Duality in String Theory

Yolanda Lozano (U. Oviedo)

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Duality: New insight into the non-perturbative regimes of both String Theory and QFT
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- In String Theory: T- and S-dualities

Define M-theory as the quantum theory whose low energy limit is eleven dim supergravity.
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(see M.P. García del Moral’s talk)
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Define it by means of the duality relations that connect its different limits

\[\text{In String and Gauge Theories:}\]

\textbf{Gauge/gravity duality} (Maldacena’97):

Some gauge theories at strong (weak) coupling are dual to weakly (strongly) coupled string theories with one extra dimension
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♦ In String and Gauge Theories:
  
  **Gauge/gravity duality** (Maldacena’97):
  
  Some gauge theories at strong (weak) coupling are dual to weakly (strongly) coupled string theories with one extra dimension

  **Holographic duality**
One of the most innovative ideas in Theoretical Physics in the recent past:

Possibility to apply string theory techniques to strongly coupled systems (quark-gluon plasma, condensed matter systems).

Possibility to learn something about quantum gravity by studying low dimensional systems with a holographic description (concrete realization of QG and space-time as emergent phenomena).
1. Duality relations
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- **T-duality:**

Perturbative transformation in String Theory. Relates large and small distances (space-time duality)
I. Duality relations

◆ T-duality:

Perturbative transformation in String Theory. Relates large and small distances (space-time duality)

In toroidal compactifications: $x \rightarrow x + 2\pi nR$

- Momentum states: $p = \frac{m}{R}$
- Winding states: $p = \frac{nR}{\alpha'}$

$\alpha' = l_s^2$; $l_s$ : string length \hspace{1cm} (l_s \rightarrow 0 \hspace{0.5cm} \text{Field Theory limit})
\[
\begin{array}{c}
R \rightarrow \frac{\alpha'}{R} \\
m \leftrightarrow n
\end{array}
\]

Symmetry of the bosonic string, maps Type IIA and Type IIB and Het SO(32) and Het \( E_8 \times E_8 \)
$R \rightarrow \frac{\alpha'}{R}$

$m \leftrightarrow n$

Symmetry of the bosonic string, maps Type IIA and Type IIB and Het SO(32) and Het $E_8 \times E_8$

11-D Supergravity

"M-Theory"

Type IIA  E8 x E8 Heterotic

Type IIB  SO(32) Heterotic

Type I
\[
\begin{aligned}
\frac{g_{st}}{R} &\rightarrow \frac{\alpha'}{R} \\
\phi &\rightarrow \text{dilaton (massless field)}
\end{aligned}
\]

Symmetry of the bosonic string, maps Type IIA and Type IIB and Het SO(32) and Het \( E_8 \times E_8 \)

\[
g_{st} \rightarrow \left( \frac{\sqrt{\alpha'}}{R} \right)^d g_{st}
\]

\( g_{st} \) : String coupling constant

\[
g_{st} = e^{\langle \phi \rangle}; \: \phi = \text{dilaton (massless field)}
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$g_{st}$ : String coupling constant

$$g_{st} = e^{\langle \phi \rangle} ; \phi = \text{dilaton}$$

(massless field)

For more general compactifications:
Abelian and non-Abelian T-duality (see later)
S-duality:

Non-perturbative transformation in String and Field Theory. Relates the strong and weak coupling regimes. Interchanges electric (elementary states) and magnetic (solitons)
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In a four dimensional Abelian Gauge Theory:

\( g = \) coupling constant

\[
\begin{align*}
g &\to \frac{1}{g} \\
\vec{E} &\to \vec{B} \\
\vec{B} &\to -\vec{E}
\end{align*}
\]

\[\rightarrow \text{Symmetry}\]
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Non-perturbative transformation in String and Field Theory. Relates the strong and weak coupling regimes. Interchanges electric (elementary states) and magnetic (solitons)

In a four dimensional Abelian Gauge Theory:

\( g \rightarrow \frac{1}{g} \)
\( E \rightarrow B \)
\( B \rightarrow -E \)

\( \tau = \frac{\theta}{2\pi} + \frac{i}{g^2} \)

generates the \( SL(2, \mathbb{Z}) \) S-duality group
• Also a symmetry of $N=4$ SYM in 4 dim (Montonen & Olive’s conjecture)
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Look at the BPS states (special class of supersymmetric states with the key property that their mass is completely determined by their charge) ⇒ Stable and protected from quantum corrections

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To test a non-perturbative duality relation:

Look at the BPS states (special class of supersymmetric states with the key property that their mass is completely determined by their charge) ⇒ Stable and protected from quantum corrections

i) Study their properties perturbatively at weak coupling in one theory

ii) Extrapolate (safely) to strong coupling

iii) Reinterpret in terms of non-perturbative configurations in the dual theory
In String Theory:

S-duality transformation:

\[
\begin{align*}
g_{st} & \rightarrow \frac{1}{g_{st}} \\
B_2 & \rightarrow C_2 \\
C_2 & \rightarrow -B_2
\end{align*}
\]

(\(B_2\) : NS-NS 2-form, \(C_2\) : RR 2-form)

(generalized Maxwell fields with 2 antisym. indices, part of the massless spectrum)
In String Theory:

S-duality transformation:

\[
g_{st} \to \frac{1}{g_{st}} + \begin{cases} 
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\end{cases}
\]

\( B_2 : \text{NS-NS 2-form, } C_2 : \text{RR 2-form} \)

(generalized Maxwell fields with 2 antisym. indices, part of the massless spectrum)

In the presence of a \( C_0 \) RR field:

\[
\lambda = C_0 + \frac{i}{g_{st}} \quad \text{transforms as} \quad \lambda \to -\frac{1}{\lambda}
\]

Together with \( \lambda \to \lambda + 1 \) they generate the \( SL(2, \mathbb{Z}) \) S-duality group
Symmetry of Type IIB, maps Type I with Heterotic $SO(32)$. Type IIA and Heterotic $E_8 \times E_8$ give different compactifications of M-theory in the strong coupling limit.
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To get deeper into these dualities in String Theory:
2. Some basic facts about String Theory
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- **Type II theories:**
  \[
  B_2, \quad C_p \quad \text{(} p \text{ odd IIA, } p \text{ even IIB) }
  \]
  
  NS-NS, RR
2. Some basic facts about String Theory

- **Type II theories:**
  
  $B_2, \ C_p, \ NS-NS, \ RR$  
  
  ( $p$ odd IIA, $p$ even IIB )

The (fundamental) string is an electric source for $B_2$. The object that couples magnetically is called and NS5-brane:

$\ast dB_2 = dB_6$
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- **Type II theories:**

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  The (fundamental) string is an electric source for \( B_2 \). The object that couples magnetically is called and NS5-brane:

  \[ *dB_2 = dB_6 \]

  A class of p-branes, called Dirichlet branes (Dp-branes) are electric sources for \( C_{p+1} \)

  All these branes are BPS objects, satisfying \( q = T : \)

  \[
  T_{F1} = \frac{1}{l_s^2} ; \quad T_{Dp} = \frac{1}{l_s^{p+1}} \frac{1}{g_{st}} ; \quad T_{NS5} = \frac{1}{l_s^6} \frac{1}{g_{st}^2}
  \]

  (non-perturbative)
They all occur as classical solitonic solutions of the low energy ($l_s \to 0$) effective actions (SUGRAs)
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**Crucial discovery** (Polchinski’95): RR charged p-branes = **Dirichlet p-branes**: (p+1) dimensional hypersurfaces on which open strings can end (previously predicted by T-duality):

![Diagram](image_url)
They all occur as **classical solitonic solutions** of the low energy \( l_s \rightarrow 0 \) effective actions \( \text{(SUGRAs)} \)

**Crucial discovery** (Polchinski’95): RR charged p-branes = **Dirichlet p-branes**: \( (p+1) \) dimensional hypersurfaces on which open strings can end (previously predicted by T-duality):

The important implication is that their dynamics can now be determined at weak coupling using open string perturbation theory!
For ex. conformal invariance of the worldsheet theory $\leftrightarrow$ Equations of motion of $S_{\text{eff}} = S_{\text{bulk}} + S_{\text{D-brane}}$ with $S_{\text{D-brane}}$ a U(1) gauge theory (U(N) for a system of N D-branes)
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- **Heterotic:** $B_2, B_6$ (F1, NS5 branes)
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- **Type I**: $C_2, C_6$ (D1, D5 branes)
- **Heterotic**: $B_2, B_6$ (F1, NS5 branes)

Type I and Heterotic SO(32) contain as well an SO(32) N=1 SYM vector multiplet ($E_8 \times E_8$ for Heterotic $E_8 \times E_8$)
Now we can go back to S-duality:

\[ g_{st} \rightarrow \frac{1}{g_{st}} \quad + \quad B_2 \rightarrow C_2 \]
\[ C_2 \rightarrow -B_2 \]
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- It relates Type I and Heterotic SO(32):

\[
\begin{align*}
C_2 & \quad B_2 \\
D1 & \rightarrow F1 \\
D5 & \quad NS5
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Now we can go back to S-duality:

\[ g_{st} \to \frac{1}{g_{st}} + B_2 \to C_2 \]

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\[
\begin{array}{c c c c c c c c c c}
C_2 & B_2 \\
D1 & \to & F1 \\
D5 & & NS5
\end{array}
\]

The strong coupling limit of Type IIA and Heterotic \(E_8 \times E_8\) is M-theory (more difficult to see..).
3. The gauge/gravity duality

Maldacena’97:

Equivalence between Type IIB string theory on $AdS_5 \times S^5$ and 4 dim $\mathcal{N} = 4$ supersymmetric SU(N) Yang-Mills theory

$AdS_5$: anti-de Sitter space in 5 dim (maximally symmetric solution of the Einstein eqs. with a negative cosmological constant)
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$AdS_5$: anti-de Sitter space in 5 dim (maximally symmetric solution of the Einstein eqs. with a negative cosmological constant)

First piece of evidence:

The symmetry group of 5 dim anti-de Sitter space matches precisely with the group of conformal symmetries of $\mathcal{N} = 4$ SYM $\Rightarrow$ AdS/CFT
Starting point: Study of $N$ coincident D3-branes in Type IIB in the large $N$ limit, based on two dual descriptions:
- As solution of the classical eqs. of motion of ST/SUGRA
- As a D-brane system
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The $U(N)$ gauge theory living on the branes becomes 4 dim $\mathcal{N} = 4$ SYM, with $g_{YM}^2 = g_s$:

Conformally invariant Topological large $N$ expansion $\sum_{g=0}^{\infty} N^{2-g} f_g(\lambda)$ with ‘t Hooft parameter $\lambda = g_{YM}^2 N$
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The U($N$) gauge theory living on the branes becomes 4 dim $\mathcal{N} = 4$ SYM, with $g^{2}_{YM} = g_s$:

- Conformally invariant
- Topological large $N$ expansion $\sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda)$ with 't Hooft parameter $\lambda = g^{2}_{YM} N$

Through the correspondence: $R^4 = 4\pi \lambda l_s^4$
\[ \lambda = g_s N \quad R^4 = 4\pi \lambda l_s^4 \quad \text{imply} \]

\[ N \to \infty, \text{ finite } \lambda \iff g_s \to 0 \Rightarrow \text{ Planar limit } \iff \text{ Classical limit of IIB ST} \]
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\[ N \to \infty, \text{ finite} \quad \lambda \iff g_s \to 0 \Rightarrow \text{Planar limit} \iff \text{Classical limit of IIB ST} \]

\[ \lambda \to \infty \iff l_s \to 0 \Rightarrow \text{Strong ‘t Hooft coupling limit} \iff \text{SUGRA limit} \]
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\[ \implies \text{Strong-weak coupling duality} \]
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⇒ Strong-weak coupling duality

Many tests:

For each field in the 5 dim bulk \( \iff \) Operator in the dual CFT

D-branes \( \iff \) Baryons, domain walls, etc.

...
Less supersymmetric extensions

Type IIB on $AdS_5 \times Y_5$ with $Y_5$ an Einstein manifold bearing 5-form flux:
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$Y_5 \leftrightarrow C_6$ with $C_6$ Ricci-flat such that $ds^2_{C_6} = dr^2 + r^2 ds^2_{Y_5}$

SUSY preserved $\iff Y_5$ is Sasaki-Einstein ($\iff C_6$ is Ricci-flat and Kähler (Calabi-Yau 3-fold))
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Examples: $Y_5 = S^5$, $C_6 = \mathbb{C}^6$

$Y_5 = T^{1,1} = \frac{SU(2) \times SU(2)}{U(1)}$, $C_6$ is the conifold

$Y_5 = Y^{p,q}$, $p, q$ coprime integers (includes quasi-regular and irregular SE manifolds)
Less supersymmetric extensions

Type IIB on $AdS_5 \times Y_5$ with $Y_5$ an Einstein manifold bearing 5-form flux:

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$\leftrightarrow$ Dual to $N=1$ SCFTs arising from a stack of D3-branes sitting at the tip of the CY cone
4. More on T-duality

4.1. D-branes from open strings
4.2. T-duality for more general backgrounds
4.3. Non-Abelian T-duality
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4.1. D-branes from open strings

Closed strings in a toroidal compactification:

Solution of the equations of motion: \( \partial_+ \partial_- X^i = 0 \) + boundary condition \( X^i(\tau, \sigma + 2\pi) = X^i(\tau, \sigma) + 2\pi n R \):

\[
X^i(\tau, \sigma) = X^i_0 + \alpha' \frac{m}{R} \tau + n R \sigma + ..
\]
T-duality transformation:

\[
R \rightarrow \frac{\alpha'}{R} \quad \iff \quad \begin{align*}
\partial_+ X^i &\rightarrow \partial_+ X^i \\
\partial_- X^i &\rightarrow -\partial_- X^i
\end{align*}
\]

\((d \rightarrow *d : \text{Hodge duality})\)
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(d \rightarrow *d : Hodge duality)

For open strings: \(\sigma \in [0, \pi]\)

Boundary:

\[
\partial \Sigma = \{\sigma = \text{constant}\}
\]

Neumann boundary conditions:

\[
\partial_\sigma X^\mu |_{\partial \Sigma} = 0
\]

(zero momentum flux at the end points)
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(d $\rightarrow$ $\ast d$ : Hodge duality)

For open strings: $\sigma \in [0, \pi]$  

Boundary:

\[\partial \Sigma = \{ \sigma = \text{constant} \}\]

Neumann boundary conditions: \[\partial_\sigma X^\mu |_{\partial \Sigma} = 0\]
(zero momentum flux at the end points)

\[\downarrow\]

Dirichlet boundary conditions: \[\dot{X}^i |_{\partial \Sigma} = 0; \ i = p + 1, \ldots, 9\]
\[ X^\mu = \{ X^m, X^i \} \quad m = 0, \ldots, p \quad \text{non-compact} \]

The end-points of the strings are fixed on a (p+1) dimensional hypersurface \( \equiv \) Dp-brane
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Open strings \(\xrightarrow{T\text{-duality}}\) Dp-branes
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The end-points of the strings are fixed on a \((p+1)\) dimensional hypersurface \(\equiv \text{Dp-brane}\)

Open strings \(\xrightarrow{T\text{-duality}}\) Dp-branes

Dynamical due to the open strings attached
4.2. T-duality for more general backgrounds

(Buscher’88; Rocek, Verlinde’92)

The fundamental string couples to the metric, NS-NS 2-form and dilaton through a non-linear sigma-model:

\[
S = \frac{1}{4\pi\alpha'} \int \left( g_{\mu\nu} \, dX^\mu \wedge \ast dX^\nu + B_{\mu\nu} \, dX^\mu \wedge dX^\nu \right) + \frac{1}{4\pi} \int R^{(2)} \phi
\]
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\]

In the presence of an **Abelian isometry**: \( \delta X^\mu = \epsilon k^\mu / \)

\[
\mathcal{L}_k g = 0, \quad \mathcal{L}_k B = d\omega, \quad i_k d\phi = 0
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In the presence of an Abelian isometry: \( \delta X^\mu = \epsilon k^\mu \)

\( \mathcal{L}_k g = 0, \mathcal{L}_k B = d\omega, i_k d\phi = 0 \)

i) Go to adapted coordinates: \( X^\mu = \{\theta, X^\alpha\} \) such that

\( \theta \to \theta + \epsilon \) and \( \partial_\theta \) (backgrounds) = 0
ii) **Gauge the isometry:** \( d\theta \rightarrow D\theta = d\theta + A \)

A non-dynamical gauge field / \( \delta A = -d\epsilon \)
ii) Gauge the isometry: \( d\theta \to D\theta = d\theta + A \)

\( A \) non-dynamical gauge field \( \delta A = -d\epsilon \)

iii) Add a Lagrange multiplier term: \( \tilde{\theta} dA \), such that

\[
\int D\tilde{\theta} \to dA = 0 \Rightarrow A \text{ exact}
\]

(in a topologically trivial worldsheet)

+ fix the gauge: \( A = 0 \to \text{Original theory} \)
ii) Gauge the isometry: \( d\theta \rightarrow D\theta = d\theta + A \)

A non-dynamical gauge field \( / \) \( \delta A = -d\epsilon \)

iii) Add a Lagrange multiplier term: \( \tilde{\theta} dA \), such that

\[
\int D\tilde{\theta} \rightarrow dA = 0 \Rightarrow A \text{ exact}
\]

(in a topologically trivial worldsheet)

+ fix the gauge: \( A = 0 \) \( \rightarrow \) Original theory

iv) Integrate the gauge field

+ fix the gauge: \( \theta = 0 \) \( \rightarrow \) Dual sigma model:

\[
\{\theta, X^\alpha\} \rightarrow \{\tilde{\theta}, X^\alpha\} \quad \text{and}
\]
\begin{align*}
\tilde{g}_{00} &= \frac{1}{g_{00}}; \quad \tilde{g}_{0\alpha} = \frac{B_{0\alpha}}{g_{00}}; \quad \tilde{g}_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{0\alpha}g_{0\beta} - B_{0\alpha}B_{0\beta}}{g_{00}} \\
\tilde{B}_{0\alpha} &= \frac{g_{0\alpha}}{g_{00}}; \quad \tilde{B}_{\alpha\beta} = B_{\alpha\beta} - \frac{g_{0\alpha}B_{0\beta} - g_{0\beta}B_{0\alpha}}{g_{00}} \\
\tilde{\phi} &= \phi - \log g_{00}
\end{align*}

Buscher’s formulae
\[ \tilde{g}_{00} = \frac{1}{g_{00}}; \quad \tilde{g}_{0\alpha} = \frac{B_{0\alpha}}{g_{00}}; \quad \tilde{g}_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{0\alpha}g_{0\beta} - B_{0\alpha}B_{0\beta}}{g_{00}} \]

\[ \tilde{B}_{0\alpha} = \frac{g_{0\alpha}}{g_{00}}; \quad \tilde{B}_{\alpha\beta} = B_{\alpha\beta} - \frac{g_{0\alpha}B_{0\beta} - g_{0\beta}B_{0\alpha}}{g_{00}} \]

\[ \tilde{\phi} = \phi - \log g_{00} \]

- Conformally invariant (to first order in \( l_s^2 \))
\[\tilde{g}_{00} = \frac{1}{g_{00}}; \quad \tilde{g}_{0\alpha} = \frac{B_{0\alpha}}{g_{00}}; \quad \tilde{g}_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{0\alpha}g_{0\beta} - B_{0\alpha}B_{0\beta}}{g_{00}}\]

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**Buscher’s formulae**

- Conformally invariant (to first order in \( l_s^2 \))
- Involutive transformation:  
  \[\tilde{S} \quad \tilde{\theta} \rightarrow \tilde{\theta} + \epsilon \quad S\]
\[
\tilde{g}_{00} = \frac{1}{g_{00}}; \quad \tilde{g}_{0\alpha} = \frac{B_{0\alpha}}{g_{00}}; \quad \tilde{g}_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{0\alpha}g_{0\beta} - B_{0\alpha}B_{0\beta}}{g_{00}}
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\]
\[
\tilde{\phi} = \phi - \log g_{00}
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Buscher’s formulae

- Conformally invariant (to first order in \( l_s^2 \))
- Involution transformation:
\[
\tilde{S} \quad \tilde{\theta} \rightarrow \tilde{\theta} + \epsilon \quad \rightarrow \quad S
\]
- \((M, \tilde{M})\) different geometries and topologies
  
  For ex. \( M = S^3 \); \( \tilde{M} = S^2 \times S^1 \)

  (Alvarez, Alvarez-Gaumé, Barbón, Y.L.’93)
- Arbitrary worldsheets? (symmetry of string perturbation theory):

\[ \Rightarrow \text{Non-trivial topologies} + \text{compact isometry orbits} \]
- **Arbitrary worldsheets?** (symmetry of string perturbation theory):

\[
\int_{\gamma} d\epsilon = 2\pi n ; \quad n \in \mathbb{Z}
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- **Arbitrary worldsheets?** (symmetry of string perturbation theory):

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To fix them:
- **Arbitrary worldsheets?** (symmetry of string perturbation theory):

⇒ Non-trivial topologies + compact isometry orbits

Large gauge transformations: \( \oint_{\gamma} d\epsilon = 2\pi n ; \quad n \in \mathbb{Z} \)

To fix them:

Multivalued Lagrange multiplier: \( \oint_{\gamma} d\tilde{\theta} = 2\pi m ; \quad m \in \mathbb{Z} \)

such that

\( \int [\text{exact}] \rightarrow dA = 0 \quad + \quad \int [\text{harmonic}] \Rightarrow A \text{ exact} \)
Arbitrary worldsheets? (symmetry of string perturbation theory):

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⇒ The gauging procedure works for all genera

(Álvarez, Álvarez-Gaumé, Barbón, Y.L.’93)
In fact, going to phase space: $L(\theta, \dot{\theta}, X^\alpha) \rightarrow H(\theta, P_\theta, X^\alpha)$
In fact, going to phase space:  \( L(\theta, \dot{\theta}, X^\alpha) \rightarrow H(\theta, P_\theta, X^\alpha) \)

The \textbf{canonical transformation} from \( \{\theta, P_\theta\} \rightarrow \{\tilde{\theta}, P_{\tilde{\theta}}\} \):

\[
\begin{align*}
P_\theta &= -\tilde{\theta}' \\
P_{\tilde{\theta}} &= -\theta'
\end{align*}
\]

generates the dual background!

(Alvarez, Alvarez-Gaumé, Y.L. ’94)
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**generates the dual background!**

(Alvarez, Alvarez-Gaumé, Y.L. ‘94)

\[ \Rightarrow \text{`Minimal` approach to Abelian T-duality} \]

In this set-up:

\[
\begin{align*}
\partial_+ X^i &\rightarrow \partial_+ X^i \\
\partial_- X^i &\rightarrow -\partial_- X^i
\end{align*}
\]

is generalized to

\[
\begin{align*}
\partial_+ \tilde{\theta} &= g_{00} \partial_+ \theta + (g_{0\alpha} - B_{0\alpha}) \partial_+ X^\alpha \\
\partial_- \tilde{\theta} &= -(g_{00} \partial_- \theta + (g_{0\alpha} + B_{0\alpha}) \partial_- X^\alpha)
\end{align*}
\]
For the open strings:

Neumann boundary conditions are generalized to

\[ g_{00} \theta' + g_{0\alpha} X^{\alpha'} - B_{0\alpha} \dot{X}^\alpha = 0 \]
\[ g_{0\alpha} \theta' + g_{\alpha\beta} X^{\beta'} + B_{0\alpha} \dot{\theta} - B_{\alpha\beta} \dot{X}^\beta = 0 \]
For the **open strings**: 

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g_{0\alpha} \theta' + g_{\alpha\beta} X^{\beta'} + B_{0\alpha} \dot{\theta} - B_{\alpha\beta} \dot{X}^{\beta} &= 0
\end{align*}
\]

and they are mapped to

\[
\begin{align*}
\dot{\theta} &= 0 \\
\tilde{g}_{0\alpha} \tilde{\theta}' + \tilde{g}_{\alpha\beta} X^{\beta'} + \tilde{B}_{0\alpha} \dot{\theta} - \tilde{B}_{\alpha\beta} \dot{X}^{\beta} &= 0
\end{align*}
\]
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and they are mapped to

\[ \dot{\tilde{\theta}} = 0 \]
\[ \tilde{g}_{0\alpha} \tilde{\theta}' + \tilde{g}_{\alpha\beta} X^{\beta'} + \tilde{B}_{0\alpha} \dot{\tilde{\theta}} - \tilde{B}_{\alpha\beta} \dot{X}^{\beta} = 0 \]

\[ \tilde{\theta} : \text{Dirichlet} \]
\[ X^{\alpha} : \text{Neumann} \]

\[ \Rightarrow \quad \tilde{\theta} : \text{D-branes} \]

(Borlaf, Y.L.'96)
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and they are mapped to

\[ \hat{\theta} = 0 \]
\[ \tilde{g}_{0\alpha} \tilde{\theta}' + \tilde{g}_{\alpha\beta} X^\beta' + \tilde{B}_{0\alpha} \dot{\theta} - \tilde{B}_{\alpha\beta} \dot{X}^\beta = 0 \]

\[ \Rightarrow \tilde{\theta} : \text{Dirichlet} \]
\[ X^\alpha : \text{Neumann} \]

\[ \Rightarrow \text{D-branes} \]

(Borlaf,Y.L.’96)

Suitable for the study of boundary conditions (also fermions)
Quantum mechanically: \( H e^{i\mathcal{F}} = \tilde{H} e^{i\mathcal{F}} \) implies:

\[
\psi_k[\tilde{\theta}] = N(k) \int \mathcal{D}\theta(\sigma) e^{i\mathcal{F}[\theta,\tilde{\theta}]} \phi_k[\theta(\sigma)]
\]

In our case:

\[
\mathcal{F} = \frac{1}{2} \int_{S^1} d\sigma (\theta'\tilde{\theta} - \theta\tilde{\theta}')
\]

- Derive global properties
- Arbitrary genus Riemann surfaces
4.3. Non-Abelian T-duality

(De la Ossa, Quevedo’93; Alvarez, Alvarez-Gaume, Y.L.’95)
4.3. Non-Abelian T-duality

(De la Ossa, Quevedo’93; Alvarez, Alvarez-Gaume, Y.L.’95)

**Non-Abelian continuous isometry:** \[ X^m \rightarrow g^m_n X^n, \quad g \in G \]
4.3. Non-Abelian T-duality

(De la Ossa, Quevedo’93; Alvarez, Alvarez-Gaume, Y.L.’95)

**Non-Abelian continuous isometry:** \( X^m \rightarrow g_n^m X^n, g \in G \)

i) **Gauge it:**

\[
dX^m \rightarrow DX^m = dX^m + A_n^m X^n
\]

\( A \in \text{Lie algebra of } G \quad A \rightarrow g(A + d)g^{-1} \)
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**Non-Abelian continuous isometry:** \( X^m \rightarrow g^m_n X^n \), \( g \in G \)

i) Gauge it: 
\[
 DX^m = dX^m + A^m_n X^n
\]
\( A \in \text{Lie algebra of } G \) 
\( A \rightarrow g(A + d)g^{-1} \)

ii) Add a Lagrange multiplier term: 
\[
 \text{Tr}(\chi F)
\]
\( F = dA - A \wedge A \)
\( \chi \in \text{Lie Algebra of } G, \chi \rightarrow g\chi g^{-1}, \text{ such that} \)
\[
 \int D\chi \rightarrow F = 0 \Rightarrow A \text{ exact}
\]
(in a topologically trivial worldsheet)

+ fix the gauge: \( A = 0 \Rightarrow \text{Original theory} \)
iii) Integrate the gauge field + fix the gauge → Dual theory
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General properties:
- Loss of symmetries in the dual $\Rightarrow$ Non-involutive
- Original and dual very different geometrically
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General properties:
- Loss of symmetries in the dual ⇒ Non-involutive
- Original and dual very different geometrically

Example: Principal chiral model with group SU(2):

Geometrically: $S^3$

$$L = Tr(g^{-1}dg \wedge *g^{-1}dg) ; \ g \in SU(2)$$

Invariant under:
$$g \rightarrow h_1 g h_2 ; \ h_1, h_2 \in SU(2)$$

Choose: $g \rightarrow hg ; \ h \in SU(2)$
\[ \tilde{L} = \frac{1}{1 + \chi^2} \left( \delta_{ij} - \epsilon_{ijk} \chi^k + \chi_i \chi_j \right) d\chi^i \wedge *d\chi^j \]

**Invariant under** \[ \chi \to h \chi h^{-1}; \ h \in SU(2) \]
\[
\tilde{L} = \frac{1}{1 + \chi^2} \left( \delta_{ij} - \epsilon_{ijk} \chi^k + \chi_i \chi_j \right) d\chi^i \wedge *d\chi^j
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Invariant under \( \chi \to h\chi h^{-1} ; \ h \in SU(2) \)

- Higher genus generalization? Set to zero \( W_\gamma = P e^{\$\gamma} A \)
\[ \tilde{L} = \frac{1}{1 + \chi^2} \left( \delta_{ij} - \epsilon_{ijk} \chi^k + \chi_i \chi_j \right) d\chi^i \wedge *d\chi^j \]

Invariant under \( \chi \to h\chi h^{-1}; \ h \in SU(2) \)

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- Global properties?
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**True symmetry in String Theory?**
Partial results about its realization as a **canonical transformation**: (Y.L.’95)
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A system of coordinates adapted to the non-commuting isometries does not exist
Partial results about its realization as a canonical transformation: (Y.L.’95)

A system of coordinates adapted to the non-commuting isometries does not exist

For principal chiral models:

\[ L = Tr(g^{-1} dg \wedge *g^{-1} dg) ; \ g \in G \]

Use Maurer-Cartan forms:

\[ g^{-1} dg \equiv \Omega_a(\theta) d\theta^a \]

Then:

\[ H = \frac{1}{4} \omega^{ak} \omega^{bk} \Pi_a \Pi_b + \Omega^k_a \Omega^k_b \theta^{a'} \theta^{b'} \]

\[ \Omega_a = \Omega^k_a T^k, \quad \omega^{ak}(\theta) \Omega^i_a(\theta) = \delta^{ki} \]
The generating functional

$$\mathcal{F}[\chi, \theta] = \int_{S^1} d\sigma \, \text{Tr} \left( \chi \, \Omega_a(\theta) \partial_\sigma \theta^a \right)$$

generates the canonical transformation that gives the dual:

$$\Pi_a = - \frac{\delta \mathcal{F}}{\delta \theta^a}$$

$$\tilde{\Pi}_i = \frac{\delta \mathcal{F}}{\delta \chi^i}$$
The generating functional

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\]

In this approach:

- Involutive
- Generalize to arbitrary genus Riemann surfaces
- Global properties
Interesting as a **solution generating technique** (Sfetsos, Thompson’10; Y.L., O Colgain, Sfetsos, Thompson’11; Itsios, Y.L., O Colgain, Sfetsos’12):
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In the context of gauge/gravity duality:

Apply it to the $AdS_5 \times S^5$ Type IIB background:
Interesting as a solution generating technique (Sfetsos, Thompson’10; Y.L., O Colgain, Sfetsos, Thompson’11; Itsios, Y.L., O Colgain, Sfetsos’12):

In the context of gauge/gravity duality:

Apply it to the $AdS_5 \times S^5$ Type IIB background:

\[
\begin{align*}
    ds^2 &= ds^2(AdS_5) + ds^2(S^5) \\
    ds^2(S^5) &= 4(d\theta^2 + \sin^2 \theta d\phi^2) + \cos^2 \theta ds^2(S^3) \\
    F_5 &= 2\text{Vol}(AdS_5) - 2\text{Vol}(S^5) \\
    F_5 &= dC_4
\end{align*}
\]
Interesting as a **solution generating technique** (Sfetsos, Thompson’10; Y.L., O Colgain, Sfetsos, Thompson’11; Itsios, Y.L., O Colgain, Sfetsos’12):

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    F_5 &= 2\text{Vol}(AdS_5) - 2\text{Vol}(S^5)
\end{align*}
\]

Dual Type IIA background:

\[
\begin{align*}
    d\tilde{s}^2 &= ds^2(AdS_5) + 4(d\theta^2 + \sin^2 \theta d\phi^2) + \frac{dr^2}{\cos^2 \theta} + \frac{r^2 \cos^2 \theta}{\cos^4 \theta + r^2} ds^2(S^2) \\
    F_2, F_4, F_6, F_8 &\neq 0
\end{align*}
\]
Less supersymmetric background related to some N=2 SCFT’s
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In general, less SUSY backgrounds with no clear dual CFT.
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Progress about the non-invertibility:
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Progress about the non-invertibility:

Abelian
(Bergshoeff, Hull, Ortín’95)
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Progress about the non-invertibility:

**Abelian**
-(Bergshoeff, Hull, Ortín’95)-

**Non-Abelian**
5. Conclusions
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Many of the most important recent developments in String Theory are based on duality relations
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Mainly focused on T-duality
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In fact, although physically they are very different, S-duality can also be formulated as $d \to \ast d$:
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In an Abelian gauge theory of 1-forms in d dim: $A_1 \leftrightarrow \tilde{A}_{d-3}$
5. Conclusions

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Mainly focused on T-duality.

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In Maxwell theory: $A_1 \leftrightarrow \tilde{A}_1$
5. Conclusions

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In String Theory?
5. Conclusions

Many of the most important recent developments in String Theory are based on duality relations

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In an Abelian gauge theory of 1-forms in d dim: $A_1 \leftrightarrow \tilde{A}_{d-3}$

In Maxwell theory: $A_1 \leftrightarrow \tilde{A}_1$

In String Theory?

Look at the worldvolume of D-branes
For a D3-brane: $A_1 \leftrightarrow \tilde{A}_1 \rightarrow$ D3-brane at strong coupling
For a D3-brane:  \( A_1 \leftrightarrow \tilde{A}_1 \rightarrow \text{D3-brane at strong coupling} \)

For a D1-brane:  \( A_1 \leftrightarrow \tilde{A}_{-1} \rightarrow q \text{ F1} \quad ((p,q) \text{ strings}) \)
For a D3-brane:  $A_1 \leftrightarrow \tilde{A}_1 \rightarrow$ D3-brane at strong coupling

For a D1-brane:  $A_1 \leftrightarrow \tilde{A}_{-1} \rightarrow q F1 \ ( (p,q) \ strings )$

For a D2-brane:  $A_1 \leftrightarrow \tilde{A}_0 \rightarrow$ M2-brane
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For a D2-brane: $A_1 \leftrightarrow \tilde{A}_0 \rightarrow$ M2-brane

....

(Y.L.’95; 96)
For a D3-brane: \( A_1 \leftrightarrow \tilde{A}_1 \rightarrow \text{D3-brane at strong coupling} \)

For a D1-brane: \( A_1 \leftrightarrow \tilde{A}_{-1} \rightarrow q \text{ F1 } \) \(((p,q) \text{ strings})\)

For a D2-brane: \( A_1 \leftrightarrow \tilde{A}_0 \rightarrow \text{M2-brane} \)

....

(Y.L.’95; 96)

Thanks!
Towards the formulation of M-theory

(See M.P. García del Moral’s talk)

Extensions and applications of gauge/gravity duality

Recent developments:

Double field formalism: Reformulate ten dim SUGRA such that the T-duality group is manifest:

Combine string fields and their T-duals: \( dX^I = (dX^\mu, d\tilde{X}^\mu) \)

Expand the tangent space from \( T\Lambda^1(M) \) to \( T\Lambda^1(M) \oplus T\Lambda^*^1(M) \)

Metric on this generalized space transforms under \( O(d, d) \)

→ Doubled geometry

Mathematics of doubled geometry being developed
Towards the formulation of M-theory

(See M.P. García del Moral’s talk)

Extensions and applications of gauge/gravity duality

Thanks!