representations there is no difference between the representation and the corresponding character the character must satisfy the Cayley table, which is enough to determine the 1-dimensional representations and therefore also the corresponding characters. The higherdimensional irreps are then deduced by using the orthogonality of the character system.

Excercise: Find the character system of D_4 and find the Clebsch-Gordon series of the highest dimensional representation with itself.

Excercise: In class we introduced the Quaternion Group $\mathbb{H}$ as a finite group of order 8 and the following Cayley table

$\mathbb{H}$	e	i	j	k	$\bar{\mathrm{e}}$	$\bar{\imath}$	$\bar{\jmath}$	$\bar{\mathrm{k}}$
e	e	i	j	k	$\bar{\mathrm{e}}$	$\bar{\imath}$	$\bar{\jmath}$	$\bar{\mathrm{k}}$
i	i	$\bar{\mathrm{e}}$	k	$\bar{\jmath}$	$\bar{\imath}$	e	$\bar{\mathrm{k}}$	j
j	j	$\bar{\mathrm{k}}$	$\bar{\mathrm{e}}$	i	$\bar{\jmath}$	k	e	$\bar{\imath}$
k	k	j	$\bar{\imath}$	$\bar{\mathrm{e}}$	$\bar{\mathrm{k}}$	$\bar{\jmath}$	i	e
$\bar{\mathrm{e}}$	$\bar{\mathrm{e}}$	$\bar{\imath}$	$\bar{\jmath}$	$\bar{\mathrm{k}}$	e	i	j	k
$\bar{\imath}$	$\bar{\imath}$	e	$\bar{\mathrm{k}}$	j	i	$\bar{\mathrm{e}}$	k	$\bar{\jmath}$
$\bar{\jmath}$	$\bar{\jmath}$	k	e	$\bar{\imath}$	j	$\bar{\mathrm{k}}$	$\bar{\mathrm{e}}$	i
$\bar{\mathrm{k}}$	$\bar{\mathrm{k}}$	$\bar{\jmath}$	i	e	k	j	$\bar{\imath}$	$\bar{\mathrm{e}}$

And we also calculated the conjugacy classes to be $\{\mathrm{e}\}$, $\{\bar{\mathrm{e}}\}$, $\{\mathrm{i}, \bar{\imath}\}$, $\{\mathrm{i}, \bar{\imath}\}$ and $\{\mathrm{k}, \bar{\mathrm{k}}\}$.

a) Construct $\mathbb{H}$'s character system.

b) Compare $\mathbb{H}$'s and D_4's character systems. Should they be equal: Can you show that $\mathbb{H} \simeq \mathrm{D}_4$ by explicitly constructing the isomorphism?

c) What irrep of $\mathbb{H}$ can be used to faithfully represent Hamilton's quaternions $(1, i, j, k)$?