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# Decomposing Productivity Growth under Production Risk

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# UNIVERSIDAD DE OVIEDO

# DEPARTAMENTO DE ECONOMÍA

## PERMANENT SEMINAR ON EFFICIENCY AND PRODUCTIVITY

# DECOMPOSING PRODUCTIVITY GROWTH UNDER PRODUCTION RISK

**Luis Orea and Alan Wall**[§]

**Efficiency Series Paper 12/2002**

**Abstract**

Production risk has generally not been taken into account when decomposing productivity growth. Interpreting productivity measures as indicators of firm production performance, we incorporate the effect of production risk on total factor productivity (TFP), extending previous work under production certainty. We outline a series of minimal characteristics an index of TFP should have in order to capture the impact of risk on producers and a simple index is proposed. Changes in total factor productivity under uncertainty are decomposed into terms related with changes in expected average productivity (technical change and changes in scale efficiency) and changes in production risk and risk preferences.

**Keywords**: Production Risk, Total Factor Productivity, Productivity Decomposition.

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## 1. Introduction

While much has been written on the measurement of productivity growth and its decomposition, very little attention has been paid in this literature to the role of production risk[1]. Productivity growth is measured as the ratio of output to inputs (either in levels or as indices of output and inputs), where output may be *observed* output or an *estimate* of output based on the estimation of a production function. Whereas many different opinions exist with regard to how productivity should be measured, it appears reasonable to state that, regardless of how it is actually measured, the objective of any productivity study, theoretical or empirical, is to provide an indicator of producer performance. If we accept this, then it appears somewhat surprising that output be expressed merely in terms of that observed or expected given a certain input usage, with no account taken of the risk associated with this output. Risk averse producers will evaluate their performance on the basis not only of the output they expect to achieve from a certain input usage but also from the risk associated with this output.

By way of example, the measurement of total factor productivity (TFP) can be interpreted as an attempt to provide an indicator of the average production performance of the producer, and two producers with the same TFP measure would therefore be regarded as having the same production performance. However, if these producers face the same degree of risk but differ in their degree of aversion to this risk, then they would have different perceptions of their production performance. Clearly, the same would apply in the case where the producers were equally risk averse but faced different levels of output risk.

On this basis, it can be argued that measures of productivity which ignore output risk and producer risk aversion provide insatisfactory indicators of producer performance in that they do not accurately reflect the welfare situation, in production terms, of the producer. It would be desirable, therefore, to have a measure of productivity which deals with these concerns. Our objective in this paper is to provide a measure of productivity which takes these welfare considerations into account. To provide a sound theoretical background for our measure, we make use of familiar concepts drawn from the literature on risk.

[1] See Morrison (1993) and Lovell (1996) for surveys of the literature.

To our knowledge, the only work which has been carried out on this issue is that of Buccola (2002). The author also draws on the risk literature to provide a measure of productivity growth under uncertainty and is closely related to the work in our study. Our focus is somewhat different however, in that Buccola (2002) constructs his index on the basis that producers maximise the expected utility of profit. Our focus is purely on *physical* production, and does not consider the role that this production plays in achieving any more general objective such as profit (revenue) maximisation in private firms or social welfare maximisation in public firms. Hence, input and output prices play no role in our analysis as our aims are to provide a simple, easily calculable index which uses only information on input and output levels and to decompose this index in order to identify the contribution of inputs and technical change to productivity under uncertainty. It is also important to state that the aim of our index is to *evaluate*, not *explain*, producer performance. Therefore, we analyse productivity growth *conditional* on the input combinations observed.

The paper proceeds as follows. In Section 2 we illustrate how productivity measures using only information on input and output levels can be decomposed when output uncertainty is ignored. Section 3 analyses the importance of incorporating uncertainty and producer risk aversion and provides a series of general characteristics which an index of productivity under uncertainty should have. An index of TFP under uncertainty is proposed in Section 4, and this index is decomposed into scale and technical change effects This section ends with a discussion of some issues surrounding the empirical calculation of the index. Section 6 concludes.

## 2. Decomposing productivity growth ignoring uncertainty

Traditional literature on productivity growth decomposition usually assumes that output is generated by the following stochastic production function:

$$y = F(x,t,v) = f(x,t) + v \tag{1}$$

where *F(x,t,v)* is the stochastic version of the production function, *f(x,t)* is the deterministic part of production, subscript *t* stands for period, *y* is the output, $x = (x_1 \ldots . x_K)$ is a vector of *K* inputs, the time trend *t* is used as an indicator of technology level, *b* is a vector of parameters to be estimated, and *v* is a random noise term. Under the classical assumptions of strict exogeneity and (conditional) homoscedasticity, we have that:[2]

$$E(v \mid x,t) = 0 \tag{2}$$

$$Var(v \mid x,t) = E(v^2 \mid x,t) = \sigma^2 \tag{3}$$

While production uncertainty exists, standard practice on productivity growth decomposition is to proceed as if the output generating process is deterministic. This is justified by the assumption in (2), as the effect of the random noise on productivity growth disappears when averaging over time or firms. Thus, the error term and hence uncertainty are ignored. Assuming that there is only one input (*x* is a scalar) in order to keep the discussion simple, total factor productivity is measured by the ratio:

$$TFP = \frac{f(x,t)}{x} = \frac{E(y \mid x,t)}{x} \tag{4}$$

Therefore, ignoring uncertainty, total factor productivity can be interpreted as an expected average productivity ratio, given (i.e. conditional on) the input level.[3] Taking logarithms and differentiating with respect to time we can express the rate of growth of total factor productivity as:

$$\frac{d\ln TFP}{dt} = \frac{d\ln E(y \mid x,t)}{dt} - \frac{d\ln x}{dt} \tag{5}$$

or in dot notation

$$\dot{TFP} = \dot{E}(y \mid x,t) - \dot{x} \tag{6}$$

[2] Here, we present the classical regression assumptions treating regressors as random. This is in contrast to the treatment in most papers, where *x* is assumed to be "fixed" or deterministic. In this case, there is no need to distinguish between conditional or unconditional notation, so assumptions (2) and (3) can be written respectively as $E(v)=0$ and $Var(v)=\sigma^2$. However, (2) and (3) are the preferred expressions, as the productivity index under uncertainty introduced later is conditional on *x*.

[3] Total factor productivity can also be measured in terms of actual output rather than expected output, i.e. as the ratio $F(x,t,v)/x$, where $F(x,t,v)$ is the random production function (1). Since this index depends explicitly on the production disturbance, it can be interpreted as an *ex post* average productivity measure. This measure reflects the "appropriate" productivity when the production disturbance is known. However, in many production problems, firms' decisions are made *ex ante*, i.e. before the realization of the production disturbance. For this reason we prefer to define productivity in terms of expected output.

Total factor productivity growth is thus defined as the rate of growth of expected output minus the rate of growth in the input usage.[4] Hence, changes in TFP capture changes in expected average productivity. As illustrated in Figure 1, expected average productivity may change due to the effect of technical change or/and the effect of non-constant returns to scale when the input level expands over time. We show in Appendix A that (4) can be decomposed as:

$$T\dot{F}P = (\varepsilon - 1)\dot{x} + \varepsilon_t \tag{7}$$

where $\varepsilon$ is the scale elasticity and $\varepsilon_t$ is the growth of output as a result of technical change. While the second term on the right-hand side measures the contribution of technical change, the first term measures changes in scale efficiency when outputs expand over time (movements along the production function) and the technology exhibits increasing or decreasing returns to scale. This term depends on the degree of returns to scale, measured as the scale elasticity minus one. Increasing and decreasing scale economies are indicated by a positive value and negative value respectively. Hence, an expansion in inputs leads to an increase (decrease) in productivity when increasing (decreasing) returns to scale exist.[5]

## 3. Incorporating risk preferences

The value attributed to production performance as expressed by the measure of the expected average rate of total factor productivity growth in (6) is defined exclusively in terms of the growth of expected output, with output risk playing no role. If we want to provide a picture of producer welfare, in the sense of how the producer perceives his production performance, use of the measure in (6) effectively assumes that producers are risk neutral: only then can a positive productivity growth according to (6) be unambiguously associated with an improved production performance. If producers are not risk neutral, then they will be concerned not only about the effects on expected output but also about risk properties when they choose input levels and/or they consider the adoption of potentially risk-reducing or risk-increasing technologies. A

[4] As shown in Appendix A, when there are multiple inputs the single input rate, $\dot{x}$, is replaced by a weighted average of input growth rates.

[5] See first part of Appendix B [to equation (B10)] for a discrete-time specification of (7), under the assumption that the deterministic production function follows a quadratic form, which can be used for calculation purposes.

growth in TFP according to (6) may therefore not necessarily be perceived as positive from the producer's perspective.[6]

Thus, the measure in (6) is somewhat incomplete when we recognise that producers may not be risk neutral. The question then arises as to how (6) can be extended to take these factors into account and thereby give a fuller picture of the productivity performance of producers. We illustrate some of the issues outlined in the above discussion using Figure 2 below.

Figure 2 represents the case of a producer who produces output with a single input, using a constant returns to scale technology where expected output is represented by $f(x)$.[7] From time $t$ to time $t+1$ the producer increases input use from $x_t$ to $x_{t+1}$. Conditional on input usage, expected output increases from $E_t(y|x_t)$ to $E_{t+1}(y|x_{t+1})$, moving from point A to point B. Now, according to (6), we see no change in the producer's productivity performance. However, the input $x$ is risk increasing, as can be seen by the fact that the conditional variance of output, and thus production risk, has increased, ie. $\text{var}_{t+1}(y|x_{t+1})=cd>ab=var_t(y|x_t)$.[8] Hence, if the producer is not risk neutral, his situation (or valuation of it) is different. In particular, if he were risk averse, he would consider that his performance in terms of productivity in time $t$ has worsened with respect to time $t+1$[9].

Alternatively, one could interpret the figure as representing the situation of two producers with different degrees of risk aversion operating at one of the points, say A. The measure in (6) assigns them the same productivity valuation ($E_t(y|x_t)/x_t$), but it is

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[6] For example, changes in input usage, or technical change, which increase expected output such that TFP rises according to (6) may lead to increases in output risk ("risk-increasing inputs") that makes a risk averse producer perceive his situation to be worse than before. Euqation (6) would therefore overstate the producer's performance (or understate it if the producer were a risk lover).

[7] Note that we assume no technical change in Figure 2.

[8] Note also that, in Figure 2, the conditional variance of output has increased both in absolute terms and relative terms (i.e. compared with the expected output).

[9] One could also interpret the figure as representing two producers with similar risk averse preferences who are operating at points A and B respectively at the same point producers in time. Their "deterministic" TFP levels according to (5) are similar but the producer at A faces a lower production risk and therefore would surely consider himself as better off.

clear that for the given production risk ($var_t(y|x_t)=ab$) the least risk averse producer would consider his situation as more favourable.[10]

The considerations just outlined provide a notion as to what properties a measure of TFP under risk should have. The natural point of comparison for such a measure is that representing the certainty case, i.e. expression (6), so the next step is to ask ourselves what properties should a measure incorporating risk have and how it would compare to the certainty measure. As a preliminary step therefore towards arriving at a credible measure of productivity under risk, on the basis of the discussion above we propose a minimal set of characteristics which we feel that a desirable total factor productivity measure under uncertainty (*TFPU*) should have.

For a given producer, let *TFP* and $T\dot{F}P$ be the *level* and rate of growth of total factor productivity under certainty. Then,

(1): *If the producer is risk neutral, then TFPU should be equal to TFP regardless of the level of risk he faces*.

(2): *If the producer is risk averse (loving), then TFPU should be less (greater) than TFP for any given level of risk.*

(3): *If producers are risk neutral, then* $T\dot{F}PU$ *should be equal to* $T\dot{F}P$ *regardless of any change in production risk that may have occurred.*

(4): *If producers are risk averse, then any decrease (increase) in production risk should imply that* $T\dot{F}PU$ *be greater (less) than* $T\dot{F}P$ *.*

(5): *If producers are risk lovers, then any increase (decrease) in production risk should imply that* $T\dot{F}PU$ *be greater (less) than* $T\dot{F}P$ *.*

(6): *If production risk remains unchanged and the risk preferences of the producers have not changed,* $T\dot{F}PU$ *should be equal to* $T\dot{F}P$ *.*

(7): *If production risk remains unchanged, then a change in a producer's preferences such that he becomes less (more) risk averse should imply that* $T\dot{F}PU$ *be greater (less) than* $T\dot{F}P$ *.*

[10] One could instead consider a single firm who operates at point A in both time periods. Faced with the same risk in both periods, if he becomes less (more) risk averse he will consider himself better (worse) off.

These properties are very straightforward. Property (1) simply says that risk neutral producers should be attributed the same productivity performance under certainty and uncertainty. Under property (2), risk averse producers are attributed a lower valuation under the uncertainty measure than under the certainty measure, reflecting the negative impact of risk on his perceived performance. Properties (3)-(7) focus on the growth rates of TFP, with (3)-(5) covering the effect of changing production risk and (6)-(7) referring to changes in risk preferences in the face of a given unchanged risk. Some general remarks can be made on these last five properties. In terms of Figure 2, a measure with these characteristics would assign a negative value to a move from point A to point B for a risk averse producer, reflecting a worsening of his productivity performance, whereas the measure under certainty would assign a value of zero implying that the performance of the producer has not changed. Also, for a producer operating at point A in *both* periods but who is more (less) risk averse in $t+1$ than in $t$, there would be a negative change in TFP under uncertainty reflecting the fact that the given unchanged risk has a more adverse effect on the producer than in the previous period. The certainty measure, on the other hand, would register no change in the producer's performance.

Clearly, many measures of productivity growth under uncertainty (TFPU) satisfying the properties above could be constructed. However, all of them can be expressed, in general terms, as a function of a total factor productivity measure under certainty (TFP), an output risk measure (expressed in terms of some function of output variance) and a risk preferences measure (based on parameters of the producers utility function). Having outlined the general characteristics that a TFPU measure should have and the components of which it should be composed, the next step therefore is to propose a specific, calculable index which satisfies the properties.

**4. Decomposition of productivity growth with production risk**

It is clear that risk averse producers will not merely maximize expected output but will take account of the impact of risk. The question then is how to adjust expected output so that a more meaningful measure of performance under uncertainty can be achieved. This can be approached by asking what level of certain output would provide the producer with the same utility as that of an uncertain prospect which has mean $E(y|x,t)$ but positive variance. The literature on risk provides an answer to this question through the concepts of certainty equivalent and risk premium.

We begin by assuming that producers have access to an input *x* (which, for notational ease, is assumed again to be a scalar) which can give rise to a conditional distribution of output with expected value $E(y|x,t)$ and variance $s^2(y|x,t)$. Then, we can define the Certainty Equivalent (CE) in output terms as the level of riskless output which would provide the producer with the same level of utility as the expected utility associated with the risky output.[11] The difference between the expected output under uncertainty and its Certainty Equivalent can be interpreted as the amount of output that the producer is willing to forego to avoid the risk, i.e. the cost of risk in output terms. This cost of risk is referred to as the Risk Premium (RP) and is defined as:

$$RP = E(y\,|\,x,t) - CE \tag{8}$$

Following Pratt (1964), the risk premium is approximated as:

$$RP \approx \frac{1}{2}\sigma^2(y\,|\,x,t)\cdot r \tag{9}$$

where $r = -U''\left[E(y\,|\,x,t)\right]/U'\left[E(y\,|\,x,t)\right]$ is the Arrow-Pratt measure of *absolute* risk aversion. In the analysis which follows, it will prove useful to consider the *proportional* risk premium, i.e. the fraction of output that the producer is willing to forego in order to avoid uncertainty. For a small variance, this is approximated by

$$\frac{RP}{E(y\,|\,x,t)} \approx \frac{1}{2}\left[\frac{\sigma(y\,|\,x,t)}{E(y\,|\,x,t)}\right]^2 \cdot r_R \tag{10}$$

where $r_R = rE(y\,|\,x,t)$ is the coefficient of *relative* risk aversion.[12] Given that the risk premium measures the perceived cost of risk to the producer, it should be subtracted from the expected output in order to provide a more meaningful performance measure. Clearly, the larger the risk premium, the stronger is the negative impact of risk on producers and the worse is his perception of his performance.[13] If the risk premium is zero, on the other hand, then risk plays no role in the producer's valuation, and he will be concerned only with the expected output.

[11] That is, $U(CE)=E[U(y|x,t)]$. Note that we are assuming the existence of a well defined utility function U(*y*).
[12] Thus, $RP/E(y|\,x,\,t)$ is proportional to the square of the coefficient of variation.
[13] From the definition of RP, this increased negative impact may be due to either an increase in risk, an increase in the producer's aversion to risk, or a combination of both factors.

In line with the above, an appropriate index of TFPU can be expressed as:

$$TFPU = \frac{E(y \mid x,t) - RP}{x} = \frac{CE}{x} \tag{11}$$

where the substitution of expected output by the certainty equivalent brings with it the incorporation of the degree of risk and risk preferences to our measure. To express this index in detail, we begin with the identity:

$$\frac{CE}{x} \equiv \frac{E(y \mid x,t)}{x} \frac{CE}{E(y \mid x,t)} \tag{12}$$

Using (8) and substituting for the proportional risk premium from (10)

$$\frac{CE}{E(y \mid x_t t)} = 1 - \frac{1}{2} \cdot \left[ \frac{ó(y \mid x,t)}{E(y \mid x,t)} \right]^2 \cdot r_R \tag{13}$$

Thus, substituting (13) into (12), yields the following index of TFP under uncertainty:

$$TFPU = \frac{E(y \mid x,t)}{x} \cdot \left[ 1 - \frac{1}{2} m^2 r_R \right] \tag{14}$$

where $m = s(y|x,t)/E(y|x,t)$ and the expression in brackets represents the certainty equivalent output expressed as a proportion of expected output. It can be seen that TFPU is expressed in terms of an adjustment to the certainty TFP index, with the direction and magnitude of this adjustment depending on the size of relative risk (as represented by the coefficient of variation, $m$) and the nature of the producer’s preferences towards risk. The latter is reflected in the sign and magnitude of the coefficient of relative risk aversion, $r_R$. As this coefficient can take values greater, less than, or equal to zero whenever the producer is risk averse, risk loving, or risk neutral respectively, the CE expressed as a proportion of expected output can be greater, less than or equal to one, and hence TFPU can be greater, less than or equal to the certainty TFP index.

It is worth emphasizing that equation (14) can still be interpreted as a ratio between output and input levels, i.e. as a traditional average productivity measure. In particular, the numerator in (14) times the expression in brackets can be viewed as an “aggregate” output that incorporates the various facets of the productive process discussed earlier.[14] That is, this “aggregate” output reflects not only the expected output level (like a certainty TFP index) but also the existence of production risk. This risk, in turn, is evaluated taking into account the degree of risk aversion of the producer. Changes in (14) will thus accurately reflect changes in producer performance as changes in risk and/or producer preferences towards this risk will be accounted for.

In accordance with the discussion surrounding Figure 2, in the presence of production uncertainty, a producer who is risk averse and has an expected output $E_t(y|x_t)$ from the input vector $x_t$ will receive a lower valuation under TFPU than he would receive under TFP, reflecting the negative influence of risk. In the absence of risk ($m=0$) or if the producer is risk neutral ($r_R = 0$), the valuation of productive performance is the same.

While the level of TFPU is in itself a valuable piece of data, more often than not the rate of growth of productivity and its decomposition is of more interest.

To analyse TFPU growth, we express (14) in logarithmic terms:

$$ln\,TFPU = \left[ln\,E(y \mid x,t) - ln\,x\right] + ln\left[1 - 1/2 \cdot m^2 r_R\right] \tag{15}$$

Differentiating (15) with respect to time,

$$T\dot{F}PU = \left[\dot{E}(y \mid x,t) - \dot{x}\right] + \frac{d\ln\left[1 - 1/2 \cdot m^2 r_R\right]}{dt} \tag{16}$$

or, substituting from (6):

$$T\dot{F}PU = T\dot{F}P + \frac{d\ln\left[1 - 1/2 \cdot m^2 r_R\right]}{dt} \tag{17}$$

Carrying out the differentiation, (17) can be expressed as:

$$T\dot{F}PU = T\dot{F}P - r_R m^2 \left(1 - 1/2 \cdot m^2 r_R\right)^{-1} \cdot \left\{1/2 \cdot \dot{r}_R + \dot{m}\right\} \tag{18}$$

The growth of TFPU is thus expressed in terms of an adjustment to TFP growth, where the adjustment takes the form of a term involving the growth of risk and changes in preferences weighted by the initial levels of risk aversion, magnitude of risk, and the expected output as a proportion of the CE.

[14] Recall that, in (14), the product of the term in brackets and the numerator is the certainty equivalent output, and hence denotes a level of output.

Briefly, it can be seen that (18) complies with the properties regarding TFP growth outlined in Section 3. If the producer is risk neutral ($r_R=0$), then the second term on the right-hand side disappears and productivity growth under uncertainty and certainty coincide. Moreover, if there is no production risk ($\mu=0$) the certainty and uncertainty measures again coincide. Assume now that the producer is risk averse so that $r_R>0$. Then, for a given production risk ($\mu>0$), an increase in risk aversion ($\dot{r}_R>0$) causes the second term on the right-hand side to be negative and $\dot{TFPU} < \dot{TFP}$. An increase in production risk ($\dot{\mu}>0$) will have the same effect as the second term will again be negative.

The decomposition above can be further extended if we specify a form for the stochastic production function according to Just and Pope (1978). These authors suggested postulates for a stochastic production function, and introduced a production function that accommodates both risk-increasing and risk-decreasing intputs. The Just and Pope production function has the general form:

$$y = f(x,t) + h(x,t)\cdot v \tag{19}$$

where again we assume strict exogeneity and conditional homoskedasticity on the random noise term ($v$).Thus, the conditional output standard error can be written as:

$$\sigma(y \mid x,t) = h(x,t)\cdot \sigma_v \quad where \quad \sigma_v = \mathrm{var}(v \mid x,t)^{1/2} \tag{20}$$

Taking logs in (20) and differentiating with respect to time, we get that the increase in (relative) production risk can be decomposed as:

$$\dot{\mu} = (\eta - \varepsilon)\cdot \dot{x} + (\eta_t - \varepsilon_t) \tag{21}$$

where $\eta$ is the elasticity of $h(x,t)$ with respect to the input, and $\eta_t$ represents the increase in variance caused by changes in technology.[15] Note that input $x$ is risk increasing (reducing) if $\eta>(<)0$. It is thus made explicit that changes in input usage not only affect expected output (as captured in the traditional productivity growth under certainty) but also may affect the variance (and hence the risk) of output, and a risk averse producer will take both considerations into account when planning input use. In addition, the fact that technical change not only affects expected output but also the

[15] This decomposition can be easily extended to account for multiple inputs as:

$$\dot{\mu} = \sum_{k=1}^{K} (\eta_k - \varepsilon_k)\cdot \dot{x}_k + (\eta_t - \varepsilon_t)$$

where $\eta_k$ and $\varepsilon_k$ are respectively the elasticity of *h(·)* and *f(·)* with respect to the *k*th input.

riskiness of output is also explicit in (21) and will again be taken into account by the producer.

Finally, introducing equations (7) and (21) into (18) we get the *overall* decomposition of total factor productivity growth under production risk:

$$T\dot{F}PU = (\varepsilon - 1)\cdot \dot{x} + \varepsilon_t - r_R m^2 \left(1 - 1/2 \cdot m^2 r_R\right)^{-1} \cdot \left\{1/2 \cdot \dot{r}_R + (\eta - \varepsilon)\cdot \dot{x} + (\eta_t - \varepsilon_t)\right\} \tag{22}$$

where the contributions of scale effects and technical change to both expected output and output risk are made explicit.

Before concluding, a few remarks on the empirical calculation of the index of TFP growth under uncertainty in (22) should be made. Clearly, a discrete approximation of the index must be made if an empirical implementation is to be carried out. In Appendix B we show how (22) can easily be expressed in discrete time under the assumption that the deterministic production function follows a quadratic form. This discrete-time approximation can be expressed as:

$$\begin{aligned}\Delta lnTFPU_{t,t+1} = {} & (\hat{\varepsilon}-1)\cdot\sum_{k=1}^{K}\bar{e}_k \cdot \Delta lnx_{kt,t+1} + \hat{\varepsilon}_t \\ & - \bar{r}_R \bar{m}^2\left(1-1/2\cdot\bar{m}^2\bar{r}_R\right)^{-1}\cdot\left\{1/2\cdot\Delta lnr_{Rt,t+1} + \sum_{k=1}^{K}\left(\hat{\eta}_k - \hat{\varepsilon}_k\right)\cdot\Delta lnx_{kt,t+1} + (\hat{\eta}_t - \hat{\varepsilon}_t)\right\}\end{aligned} \tag{23}$$

where $\Delta$ stands for first differences over time; bars over variables represent their arithmetic mean over time; and the remaining variables, with hats, represent the discrete time counterparts of the elasticities and derivatives in (22).

Calculation of the index then basically boils down to obtaining estimates of three components: expected output, the variance of output (and hence the standard error), and producer risk preferences. Estimation of the Just-Pope stochastic production function will provide estimates of the two first components and it only remains therefore to find estimates of the coefficient of relative risk aversion. The most problematic is the coefficient of relative risk aversion as this generally requires the estimation of a utility function, the form of which will depend on the assumptions made regarding the nature of producers' preferences and in particular how these vary with increases in output.

The easiest way to approach this is to work under the assumption that preferences exhibit constant relative risk aversion (CRRA). Under this assumption, the coefficient of relative risk aversion is independent of the level of output. That is, the proportion of output that producers are willing to sacrifice to avoid risk is constant for all levels of output.[16] As this coefficient is independent of output, estimates of it can be taken from other studies. Simulation exercises can therefore be carried out to estimate the index under different assumptions/conjectures on the degree of risk aversion of producers. More precise assumptions could be made on the basis of a more in depth knowledge of the firms in the sector under review: for example, if we had access to insurance data, we could pinpoint the degree of risk aversion of the producers with more accuracy, perhaps even discriminating between producers depending on the quality of information. Alternatively, a detailed analysis of the relative usage of inputs could provide information about producer preferences, in that risk averse producers will overuse (underuse) risk reducing (increasing) inputs. An analysis of the intensity with which producers use these different categories of inputs may then provide clues about their degree of risk aversion. In any case, $r_R$ will simply be replaced by the appropriate value of $\theta$ and $\dot{r}_R$ will denote any exogenous changes in $\theta$ over the sample period that we may believe appropriate. If CRRA is not assumed, the $r_R$ will be a function of y and $\dot{r}_R$ will be a function of how $y$ changes over time, and their calculation will depend on the alternative functional form given to the utility function. This functional form will then have to be chosen depending on the risk assumptions seen as appropriate by the researcher.

## 5. Conclusions

Focusing on physical production, in this paper we extend previous work on productivity measurement in order to incorporate the impact of risk on productivity growth, thereby providing a more accurate picture of production performance. In this context, we outline certain desirable characteristics that we believe an index of total factor productivity under uncertainty should have if it is to capture the impact of risk on the production performance perceived by the producer. Drawing on familiar concepts from the literature on the measurement of risk aversion, a total factor productivity index under uncertainty is proposed. This index depends on data on input and output levels and preferences, and is decomposed in order to isolate the contributions of changes in

[16] The CRRA assumption is consistent with the producer's utility function $U(y)=y^{1-q}/(1-q)$, from which it is easily seen that $r_R=q$ . It is worth noting that the coefficient of absolute risk aversion under this form is $r=q\ /y$. Thus, CRRA implies decreasing absolute risk aversion in that the producer's absolute risk aversion coefficient decreases as output grows.

scale (inputs) and technical change on both the expected and risky facets of production process. We finish with some suggestions as to how the index can calculated, and it is shown that if the assumption of constant relative risk aversion is made, the index is easily calculable.

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## Appendix A. Decomposing TFP growth under certainty

As noted in Section 1, the standard framework for estimating (decomposing) productivity change under certainty is derived from the deterministic production function:

$$y = f(x,t) \tag{A1}$$

As customary, a total factor productivity index can be obtained by logarithmically differentiating (A1) to obtain

$$\dot{y} = \sum_{k=1}^{K} \boldsymbol{e}_k \, \dot{x}_k + \boldsymbol{e}_t \tag{A2}$$

where $e_k$ is the elasticity of output with respect to input $k$ and $e_t$ is the rate of technical change, that is:

$$\varepsilon_k = \frac{\partial \ln f(x,t)}{\partial \ln x_k} \quad , \quad k = 1,\ldots,K \tag{A3}$$

$$\varepsilon_t = \frac{\partial \ln f(x,t)}{\partial t} \tag{A4}$$

Taking into account that $\dot{y} = \dot{f}(x,t) = \dot{E}(y \mid x,t)$, we can rewrite expression (A2) as

$$\dot{E}(y \mid x,t) - \sum_{k=1}^{K} \boldsymbol{e}_k \dot{x}_k = \boldsymbol{e}_t \tag{A5}$$

The left-hand side can be viewed as an index of total factor productivity, defined as the difference between the growth of (expected) output and the weighted average rates of growth of inputs. Using input elasticities as weights for aggregating the rate of growth of inputs, (A5) measures exclusively the effect of technical change on (expected) output. The expression above can be extended to allow for the effect of non-constant returns to scale. This can be accomplished by aggregating the growth of inputs using input elasticities shares rather than input elasticities. Defining the elasticitiy share of the $k$th input, $e_k$, as

$$e_k = \frac{\boldsymbol{e}_k}{\boldsymbol{e}} = \frac{\boldsymbol{e}_k}{\sum_{k=1}^{K} \boldsymbol{e}_k} \tag{A6}$$

total factor productivity growth in (6) can be rewritten in the case of multiple inputs as:

$$\dot{TFP} = \dot{E}(y \mid x,t) - \sum_{k=1}^{K} e_k \dot{x}_k \tag{A7}$$

Using expression (A2), and after some manipulation, equation (A7) can then be decomposed into two terms:

$$\dot{TFP} = (\varepsilon - 1)\sum_{k=1}^{K} e_k \dot{x}_k + \varepsilon_t = (\varepsilon - 1)\dot{X} + \varepsilon_t \qquad \text{(A8)}$$

Note, finally, that the TFP growth rate in (A8) collapses to

$$\dot{TFP} = (\varepsilon - 1)\dot{x} + \varepsilon_t \qquad \text{(A9)}$$

when output is produced using a unique input.

## Appendix B. Discrete-time specification of the overall TFPU decomposition

We begin with the discrete-time counterpart of the TFP growth decomposition in (A8), which is the multiple-input version of (7). Assume that the deterministic production function follows a quadratic form, that is:

$$E_t(y \mid x_t, t) = f(x_t, t) = \boldsymbol{b}_0 + \sum_{k=1}^{K} \boldsymbol{b}_k x_{x_{kt}} + \frac{1}{2}\sum_{k=1}^{K}\sum_{h=1}^{K} \boldsymbol{b}_{kh} x_{x_{kt}} x_{ht} + \boldsymbol{b}_t t + \boldsymbol{b}_{tt} t^2 + \sum_{k=1}^{K} \boldsymbol{b}_{kt} x_{x_{kt}} t \qquad \text{(B1)}$$

Since $f(x_t,t)$ is Quadratic it is possible to apply Diewert's (1976) Quadratic Identity Lemma. Using this identity, changes in (expected) output from time *t* to time *t+1* can be written as:

$$f(t+1) - f(t) = \sum_{k=1}^{K} \frac{1}{2}\left[\frac{\partial f(t)}{\partial x_k} + \frac{\partial f(t+1)}{\partial x_k}\right] \cdot \left(x_{kt+1} - x_{kt}\right) + \frac{1}{2}\left[\frac{\partial f(t)}{\partial t} + \frac{\partial f(t+1)}{\partial t}\right] \qquad \text{(B2)}$$

where $f(t)$ is short for $f(x_t,t)$. This difference can be written in terms of rate of growth as:

$$\frac{\text{Ä}f_{t,t+1}}{\bar{f}} = \sum_{k=1}^{K} \mathring{\text{â}}_k \cdot \frac{\text{Ä}x_{kt,t+1}}{\bar{x}_k} + \mathring{\text{â}}_t \qquad \text{(B3)}$$

where $\Delta$ stands for first differences, bars represent the arithmetic mean over the periods *t* and *t + 1*, and

$$\mathring{\text{â}}_k = \frac{1}{2}\left(\frac{\partial f(t)}{\partial x_k} + \frac{\partial f(t+1)}{\partial x_k}\right) \frac{\bar{x}_k}{\bar{f}} \qquad \text{(B4)}$$

$$\mathring{\text{â}}_t = \frac{1}{2}\left(\frac{\partial f(t)}{\partial t} + \frac{\partial f(t+1)}{\partial t}\right) \cdot \frac{1}{\bar{f}} \qquad \text{(B5)}$$

Taking into account that

$$\Delta \ln E_{t,t+1}(y \mid x) \equiv \Delta \ln f_{t,t+1} \approx \frac{\Delta f_{t,t+1}}{\bar{f}} \qquad \text{(B6)}$$

$$\Delta \ln x_{kt,t+1} \approx \frac{\Delta x_{kt,t+1}}{\bar{x}}, \qquad \text{(B7)}$$

and defining the input elasticities shares as:

$$\widehat{e}_k = \frac{\mathring{\text{â}}_k}{\mathring{\text{â}}} \quad , \quad \mathring{\text{â}} = \sum_{k=1}^{K} \mathring{\text{â}}_k \qquad \text{(B8)}$$

total factor productivity growth can be rewritten in the case of multiple inputs as:

$$\Delta lnTFP_{t,t+1} = \Delta ln\, E_{t,t+1}(\, y \mid x\,) - \sum_{k=1}^{K} \hat{\varepsilon}_k \cdot \Delta ln\, x_{kt,t+1} \qquad \text{(B9)}$$

Using expression (B3), and after some manipulation, equation (B13) can be decomposed into two terms:

$$\Delta lnTFP_{t,t+1} = (\,\hat{\beta} - 1) \cdot \sum_{k=1}^{K} \hat{\varepsilon}_k \cdot \Delta ln\, x_{kt,t+1} + \hat{\beta}_t \qquad \text{(B10)}$$

which is the discrete counterpart of (7) when there are multiple inputs.

Next, we suggest a discrete specification for the rate of growth of the production risk measure in (21) in the case of several inputs. In the Just and Pope formulation, the rate of growth of the output standard error from period t to t+1 can be written as:

$$\Delta \ln \sigma_{t,t+1}(y \mid x) = \ln h(t+1) - \ln h(t) \qquad \text{(B11)}$$

where *h(t)* is short for $h(x_t,t)$. We follow Harvey (1976) and assume an exponential specification for the variance function in order to ensure positive output variances. In particular, this variance function is of the form:

$$h(t) = \exp\left\{ \alpha_0 + \sum_{k=1}^{K} \alpha_k x_{kt} + \frac{1}{2} \sum_{k=1}^{K} \sum_{h=1}^{K} \alpha_{kh} x_{kt} x_{ht} + \alpha_t t + \alpha_{tt} t^2 + \sum_{k=1}^{K} \alpha_{kt} x_{kt} t \right\} \qquad \text{(B12)}$$

Applying again the Quadratic Identity Lemma to the logarithm of (B12), changes in ln*h(·)* from time *t* to time *t+1* can be written as:

$$\ln h(t+1) - \ln h(t) = \sum_{k=1}^{K} \frac{1}{2} \left[ \frac{\partial \ln h(t)}{\partial x_k} + \frac{\partial \ln h(t+1)}{\partial x_k} \right] \cdot \left( x_{kt+1} - x_{kt} \right) + \frac{1}{2} \left[ \frac{\partial \ln h(t)}{\partial t} + \frac{\partial \ln h(t+1)}{\partial t} \right] \qquad \text{(B13)}$$

Inserting (B13) in (B11) and using (B7), the rate of growth of the output standard error can be written as:

$$\Delta ln\, \sigma_{t,t+1}(\, y \mid x\,) = \sum_{k=1}^{K} \hat{\eta}_k \cdot \Delta ln\, x_{kt,t+1} + \hat{\eta}_t \qquad \text{(B14)}$$

where

$$\hat{\eta}_k = \frac{1}{2} \left( \frac{\partial\, ln\, h(\, t\,)}{\partial x_k} + \frac{\partial\, ln\, h(\, t+1)}{\partial x_k} \right) \cdot \overline{x}_k \qquad \text{(B15)}$$

$$\hat{\eta}_t = \frac{1}{2}\left(\frac{\partial \ln h(t)}{\partial t} + \frac{\partial \ln h(t+1)}{\partial t}\right) \tag{B16}$$

Using (B3) to (B7), and (B14), the discrete counterpart of (21) then becomes:

$$\Delta \ln m_{t,t+1} = \sum_{k=1}^{K} \left(\hat{\eta}_k - \hat{\varepsilon}_k\right) \cdot \Delta \ln x_{kt,t+1} + (\hat{\eta}_t - \hat{\varepsilon}_t) \tag{B21}$$

We finish with the discrete-time counterpart of the decomposition introduced in equation (22), but allowing for multiple inputs. This decomposition can be expressed in discrete terms by first substituting the certainty TFP decomposition with (B14) and the increase in production risk with (B21), and then substituting the remainder instantaneous variables in (22) by the arithmetic mean over two time periods. This yields:

$$\begin{aligned}\Delta \ln TFPU_{t,t+1} = {} & (\hat{\varepsilon} - 1) \cdot \sum_{k=1}^{K} \bar{e}_k \cdot \Delta \ln x_{kt,t+1} + \hat{\varepsilon}_t \\ & - \bar{r}_R \bar{m}^2 \left(1 - 1/2 \cdot \bar{m}^2 \bar{r}_R\right)^{-1} \cdot \left\{1/2 \cdot \Delta \ln r_{Rt,t+1} + \sum_{k=1}^{K} (\hat{\eta}_k - \hat{\varepsilon}_k) \cdot \Delta \ln x_{kt,t+1} + (\hat{\eta}_t - \hat{\varepsilon}_t)\right\}\end{aligned} \tag{B22}$$

**Figure 1**. Productivity decomposition under certainty

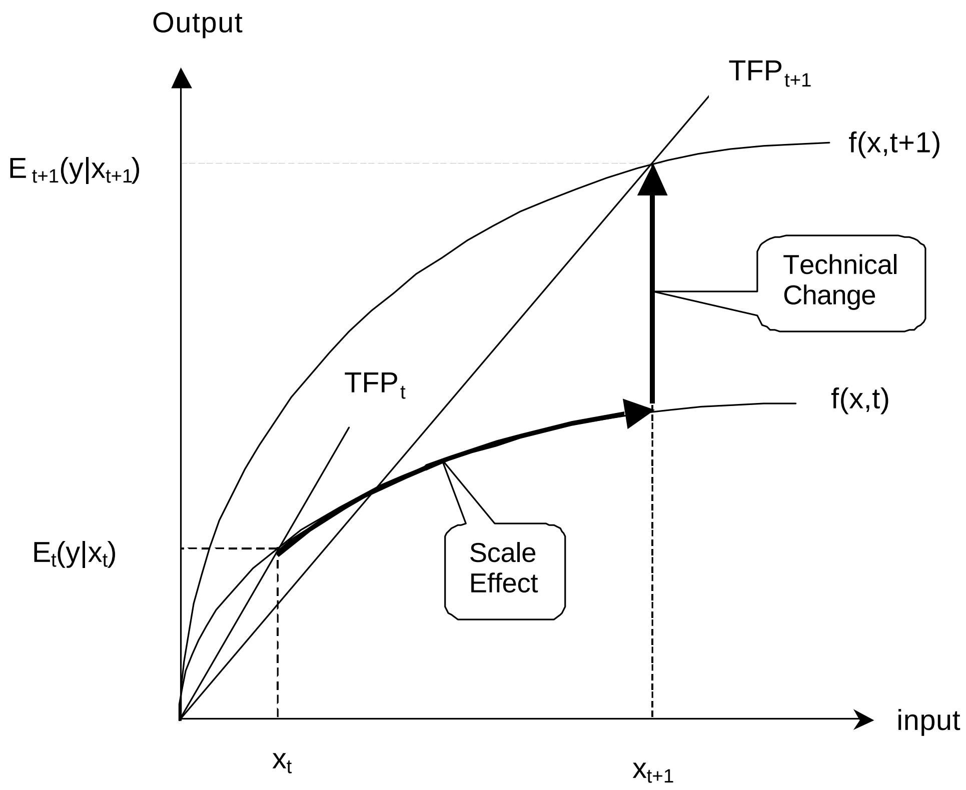


**Figure 2**. Productivity and risk